Discussiones Mathematicae
Graph Theory 35 (2015) 387–398
doi:10.7151/dmgt.1794

GRAPHS WITH 4-RAINBOW INDEX 3 AND $n-1$

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Abstract

Let G be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A tree T in G is called a *rainbow tree* if no two edges of T receive the same color. For a vertex set $S \subseteq V(G)$, a tree that connects S in G is called an S-*tree*. The minimum number of colors that are needed in an edge-coloring of G such that there is a rainbow S-tree for every set S of k vertices of $V(G)$ is called the k-*rainbow index* of G, denoted by $rx_k(G)$. Notice that a lower bound and an upper bound of the k-rainbow index of a graph with order n is $k-1$ and $n-1$, respectively. Chartrand *et al.* got that the k-rainbow index of a tree with order n is $n-1$ and the k-rainbow index of a unicyclic graph with order n is $n-1$ or $n-2$. Li and Sun raised the open problem of characterizing the graphs of order n with $rx_k(G) = n-1$ for $k \geq 3$. In early papers we characterized the graphs of order n with 3-rainbow index 2 and $n-1$. In this paper, we focus on $k = 4$, and characterize the graphs of order n with 4-rainbow index 3 and $n-1$, respectively.

Keywords: rainbow S-tree, k-rainbow index.

2010 Mathematics Subject Classification: 05C05, 05C15, 05C75.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [1]. Let G be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A path of G is a *rainbow path* if any two edges of the path have distinct colors. G is *rainbow connected* if any two vertices of G are connected by a rainbow path. The minimum number of colors required to make G rainbow connected is called its *rainbow connection number*, denoted by $rc(G)$. Results on the rainbow connectivity can be found in [2, 3, 4, 5, 6, 10, 11].

These concepts were introduced by Chartrand *et al.* in [4]. In [7], they generalized the concept of rainbow path to rainbow tree. A tree T in G is called a *rainbow tree* if no two edges of T receive the same color. For $S \subseteq V(G)$, a *rainbow S-tree* is a rainbow tree that connects S. Given a fixed integer k with $2 \leq k \leq n$, the edge-coloring c of G is called a *k-rainbow coloring* of G if, for every set S of k vertices of G, there exists a rainbow S-tree, and we say that G is *k-rainbow connected*. The *k-rainbow index $rx_k(G)$* of G is the minimum number of colors that are needed in a *k-rainbow coloring* of G. Clearly, when $k = 2$, $rx_2(G)$ is nothing new but the rainbow connection number $rc(G)$ of G. For every connected graph G of order n, it is easy to see that $rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$.

The *Steiner distance $d_G(S)$* of a set S of vertices in G is the minimum size (number of edges) of a tree in G that connects S. Such a tree is called a *Steiner S-tree* or simply an *S-tree*. The *k-Steiner diameter $sdiam_k(G)$* of G is the maximum Steiner distance of S among all sets S with k vertices in G. Then there is a simple upper bound and lower bound for $rx_k(G)$.

Observation 1.1 [7]**.** *For every connected graph G of order $n \geq 3$ and each integer k with $3 \leq k \leq n$, we have $k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1$.*

It is easy to get the following observations.

Observation 1.2 [7]**.** *Let G be a connected graph of order n containing two bridges e and f. For each integer k with $2 \leq k \leq n$, every k-rainbow coloring of G must assign distinct colors to e and f.*

Observation 1.3 [8]**.** *Let G be a connected graph of order n, and H be a connected spanning subgraph of G. Then $rx_k(G) \leq rx_k(H)$.*

The following is an immediate consequence of the observations above. Namely, trees attain the upper bound of k-rainbow index, regardless of the value of k.

Proposition 1.4 [7]**.** *Let T be a tree of order $n \geq 3$. For each integer k with $3 \leq k \leq n$, $rx_k(T) = n - 1$.*

In [7], they also showed that the k-rainbow index of a unicyclic graph is $n-1$ or $n-2$.

Theorem 1.5 [7]. *If G is a unicyclic graph of order $n \geq 3$ and girth $g \geq 3$, then*

$$rx_k(G) = \begin{cases} n-2, & k = 3 \text{ and } g \geq 4; \\ n-1, & g = 3 \text{ or } 4 \leq k \leq n. \end{cases} \tag{1}$$

Notice that a lower bound and an upper bound of the k-rainbow index of a graph with order n is $k-1$ and $n-1$, respectively. In [10], the authors raised an open problem: for $k \geq 3$, characterize the graphs of order n with $rx_k(G) = n-1$. It is not easy to settle down the problem for general k. In [8] and [12], we characterized the graphs of order n with 3-rainbow index 2 and $n-1$, respectively. In this paper we mainly deal with the 4-rainbow index of graphs with order n. More specifically, characterize the graphs of order n whose 4-rainbow index is 3 and $n-1$, respectively.

2. Characterization of Graphs with $rx_4(G)$= 3

First we give a necessary and sufficient condition for $rx_4(G) = 3$. Note that if a connected graph of order 4 has three colors, then it has a rainbow spanning tree. Thus, the following lemma holds.

Lemma 2.1. *Let G be a connected graph of order n $(n \geq 4)$. Then $rx_4(G) = 3$ if and only if each induced subgraph of G with order 4 is connected and has three different colors.*

Next we give some necessary conditions for $rx_4(G) = 3$. By Lemma 2.1, it is easy to get the following proposition.

Proposition 2.2. *Let G be a graph of order n with $rx_4(G) = 3$, where $n \geq 5$. Then $\delta(G) \geq n-3$ and $\Delta(\overline{G}) \leq 2$. In other words, $\overline{G}$ is the union of some paths (may be trivial) and cycles.*

For fixed integers p, q, an edge-coloring of a complete graph K_n is called a (p,q)-*coloring* if the edges of every $K_p \subseteq K_n$ are colored with at least q distinct colors. Clearly, $(p,2)$-colorings are the classical Ramsey colorings without monochromatic K_p as subgraphs. Let $f(n,p,q)$ be the minimum number of colors needed for a (p,q)-coloring of K_n. In [9], Erdős and Gyárfás got that $f(10,4,3) = 4$, and so the following proposition holds.

Proposition 2.3. *Let G be a graph of order n with $rx_4(G) = 3$. Then $n \leq 9$.*

By Lemma 2.1 and Theorem 1.5, we get the following proposition.

Proposition 2.4. *Let G be a connected graph of order n $(n \geq 4)$ with $rx_4(G) = 3$. Then $\overline{G}$ contains neither C_4 nor C_5.*

When G is a graph of order 4, it is obvious that $rx_4(G) = 3$ if and only if G is connected. Hence, for the remaining values of n with $5 \leq n \leq 9$ we distinguish five cases.

Lemma 2.5. *Let G be a connected graph of order 5. Then $rx_4(G) = 3$ if and only if $\overline{G}$ is a subgraph of P_5 or $K_2 \cup K_3$.*

Proof. Let G be a graph with $rx_4(G) = 3$. By Proposition 2.2, it is easy to check that if $\overline{G}$ is not a subgraph of P_5 or $K_2 \cup K_3$, then $\overline{G}$ is isomorphic to C_4 or C_5, a contradiction by Proposition 2.4.

Conversely, by Observation 1.3, we need to provide an edge-coloring $C : E \to \{1, 2, 3\}$ of G when $\overline{G}$ is isomorphic to P_5 or $K_2 \cup K_3$. Suppose $\overline{G}$ is isomorphic to P_5, denote $V(\overline{G}) = \{v_1, \ldots, v_5\}$ and $E(\overline{G}) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5\}$. Set $c(v_1v_3) = 2$, $c(v_1v_4) = 1$, $c(v_1v_5) = 3$, $c(v_2v_4) = 3$, $c(v_2v_5) = 2$, $c(v_3v_5) = 1$. Suppose $\overline{G}$ is isomorphic to $K_2 \cup K_3$, denote $V(\overline{G}) = \{v_1, \ldots, v_5\}$ and $E(\overline{G}) = \{v_1v_2, v_2v_3, v_1v_3, v_4v_5\}$. Set $c(v_1v_4) = 1$, $c(v_1v_5) = 2$, $c(v_2v_4) = 2$, $c(v_2v_5) = 3$, $c(v_3v_4) = 3$, $c(v_3v_5) = 1$. It is easy to show that the two edge-colorings make G 4-rainbow connected. ■

Lemma 2.6. *Let G be a graph of order 6. Then $rx_4(G) = 3$ if and only if $\overline{G}$ is a subgraph of C_6 or $2K_3$.*

Proof. Let G be a graph with $rx_4(G) = 3$. By Proposition 2.2, if $\overline{G}$ is not a subgraph of C_6 or $2K_3$, then $\overline{G}$ contains C_4 or C_5, a contradiction by Proposition 2.4.

Conversely, by Observation 1.3, we need to provide an edge-coloring $C : E \to \{1, 2, 3\}$ of G when $\overline{G}$ is isomorphic to C_6 or $2K_3$. Suppose $\overline{G}$ is isomorphic to C_6, denote $V(\overline{G}) = \{v_1, \ldots, v_6\}$ and $E(\overline{G}) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$. Set $c(v_1v_3) = 2$, $c(v_1v_4) = 3$, $c(v_1v_5) = 1$, $c(v_2v_4) = 1$, $c(v_2v_5) = 2$, $c(v_2v_6) = 3$, $c(v_3v_5) = 3$, $c(v_3v_6) = 1$, $c(v_4v_6) = 2$. Suppose $\overline{G}$ is isomorphic to $2K_3$, denote $V(\overline{G}) = \{v_1, \ldots, v_6\}$ and $E(\overline{G}) = \{v_1v_2, v_1v_3, v_2v_3, v_4v_5, v_4v_6, v_5v_6\}$. Set $c(v_1v_4) = 3$, $c(v_1v_5) = 2$, $c(v_1v_6) = 1$, $c(v_2v_4) = 1$, $c(v_2v_5) = 3$, $c(v_2v_6) = 2$, $c(v_3v_4) = 2$, $c(v_3v_5) = 1$, $c(v_3v_6) = 3$. It is easy to show that the two edge-colorings make G 4-rainbow connected. ■

It is a tedious work to check whether a graph is 4-rainbow connected when $7 \leq n \leq 9$. Hence we introduce an algorithm with the idea of backtracking to deal with such cases. Given a graph $G = (V(G), E(G))$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$, we color $E(G)$ with colors {1,2,3} in a proper order: at the beginning, consider the edge of the subgraph induced by $\{v_1, v_2\}$, namely the edge v_1v_2, and color it with 1 initially. Once all edges of the subgraph induced by $\{v_1, v_2, \ldots, v_s\}$ are colored, we come to deal with the new edges of the larger subgraph by adding v_{s+1} to the former one. For a new edge e, we color it with 1, 2 or 3, and if the subgraph induced by the vertices incident with already colored edges is 4-rainbow connected, we go on to the next edge of e. Otherwise if all 1, 2 and 3 are not available, we go back to the former edge of e and give it a new color and repeat the procedure. Clearly, the procedure always terminates. We should point out that the algorithm has a good performance when $n \leq 9$, although the time complexity is not polynomial. In fact, we need the algorithm only to test whether four graphs have 4-rainbow colorings in the following three lemmas.

Algorithm The 4-rainbow coloring of a graph

Input: a graph $G = (V, E)$ with $V = \{v_1, v_2, \dots, v_n\}$, $E = \{e_1, e_2, \dots, e_m\}$.
Output: give a 4-rainbow coloring *colorlist*$[m]$ of G, or verify that G has no 4-rainbow coloring.

1. reorder the edge sequence $e_1, e_2, \dots, e_m$, to make sure $E(G[v_1, \dots, v_t])$ $= \{e_1 \dots, e_s\}$, where s denotes the number of edges of $G[v_1, \dots, v_t]$, where $1 \leq t \leq n$.
2. fix the color of e_1 with 1. Initialize $i = 2$ and *colorlist* $= [1, 0, 0, \dots, 0]$;
3. while $i \geq 2$
    if $i > m$
        show *colorlist*; stop;
        colorlist$[i]$ = *colorlist*$[i] + 1$;
    if *colorlist*$[i] > 3$
        colorlist$[i] = 0$; $i - -$;
    else if **Boolean CHECK**(e_i)
        $i + +$;
4. there is no 4-rainbow coloring; stop.

Boolean CHECK(e_s)
Input: a graph $G = (V, E)$ with $V = \{v_1, v_2, \dots, v_n\}$, $E = \{e_1, e_2, \dots, e_m\}$ with the order described above. Set $e_s = (v_p, v_q)$, where $p < q$. Give a coloring of the first s edges of $E(G)$.
Output: determine whether the given coloring is not 4-rainbow.

1. for $i = 1$ up to $q - 2$ and $i \neq p$
    for $j = i + 1$ up to $q - 1$ and $j \neq p$
        if all edges of the induced subgraph $G[v_i, v_j, v_p, v_q]$ are colored but $G[v_i, v_j, v_p, v_q]$ is not 4-rainbow colored
            return $false$; stop;
2. return $true$; stop.

Lemma 2.7. *Let G be a graph of order 7. Then $rx_4(G) = 3$ if and only if $\overline{G}$ is a subgraph of C_6 or $2K_2 \cup K_3$ or $P_5 \cup K_2$ or $2K_3$.*

Proof. Let G be a graph with $rx_4(G) = 3$. By Proposition 2.2, if $\overline{G}$ is not a subgraph of C_6 or $2K_2 \cup K_3$ or $P_5 \cup K_2$ or $2K_3$, then by Proposition 2.4, $\overline{G}$ is isomorphic to $P_4 \cup P_3$ or $P_4 \cup K_3$ or P_7 or C_7. By Observation 1.3, we need only to verify that $rx_4(G) \neq 3$ when $\overline{G}$ is isomorphic to $P_4 \cup P_3$. By the algorithm, $rx_4(G) \neq 3$.

Conversely, by Observation 1.3 again, we need to provide an edge-coloring of G when $\overline{G}$ is isomorphic to C_6 or $2K_2 \cup K_3$ or $P_5 \cup K_2$ or $2K_3$. The four colorings are shown in Figure 1. It is easy to show that these four colorings make G 4-rainbow connected. ■

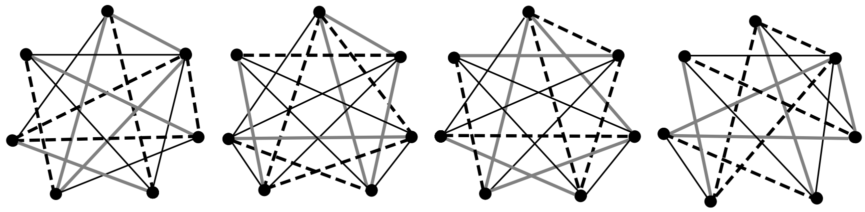

Figure 1. Graphs for Lemma 2.7 (lines of the same type have the same color).

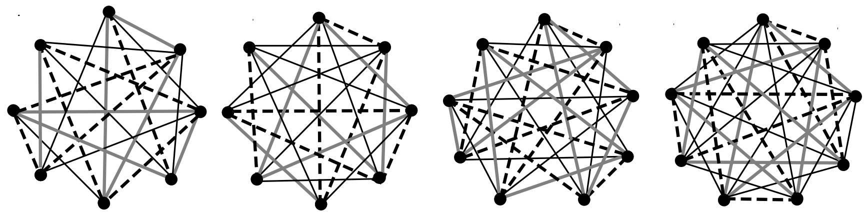

Figure 2. Graphs for Lemmas 2.8 and 2.9.

Lemma 2.8. *Let G be a graph of order 8. Then $rx_4(G) = 3$ if and only if $\overline{G}$ is a subgraph of $K_2 \cup 2K_3$ or $P_6 \cup K_2$.*

Proof. Let G be a graph with $rx_4(G) = 3$. By Proposition 2.2, if $\overline{G}$ is not a subgraph of $K_2 \cup 2K_3$ or $P_6 \cup K_2$, then by Proposition 2.4, it is easy to check that either $\overline{G}$ contains $P_4 \cup P_3 \cup K_1$ or $\overline{G}$ is isomorphic to $C_6 \cup 2K_1$. By Observation 1.3, we need to verify that $rx_4(G) \neq 3$ when $\overline{G}$ is isomorphic to $P_4 \cup P_3 \cup K_1$ or $\overline{G}$ is isomorphic to $C_6 \cup 2K_1$. If $\overline{G}$ is isomorphic to $P_4 \cup P_3 \cup K_1$, then by Lemma 2.7, $rx_4(G) \neq 3$. If $\overline{G}$ is isomorphic to $C_6 \cup 2K_1$, by the algorithm, $rx_4(G) \neq 3$.

Conversely, by Observation 1.3 again, we need to provide an edge-coloring of G when $\overline{G}$ is isomorphic to $K_2 \cup 2K_3$ or $P_6 \cup K_2$. The two edge-colorings are shown in the first two graphs of Figure 2. It is easy to show that the two edge-colorings make G 4-rainbow connected. ■

Lemma 2.9. *Let G be a graph of order 9. Then $rx_4(G) = 3$ if and only if $\overline{G}$ is a subgraph of $3K_3$ or $P_3 \cup 3K_2$.*

Proof. Let G be a graph with $rx_4(G) = 3$. By Proposition 2.2, if $\overline{G}$ is not a subgraph of $3K_3$ or $P_3 \cup 3K_2$, then by Proposition 2.4, it is easy to check that either $\overline{G}$ contains P_4 or $\overline{G}$ is isomorphic to $K_3 \cup 3K_2$. By Observation 1.3, we need to verify that $rx_4(G) \neq 3$ when $\overline{G}$ is isomorphic to P_4 or $K_3 \cup 3K_2$, by the algorithm, in each case, $rx_4(G) \neq 3$.

Conversely, by Observation 1.3 again, we need only to provide an edge-coloring of G when $\overline{G}$ is isomorphic to $3K_3$ or $P_3 \cup 3K_2$. The two edge-colorings are shown in the last two graphs of Figure 2. It is easy to show that the two edge-colorings make G 4-rainbow connected. ■

Combining the preceding five lemmas, we are ready to characterize the graphs whose 4-rainbow index is 3.

Theorem 2.10. *Let G be a connected graph of order $n \geq 4$. Then $rx_4(G) = 3$ if and only if G is one of the following graphs:*
(1) *G is a connected graph of order 4;*
(2) *G is of order 5 and $\overline{G}$ is a subgraph of P_5 or $K_2 \cup K_3$;*
(3) *G is of order 6 and $\overline{G}$ is a subgraph of C_6 or $2K_3$;*
(4) *G is of order 7 and $\overline{G}$ is a subgraph of C_6 or $2K_2 \cup K_3$ or $P_5 \cup K_2$ or $2K_3$;*
(5) *G is of order 8 and $\overline{G}$ is a subgraph of $K_2 \cup 2K_3$ or $P_6 \cup K_2$;*
(6) *G is of order 9 and $\overline{G}$ is a subgraph of $3K_3$ or $P_3 \cup 3K_2$.*

3. Characterization of Graphs with $rx_4(G) = n-1$

First of all, we need some notation and basic results.

Definition 3.1. Let G be a connected graph with n vertices and m edges. Define the *cyclomatic number* of G as $c(G) = m - n + 1$. A graph G with $c(G) = k$ is called a *k-cyclic* graph. According to this definition, if a graph G meets $c(G) = 0$, 1, 2 or 3, then G is called acyclic (or a tree), unicyclic, bicyclic, or tricyclic, respectively.

Definition 3.2. For a subgraph H of a connected graph G and $v \in V(G)$, let $d(v, H) = \min\{d_G(v, x) : x \in V(H)\}$.

Let G be a connected graph. To *contract* an edge $e = uv$ is to delete e and replace its ends by a single vertex incident to all the edges which were incident to either u or v. Let $G^{'}$ be the graph obtained by contracting some edges of G and suppose that the resulting graph $G^{'}$ is a simple graph. Given a rainbow coloring of $G^{'}$, when it comes back to G, every modified edge takes the following operation: assign the color of uv to uw and a new color to the edge wv if an edge uv of $G^{'}$ is expanded into two edges uw, wv between the ends of the contracted edge. Then G can be made to be 4-rainbow connected if G' is 4-rainbow connected. Hence, the following lemma holds.

Lemma 3.3. *Let G be a connected graph, and $G^{'}$ be a connected graph by contracting some edges of G. Then $rx_4(G) \leq rx_4(G^{'}) + |V(G)| - |V(G^{'})|$.*

The Θ-*graph* is a graph consisting of three internally disjoint paths with common end vertices and of lengths a, b, and c, respectively, such that $a \leq b \leq c$. It follows that if a Θ-graph has order n, then $a + b + c = n + 1$.

Let G be a connected graph of order n, to *subdivide* an edge e is to delete e, add a new vertex x, and join x to the ends of e. We will first give some sufficient conditions to make sure that the 4-rainbow index of G never attains the upper bound $n - 1$.

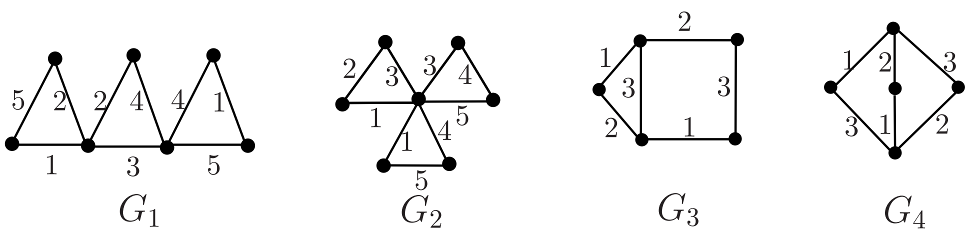

Figure 3. Graphs for Lemma 3.4.

Lemma 3.4. *Let G be a connected graph of order n. If G contains three edge-disjoint cycles, or a Θ-graph of order at least 5 as subgraphs, then $rx_4(G) \leq n-2$.*

Proof. Consider two graphs G_1, G_2 in Figure 3, and by checking the given edge-coloring in the figure, we have $rx_4(G_i) \leq |V(G_i)|-2$, $i = 1, 2$. Thus, if G contains three edge-disjoint cycles C_1, C_2, C_3,then we can extend the three triangles of G_1 or G_2 to C_1, C_2 and C_3 respectively by a sequence of operations of subdivision. Then add to the cycles an additional set of edges, to get a spanning subgraph G' of G. By Observation 1.3 and Lemma 3.3, we have $rx_4(G) \leq rx_4(G') \leq rx_4(G_i) + |V(G^{'})| - |V(G_i)| \leq n - 2$.

Let $\mathcal{G}$ be the set of Θ-graphs whose order is exactly 5. Then $\mathcal{G} = \{G_3, G_4\}$ (see Figure 3). By checking the given edge-coloring, we have $rx_4(G_i) \leq |V(G_i)| - 2$, $i = 3, 4$. Similarly, $rx_4(G) \leq n - 2$ follows. ∎

A graph G is a *cactus* if every edge is part of at most one cycle in G.

Lemma 3.5. *Let G be a cactus of order n and $c(G) = 2$. Then $rx_4(G) = n-1$.*

Proof. Let the two cycles of G be C^1 and C^2, where $C^1 = v_1v_2\cdots v_\ell v_1$, $C^2 = v'_1v'_2\cdots v'_{\ell'}v'_1$, the unique path connecting the two cycles be $v_iPv'_j$, where the two end-vertices v_i and v'_j may coincide. Suppose we have a color set C and $|C| = n-2$. Set $C = \{1, 2, \ldots, n-2\}$ and E_i is the set of edges colored with i, $c_i = |E_i|$, $1 \le i \le n-2$. Without loss of generality, we always set $c_1 \ge c_2 \ge \cdots \ge c_{n-2}$. Notice that $\sum_{i=1}^{n-2} c_i = n+1$. We distinguish the following cases.

Case 1. $c_1 = 4$, $c_2 = c_3 = \cdots = c_{n-2} = 1$. We have the following claim.

Claim 1. *No three edges of C^1 or C^2 have the same color.*

Proof. Suppose $c(v_1v_2) = c(v_pv_{p+1}) = c(v_qv_{q+1})$, where v_1v_2, v_pv_{p+1}, v_qv_{q+1} are three distinct edges. Let $S = \{v_1, v_p, v_q\}$. It is easy to check that any tree connecting S contains at least two edges of v_1v_2, v_pv_{p+1} and v_qv_{q+1}, this contradiction proves the claim. □

By Observation 1.2 and Claim 1, at least 3 edges of E_1 exist on cycles and each cycle has at most two of them. Suppose v_1v_2 and v_pv_{p+1} of C^1 have color 1, we distinguish two subcases: (1) there is a cut edge uu' in E_1. Suppose $d(u, C^1) \ge d(u', C^1)$ and $d(u, v_i) = d(u, C^1)$, where $2 \le i \le p$. Any tree connecting v_1 and u contains at least two edges colored with 1. (2) no cut edge has color 1. Then at least two edges, say $v'_1v'_2$ and $v'_qv'_{q+1}$ of C^2 have color 1, and the end-vertices of the path connecting C^1 and C^2 are v_i and v'_j, where $2 \le i \le p$, $2 \le j \le q$. Again, any tree connecting v_1 and v'_1 contains at least two edges in E_1.

Case 2. $c_1 = 3, c_2 = 2, c_3 = \cdots = c_{n-2} = 1$. We also have the following claim.

Claim 2. *No four edges of a cycle can have only two colors.*

Proof. Suppose otherwise four edges, v_1v_2, v_pv_{p+1}, v_qv_{q+1}, v_rv_{r+1} of C^1 have color a or b, where $a, b \in C$. Set $S = \{v_1, v_p, v_q, v_r\}$. It is easy to check that any tree connecting S contains at least three of the four edges above. By the Pigeon Hole Principle, one of the two colors occurs at least twice, a contradiction. □

By Claim 2, at most three edges of $C^i (i = 1, 2)$ can have colors 1 and 2. Notice that $|E_1 \cup E_2| = 5$. Since no two cut edges can have the same color, there are the following possibilities:

(1) three edges of $E_1 \cup E_2$ are in a cycle, say C^1. Then there exist cut edges in $E_1 \cup E_2$, or the other two edges of $E_1 \cup E_2$ are both in C^2. Similar to Case 1, we can choose three vertices such that no rainbow tree connects them.

(2) two edges of $E_1 \cup E_2$ are in each cycle. Then a cut edge uu' exists in $E_1 \cup E_2$. There are two situations according to the positions of uu' and the other four edges of $E_1 \cup E_2$ in cycles. We can always find three vertices such that any tree connecting them contains at least three edges of $E_1 \cup E_2$. (3) two edges of $E_1 \cup E_2$ are in one cycle, and other two of them are cut edges. The argument is similar, and it also produces a contradiction.

Case 3. $c_1 = c_2 = c_3 = 2, c_4 = \cdots = c_{n-2} = 1$. In a number of subcases similar to those in Cases 1 and 2, a set S of vertices can be found such that a tree connecting them contains at least four edges from $E_1 \cup E_2 \cup E_3$. So by the Pigeon Hole Principle again, one of the three colors occurs at least twice.

By the analysis above, all the possibilities of an $(n-2)$-coloring lead to a contradiction, thus we have $rx_4(G) \geq n-1$. On the other hand, by Observation 1.1, it follows that $rx_4(G) = n-1$. ■

To characterize all the graphs with 4-rainbow index $n-1$, we need to introduce more graphs. Let $\mathcal{G}_1$ be the set of graphs by identifying each vertex of K_4 with an end-vertex of an arbitrary path, and $\mathcal{G}_2$ be the set of graphs by identifying each vertex of $K_4 - e$ with the root of an arbitrary tree.

Lemma 3.6. *Let G be a connected graph of order n. If $G \in \mathcal{G}_1 \cup \mathcal{G}_2$, then $rx_4(G) = n-1$.*

Proof. Suppose $G \in \mathcal{G}_1$, and v_1, v_2, v_3 and v_4 are the four pendant vertices of G. We have $d_G(\{v_1, v_2, v_3, v_4\}) = n-1$. Combining with Observation 1.1, we have $rx_4(G) = n-1$. Let $G \in \mathcal{G}_2$. Denote by H the induced subgraph $K_4 - e$ of G, where $E(H) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_2v_4\}$ and denote by T_i the tree rooted at v_i, $i = 1, 2, 3, 4$. We have the following claim.

Claim 3. *No three edges of H share colors with the cut edges.*

Proof. Let $v_i'v_i'', 1 \leq i \leq 3$, be the cut edges whose colors exist in H. We may assume that $d(v_i', H) \geq d(v_i'', H)$. Notice that the deletion of any three edges of H disconnects G, and we will get some components. Let v be an arbitrary vertex of H in the component different from the one containing v_1'. Set $S = \{v, v_1', v_2', v_3'\}$. There is no rainbow tree connecting S, which verifies Claim 3. □

Now we are aiming to prove that H needs at least three new colors different from the $n-4$ colors of cut edges to make sure that G is 4-rainbow connected. Then we get the conclusion $rx_4(G) = n-1$. Since $rx_4(H) = 3$ and by Claim 3, one or two edges of H have the color of cut edges. Assume first that the colors of cut edges $v_1'v_1''$, $v_2'v_2''$ appear in H. Suppose $d(v_i', H) \geq d(v_i'', H)$, $i = 1, 2$. Since the deletion of two edges incident to a vertex of degree two disconnects H, the position of the two edges of H having the colors of cut edges may have the following possibilities: v_1v_4, v_2v_4 or v_1v_4, v_3v_4 or v_1v_2, v_3v_4. Notice that the remaining three edges can only have new colors. If only two colors are used, then at least two edges of H have the same color. It is easy to find two vertices v_i, v_j of H, such that no rainbow tree connects S, where $S = \{v_1', v_2', v_i, v_j\}$. Assume then only one edge of H has the color of cut edge, say $v_1'v_1''$ of T_i. Suppose $d(v_1', H) \geq d(v_1'', H)$. Then any tree connecting v_1' and the three vertices of H except v_i makes use of at least three edges of H, namely at least three new distinct colors are needed in H. Thus the result follows. ■

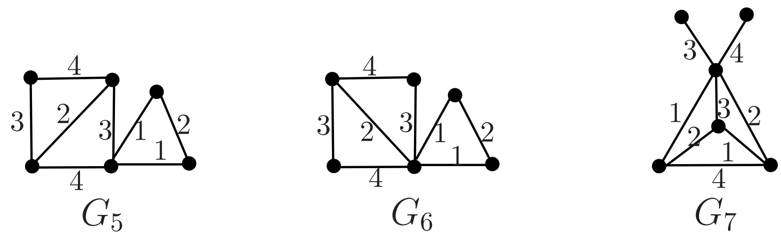

Figure 4. Graphs for Theorem 3.7.

Now we are prepared to characterize the graphs of order n whose 4-rainbow index is $n-1$.

Theorem 3.7. *Let G be a graph of order n. Then $rx_4(G) = n-1$ if and only if G is a tree, or a unicyclic graph, or a cactus with $c(G) = 2$, or $G \in \mathcal{G}_1 \cup \mathcal{G}_2$.*

Proof. By Lemma 3.3, 3.4, 3.5, 3.6, we only need to prove the necessity. Let G be a graph with $rx_4(G) = n-1$. By Proposition 1.4, Theorem 1.5, Lemma 3.4 and Lemma 3.5, we know that if G is not a tree or a unicyclic graph or a cactus with $c(G) = 2$, then G contains a K_4 or $K_4 - e$ as an induced subgraph. Now suppose that G contains a K_4 or $K_4 - e$ but $G \notin \mathcal{G}_1 \cup \mathcal{G}_2$. Consider the three graphs G_5, G_6, G_7 (see Figure 4). By checking the given coloring in Figure 4, we have $rx_4(G_i) \leq n-2$, $i = 5, 6, 7$. Thus we can extend G_5, G_6 or G_7 to get a spanning subgraph G' of G, then $rx_4(G) \leq rx_4(G') \leq n-2$, a contradiction. ■

Acknowledgement

The authors are very grateful to the referees for their valuable comments and suggestions which helped to improve the presentation of the paper. This work was supported by NSFC No.11371205 and PCSIRT.

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Received 14 January 2014
Revised 22 May 2014
Accepted 16 June 2014