# ON •-LINE SIGNED GRAPHS $L_{\bullet}(S)$ 

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#### Abstract

A signed graph (or sigraph for short) is an ordered pair $S=\left(S^{u}, \sigma\right)$, where $S^{u}$ is a graph, $G=(V, E)$, called the underlying graph of $S$ and $\sigma$ : $E \rightarrow\{+,-\}$ is a function from the edge set $E$ of $S^{u}$ into the set $\{+,-\}$. For a sigraph $S$ its $\bullet$-line sigraph, $L_{\bullet}(S)$ is the sigraph in which the edges of $S$ are represented as vertices, two of these vertices are defined adjacent whenever the corresponding edges in $S$ have a vertex in common, any such $L$-edge $e e^{\prime}$ has the sign given by the product of the signs of the edges incident with the vertex in $e \cap e^{\prime}$. In this paper we establish a structural characterization of --line sigraphs, extending a well known characterization of line graphs due to Harary. Further we study several standard properties of •-line sigraphs, such as the balanced •-line sigraphs, sign-compatible •-line sigraphs and $\mathcal{C}$-sign-compatible $\bullet$-line sigraphs.


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## 1. Introduction

For standard terminology and notation in graph theory we refer the reader to Harary [8] and West [19], and Zaslavsky [21, 22] for sigraphs. Throughout the paper, we consider finite, undirected graphs with no loops or multiple edges.

A signed graph (or sigraph for short; see [7]) is an ordered pair $S=\left(S^{u}, \sigma\right)$, where $S^{u}$ is a graph $G=(V, E)$, called the underlying graph of $S$ and $\sigma: E \rightarrow$ $\{+,-\}$ is a function from the edge set $E$ of $S^{u}$ into the set $\{+,-\}$, called the signature of $S$. The edges of $S$ with positive and negative signs are called positive edges and negative edges, respectively. In a pictorial representation of a sigraph $S$, when $S$ is small enough, its positive edges are shown as bold oriented line segments and negative edges as broken line segments. The positive (negative) degree of a vertex $v \in V(S)$ denoted by $d^{+}(v)\left(d^{-}(v)\right)$ is the number of positive (negative) edges incident with the vertex $v$ and $d(v)=d^{+}(v)+d^{-}(v)$. The edge degree $d_{e}\left(e_{j}\right)$ of an edge $e_{j}$ in a sigraph $S$ is the total number of edges adjacent to $e_{j}$ in $S$. If the end vertices of the edge $e_{j}$ are $u$ and $v$, then edge-degree of $e_{j}$ is defined as the number $d_{e}\left(e_{j}\right)=d(u)+d(v)-2$. A vertex is called pendent if its degree is one.

A sigraph is all-positive (all-negative) if all its edges are positive (negative); further, it is said to be homogeneous if it is either all-positive or all-negative and heterogeneous otherwise.

A marked sigraph is an ordered pair $S_{\mu}=(S, \mu)$ where $S=\left(S^{u}, \sigma\right)$ is a sigraph and $\mu: V(S) \rightarrow\{+,-\}$ is a function from the vertex set $V(S)$ of $S$ into the set $\{+,-\}$, called a marking of $S$. In particular, a sigraph $S=\left(S^{u}, \sigma\right)$ has a canonical marking or $\mathcal{C}$-marking, $\mu_{\sigma}$, defined for each vertex $v \in V(S)$ by $\mu_{\sigma}(v)=\prod_{e \in E_{v}} \sigma(e)$.

The line graph $L(G)$ of a graph $G$ is that graph whose vertex set can be put in one-to-one correspondence with the edge set of $G$, such that two $L$-vertices of $L(G)$ are adjacent if and only if the corresponding edges of $G$ are adjacent. The edges of the line graph $L(G)$ are called $L$-edges. The line graphs were first studied by Whitney [20] and the first characterization of line graphs in terms of complete subgraphs was obtained by Krausz [11]. In the literature, we find that different authors gave different name to line graphs; particularly, line graphs are termed as derivative (see [14]), interchange graph ([13]), adjoint ([12]), derived graph ([4]) and covering graph (see [10]). Harary and Norman [9] finally fixed the terminology by calling it a 'line graph'. A forbidden subgraph characterization of line graphs was established by Beineke [5]. The following theorem is the well known characterization of a line graph given in most of the standard text-books on graph theory (e.g., see Harary [8], Ch. 8, p. 74), originally due to Beineke [4].

Theorem 1 [8]. The following statements are equivalent:
(a) $G=(V, E)$ is a line graph.
(b) The edges of $G$ can be partitioned into some of its complete subgraphs in such a way that no vertex lies in more than two of the subgraphs.
(c) $G$ does not have $K_{1,3}$ as an induced subgraph, and if two odd triangles have a common edge then the subgraph induced by their vertices is $K_{4}$.
(d) None of the nine subgraphs shown in Figure 1 is an induced subgraph of $G$.

A triangle is said to be odd if there is a vertex in the graph adjacent to an odd number of vertices of the triangle.


Figure 1. Beineke's nine forbidden subgraphs for a line graph.
The $\bullet$-line sigraph $L_{\bullet}(S)$ of a sigraph $S$ is the line graph of $S=\left(S^{u}, \sigma\right)$, with each $L$-edge $e e^{\prime}\left(e, e^{\prime} \in E(S)\right)$ signed with $\mu_{\sigma}\left(e \cap e^{\prime}\right)$. There are two more notions of a 'signed line graph' of a given sigraph $S=\left(S^{u}, \sigma\right)$ in the literature, viz., $L(S)$ and $L_{\times}(S)$, both have $L\left(S^{u}\right)$ as their underlying graph; only the rule to assign signs to the edges of $L\left(S^{u}\right)$ differ. An $L$-edge $e e^{\prime}$ in $L(S)$ is negative if and only if both the edges $e$ and $e^{\prime}$ in $S$ are negative [3] and an $L$-edge $e e^{\prime}$ in $L_{\times}(S)$ has the product $\sigma(e) \sigma\left(e^{\prime}\right)$ as its sign [6].

## 2. Balance in •-Line Sigraphs

The sign of a cycle $Z$ in a sigraph $S$ is the product of the signs of all its edges and is denoted by $\theta(Z)$. A cycle in a sigraph $S$ is said to be positive (negative) if its sign is positive (negative). A sigraph $S$ is said to be balanced if and only if all its cycles are positive.

Harary [7] derived the following structural criterion called partition criterion for balance in sigraphs.

Theorem 2 [7]. A sigraph $S$ is balanced if and only if its vertex set $V(S)$ can be partitioned into two subsets $V_{1}$ and $V_{2}$, one of them possibly empty, such that every positive edge joins two vertices in the same subset and every negative edge joins two vertices from different subsets.

The following important lemma on balanced sigraphs is given by Zaslavsky.
Lemma 3 [23]. A sigraph in which every chordless cycle is positive, is balanced.
Now, the following theorem gives us the solution for $L_{\bullet}(S)$ to be balanced.
Theorem 4. For a sigraph $S, L_{\bullet}(S)$ is balanced if and only if the following conditions hold:
(i) for every cycle $Z$ in $S, Z$ has even number of negatively marked vertices and
(ii) for $v \in V(S)$, if $d(v)>2$, then $d^{-}(v) \equiv 0(\bmod 2)$.

Proof. Necessity: Suppose $L_{\bullet}(S)$ is balanced. Then, by definition of $L_{\bullet}(S)$, every $L$-cycle $Z^{\prime}$ in $L_{\bullet}(S)$ contains an even number of negative edges. Suppose there is any cycle in $S$ that has odd number of negatively marked vertices. Then, by the definition of $L_{\bullet}(S)$, there are odd number of negative $L$-edges in $L_{\bullet}(S)$, a contradiction to the hypothesis. Thus, (i) follows. Now, by the definition of $L_{\bullet}(S)$, any three edges of $S$ incident with $v$ are the $L$-vertices of an all-negative $L$-triangle in $L_{\bullet}(S)$, contrary to the hypothesis. Thus condition (ii) is necessary. Hence, both conditions are necessary.

Sufficiency: Suppose conditions (i) and (ii) hold for a given sigraph $S$. We shall show that $L_{\bullet}(S)$ is balanced. If $S$ is all-positive then, by definition, $L_{\bullet}(S)$ is also all-positive and hence, it is trivially balanced. Now, suppose that $L_{\bullet}(S)$ is not balanced. Then, we may assume the negative $L$-cycle $Z_{k}^{\prime}$ is of least possible length with this property. Let it be $\left(e_{1}, e_{2}, \ldots, e_{k}, e_{1}\right)$, where each $e_{i}$ is an $L$ vertex. Suppose it has a chord $e_{i} e_{j}$, then one of the $L$-cycles $\left(e_{i}, e_{i+1}, \ldots, e_{j}, e_{i}\right)$ or $\left(e_{j}, e_{j+1}, \ldots, e_{k}, e_{1}, \ldots, e_{i}, e_{j}\right)$ is negative, contrary to the least length assumption. So we may assume $Z_{k}^{\prime}$ chordless. Now for any graph $G$, a chordless $L$-cycle of $L(G)$ must consist either of three $L$-vertices corresponding to edges of $G$ incident with a single vertex, or of an $L$-cycle whose $L$-vertices are the edges of a chordless cycle of $G$. The result follows.

## 3. Sign-Compatibility and Canonical Sign-Compatibility of $L_{\bullet}(S)$

A sigraph $S=\left(S^{u}, \sigma\right)$ is sign-compatible [16] if it has a vertex marking $\mu$ such that each edge $e=v w$ has $\sigma(e)=-$ if and only if $\mu(v)=\mu(w)=-$. If the canonical marking $\mu_{\sigma}$ has this property, then $S$ is said to be canonically sign-compatible (or $\mathcal{C}$-sign-compatible.)

The conditions for a sigraph $S$ to have these properties are known (Theorems 5,6 and 8 below). In this section we establish the conditions for a $\bullet$-line sigraph $L_{\bullet}(S)$ to have each of these properties.

Theorem 5 [17]. A sigraph $S$ is sign-compatible if and only if there is a subset $W$ of $V(S)$ whose induced subsigraph has for its edge set exactly the negative edges of $S$.

Theorem 6 [17]. A sigraph $S$ is sign-compatible if and only if $S$ does not contain a subsigraph isomorphic to either of the two sigraphs, $S_{1}$ formed by taking the path $P_{4}=(x, u, v, y)$ with both the edges $x u$ and vy negative and the edge $u v$ positive and $S_{2}$ formed by taking $S_{1}$ and identifying the vertices $x$ and $y$ Figure 2.


Figure 2. Acharya and Sinha forbidden subsigraphs for a sign-compatible sigraph.
Now, we present the condition for a $\bullet$-line sigraph to be sign-compatible.
Theorem 7. Let $S=\left(S^{u}, \sigma\right)$ be a canonically-marked sigraph. Then $L \bullet(S)$ is sign-compatible if and only if $S$ has the following property: Let $e_{i}, e_{j}, e_{k}, e_{l} \in E(S)$ such that there are vertices $v, w, x$ (not necessarily distinct) with $e_{i} \cap e_{j}=v$, $e_{k} \cap e_{i}=w$ and $e_{l} \cap e_{j}=x$ and $\mu_{\sigma}(v)=+$. Then, either $\mu_{\sigma}(w)=+$ or $\mu_{\sigma}(x)=+$.

Proof. Necessity: Suppose $L_{\bullet}(S)$ is sign-compatible. Then, by Theorem 6, $L_{\bullet}(S)$ does not contain a subsigraph isomorphic to $S_{1}$ or $S_{2}$ in Figure 2.

Let $e_{i}$ and $e_{j}$ be two adjacent edges in $S$ and $v$ be the common vertex between them and $\mu_{\sigma}(v)=+$. Now, suppose that $e_{k}$ is adjacent with $e_{i}$ and $e_{l}$ is adjacent with $e_{j}$ and $w$ and $x$ are common vertices between them respectively. If possible, suppose the condition is false. Then, $\mu_{\sigma}(v)=+$ and $\mu_{\sigma}(w)=\mu_{\sigma}(x)=-$. If $e_{k}=e_{l}$ then, by the definition of $L_{\bullet}(S)$, we have an $L$-triangle with two negative and one positive $L$-edges in $L \bullet(S)$. Thus, we have a subsigraph isomorphic to $S_{2}$ in Figure 2 in $L_{\bullet}(S)$, a contradiction to the hypothesis. Now, suppose $e_{k} \neq e_{l}$, then we have an $L$-path $P_{4}^{\prime}=\left(e_{k}, e_{i}, e_{j}, e_{l}\right)$ in $L_{\bullet}(S)$ such that $e_{i} e_{j}$ is a positive $L$-edge while $e_{k} e_{i}$ and $e_{j} e_{l}$ are negative $L$-edges. Thus, we have a subsigraph
isomorphic to $S_{1}$ in Figure 2 in $L_{\bullet}(S)$, a contradiction to the hypothesis. Thus, in both the conditions we have a contradiction. Hence, the conditions are necessary.

Sufficiency: Suppose the condition in the statement of the theorem holds for a sigraph $S$. We want to show that $L_{\bullet}(S)$ is sign-compatible. Suppose to the contrary that $L_{\bullet}(S)$ contains a subsigraph isomorphic to $S_{1}$ or $S_{2}$.

Case I: Suppose $L_{\bullet}(S)$ contains a subsigraph, say $P_{4}{ }^{\prime}$, isomorphic to $S_{1}$. Let $P_{4}{ }^{\prime}=\left(e_{i}, e_{j}, e_{k}, e_{l}\right)$ be such that $e_{i} e_{j}$ and $e_{k} e_{l}$ are negative $L$-edges and $e_{j} e_{k}$ is a positive $L$-edge in $L_{\bullet}(S)$. Then, by the definition of $L_{\bullet}(S)$, there exists a vertex between $e_{j}$ and $e_{k}$ in $L_{\bullet}(S)$ such that it is positively marked in $S$ and the common vertices between $e_{i}, e_{j}$ and $e_{k}, e_{l}$ are marked negatively in $S$, a contradiction to the hypothesis. Thus, $L_{\bullet}(S)$ does not contain a subsigraph isomorphic to $S_{1}$.

Case II: Now, let $L_{\bullet}(S)$ contain a subsigraph isomorphic to $S_{2}$. Then this $L$-triangle is either due to the edges of a triangle or due to a vertex $v \in V(S)$ in $S$ with $d(v) \geq 3$. Let $Z^{\prime}$ be an $L$-triangle in $L_{\bullet}(S)$ which is isomorphic to $S_{2}$.

Case II(a): If all the vertices of $Z^{\prime}$ are due to the adjacent edges of a single triangle $Z$ in $S$, then by the definition of $L_{\bullet}(S)$, we have a triangle $Z$ in $S$ such that its two vertices are marked negatively while one vertex is marked positively, a contradiction to the hypothesis.

Case II(b): Now, since this triangle is not due to any triangle of $S$, therefore $Z^{\prime}$ must contain an $L$-vertex, say $e_{p}$, which corresponds to an edge $e_{p}$ incident with a vertex $v$ with $d(v) \geq 3$ in $S$. Then either $\mu_{\sigma}(v)=+$ or $\mu_{\sigma}(v)=-$. Then, by the definition of $L_{\bullet}(S)$, such $L$-triangle is either all-positive or all-negative, a contradiction to the hypothesis. Hence, in all the conditions $L_{\bullet}(S)$ does not contain a subsigraph isomorphic to $S_{2}$. Hence, by Theorem $6, L_{\bullet}(S)$ is signcompatible.

The following characterization of the $\mathcal{C}$-sign-compatible sigraph is given by the authors in [18]. This theorem is useful for our further investigation of $\mathcal{C}$-signcompatible $L_{\bullet}(S)$.

Theorem 8 [18]. A sigraph $S=\left(S^{u}, \sigma\right)$ is $\mathcal{C}$-sign-compatible if and only if the following conditions hold in $S$ :
(i) for every vertex $v \in V(S)$ either $d^{-}(v)=0$ or $d^{-}(v) \equiv 1(\bmod 2)$, and
(ii) for every positive edge $e_{k}=v_{i} v_{j}$ in $S, d^{-}\left(v_{i}\right)=0$ or $d^{-}\left(v_{j}\right)=0$.

Now, the following theorem determines the condition for $L_{\bullet}(S)$ to be $\mathcal{C}$-signcompatible.

Theorem 9. For a given sigraph $S=\left(S^{u}, \sigma\right), L \bullet(S)$ is $\mathcal{C}$-sign-compatible if and only if the following conditions hold in $S$ :
(a) for each edge e of $S$, the number of edges that are adjacent with e and incident with a negative vertex that is also adjacent with $e$, is zero or odd, and
(b) for every positively marked vertex in $S$, say $v_{i}$, if there are two negatively marked vertices, say $v_{j}$ and $v_{k}$, adjacent with $v_{i}$, then there is no other vertex adjacent with $v_{j}$ or there is no other vertex adjacent with $v_{k}$.

Proof. Necessity: Let •-line sigraph $L_{\bullet}(S)$ be $\mathcal{C}$-sign-compatible. Then, conditions (i) and (ii) of Theorem 8 hold for $L_{\bullet}(S)$.

Let $\theta$ denote the signature of $L_{\bullet}(S)$. Thus, for any edge $e=v w$ of $S$, the $L$-vertex $e$ of $L_{\bullet}(S)$ has $\mu_{\theta}(e)$ equal to the number of edges of $S$ that are adjacent with $e$ and incident with $v$ (if $v$ is negative) or $w$ (if $w$ is negative). If this number is even and positive, then $L_{\bullet}(S)$ is not $\mathcal{C}$-sign-compatible. Hence condition (a) is necessary.

Now, suppose there is a positively marked vertex $v_{i}$ in $S$ and two negatively marked vertices $v_{j}$ and $v_{k}$ such that $v_{j}$ and $v_{k}$ are adjacent with $v_{i}$. Suppose $v_{j}$ and $v_{k}$ are adjacent with some vertices. Then, one possibility is that $v_{j}$ and $v_{k}$ are adjacent with each other. In this case, we have an $L$-triangle ( $v_{i} v_{j}, v_{j} v_{k}, v_{k} v_{i}, v_{i} v_{j}$ ) with one positive $L$-edge $v_{i} v_{j} v_{k} v_{i}$ and two negative $L$-edges in $L_{\bullet}(S)$. So there exists a positive $L$-edge in $L_{\bullet}(S)$ such that it does not satisfy condition (ii) of Theorem 8. Thus, by Theorem $8, L_{\bullet}(S)$ is not $\mathcal{C}$-sign-compatible, a contradiction to the hypothesis. Hence, $v_{j}$ and $v_{k}$ are not adjacent with each other in $L_{\bullet}(S)$.

Now, suppose $v_{l}$ is a vertex adjacent with $v_{j}$ and $v_{m}$ is a vertex adjacent with $v_{k}$ respectively in $S$. Then, by the definition of $L_{\bullet}(S)$, we have an $L$-path $p_{4}=\left(v_{l} v_{j}, v_{j} v_{i}, v_{i} v_{k}, v_{k} v_{m}\right)$ in $L_{\bullet}(S)$ such that $L$-edge $v_{j} v_{i} v_{i} v_{k}$ is a positive edge while $L$-edges $v_{l} v_{j} v_{j} v_{i}$ and $v_{i} v_{k} v_{k} v_{m}$ are negative edges. Thus, again there exists a positive $L$-edge in $L \bullet(S)$ such that it does not satisfy condition (ii) of Theorem 8. So by the same argument as above, we get a contradiction to the hypothesis. Hence, condition (b) is necessary.

Sufficiency: Suppose conditions (a) and (b) hold for a given sigraph $S$. We shall show that $L_{\bullet}(S)$ is $\mathcal{C}$-sign-compatible. Suppose on contrary that $L_{\bullet}(S)$ is not $\mathcal{C}$-sign-compatible. Then, by Theorem 8 condition (i) or condition (ii) is not satisfied for $L_{\bullet}(S)$.

Suppose condition (i) of Theorem 8 is not satisfied for $L_{\mathbf{\bullet}}(S)$ i.e., there is an $L$-vertex $e \in V\left(L_{\bullet}(S)\right)$ such that neither $d^{-}(e)=0$ nor $d^{-}(e) \equiv 1(\bmod 2)$. This shows that there are even number of edges adjacent with $e$ in $S$ such that these edges are incident with vertices with negative marking, a contradiction to condition (a).

Now, suppose condition (ii) of Theorem 8 is not satisfied i.e., for any positive edge $k=e e^{\prime}$ there are negative $L$-edges on $e$ and $e^{\prime}$ in $L_{\bullet}(S)$. Suppose this positive $L$-edge lies on an $L$-path $P_{4}^{\prime}=\left(e_{i}, e_{j}, e_{k}, e_{l}\right)$ such that $e_{j} e_{k}$ is a positive $L$-edge while $e_{i} e_{j}$ and $e_{k} e_{l}$ are negative $L$-edges in $L_{\bullet}(S)$. Then, by the definition
of $L_{\bullet}(S)$, we have a path $p_{5}=(u, v, w, x, y)$ in $S$ such that $e_{i}=u v, e_{j}=v w$, $e_{k}=w x$ and $e_{l}=x y$. Clearly, $e_{j} e_{k}$ is a positive $L$-edge in $L_{\bullet}(S)$ so, by the definition of $L_{\bullet}(S)$, common vertex $w$ between $e_{j}$ and $e_{k}$ is surely positively marked in $S$. Similarly, $e_{i} e_{j}$ and $e_{k} e_{l}$ are negative $L$-edges in $L_{\bullet}(S)$, so their common vertices $v$ and $x$, respectively, receive negative mark in $S$. Thus, for a positively marked vertex $w$ in $S$, two negative vertices $v$ and $x$ are adjacent with $w$ in $S$. Also, $u$ and $y$ are adjacent with $v$ and $x$ in $S$, a contradiction to (a).

This positive $L$-edge can be in a triangle also. Let $Z^{\prime}$ be such $L$-triangle in $L \bullet(S)$. Now, this $L$-triangle is either due to the edges of a triangle of $S$ or due to a triangle $v \in V(S)$ in $S$ with $d(v) \geq 3$.

Case I: If all the vertices of $Z^{\prime}$ are due to the adjacent edges of a single triangle $Z$ in $S$, then by the definition of $L_{\bullet}(S)$, we have triangle $Z$ in $S$ with one and two negatively marked vertices in $S$. Thus, again we have a contradiction to (a).

Case II: Now, since this $L$-triangle is not due to any triangle of $S$, therefore $Z^{\prime}$ must contain an $L$-vertex, say $e_{p}$, such that it is incident with a vertex $v \in V(S)$ with $d(v) \geq 3$ in $S$. Then, by the definition of $L_{\bullet}(S)$, either such $L$-triangle is all-positive or all-negative. So there is no such positive $L$-edge in $L_{\bullet}(S)$. Hence, by Theorem $8, L_{\bullet}(S)$ is $\mathcal{C}$-sign-compatible. This completes the proof.

## 4. Existential Characterization of $L$ • $(S)$

In this section we establish the characterization of the -line sigraph. While the characterization problem has been solved for 'line sigraph' (i.e., sigraph $S$ for which there exists a sigraph $H$ such that $L(H) \cong S$ ) as in [2], the same remains to be solved for the $\times$-line sigraph' (i.e., sigraph $S$ for which there exists a sigraph $H$ such that $L_{\times}(H) \cong S$ ) as well as for the ' $\bullet$-line sigraph' (i.e., sigraph $S$ for which there exists a sigraph $H$ such that $\left.L_{\bullet}(H) \cong S\right)$.

Hence, for any given isolate-free sigraph $S$, consider the sigraph equation

$$
\begin{equation*}
L_{\bullet}(H) \cong S \tag{1}
\end{equation*}
$$

where any sigraph $H$ satisfying (1) (i.e., a 'solution' of (1)) will be called an $L_{\bullet}$-root of $S$ [1]. By the definition of $\bullet$-line sigraph it is clear that

$$
L_{\bullet}\left(S^{u}\right) \cong L\left(S^{u}\right),
$$

so Theorem 1 is a characterization of $\bullet$-line graphs also. We have the following important observation by Sampatkumar, which is useful in the upcoming theorem.

Remark 10 [15]. Every canonically marked sigraph contains an even number of negative vertices.

Now, we give two following lemmas which are essential for the characterization of $\bullet$-line graphs.

Lemma 11. Let $S$ be a connected graph and let $U$ be any even subset of $V(S)$. Then there is a signature $\sigma$ for $S$ such that $\mu_{\sigma}(v)=-i$ if and only if $v \in U$.

Proof. Let $|U|=2 k$. If $k=0$ then we have the all-positive signature; now inductively suppose the statement true for even subsets of size $2,4, \ldots, 2(k-1)$. Let $u, v$ be any two elements of $U$. By the inductive hypothesis there is a signature $\sigma$ such that $\mu_{\sigma}(w)=-$ if and only if $w \in U-\{u, v\}$ (or $w$ could be anything other than $u$ or $v$ ). There is a path from $u$ to $v$; change the sign of every edge on the path, giving a signature $\tau$. It is clear that $\mu_{\tau}(w)=-$ if and only if $w \in U$.

Lemma 12. Let $S=\left(S^{u}, \sigma\right)$ be the $\bullet$-line graph of sigraph $T=\left(T^{u}, \tau\right)$ and extend $T^{u}$ to a graph $\hat{T}^{u}$ by adding a new vertex $t$ and an edge $e=s t$, where $s \in V\left(T^{u}\right)$ (so that $t$ is pendent). Now extend $\sigma$ to a signature $\hat{\sigma}$ on $L\left(S^{u}\right)$ as follows. The new L-edges are $e_{1} e, e_{2} e, \ldots, e_{d} e$ where the $e_{i}$ are the edges of $T^{u}$ incident with s; give all these the same sign (either + or -$)$. Then $\hat{S}=\left(\hat{S}^{u}, \hat{\sigma}\right)$ is $a \bullet$-line graph.
Proof. Extend $\tau$ to a signature on $\hat{T}^{u}$, by appropriately signing st so that the canonical marking $\hat{\mu}(s)$ of $V(T)$ agrees with the $\hat{\sigma}\left(e_{i} e\right)$.

Now, the following theorem gives us the solution of (1).
Theorem 13. A given connected sigraph $S=\left(S^{u}, \sigma\right)$ is a $\bullet$-line graph if and only if following conditions hold in $S$ :
(1) $S^{u}$ is a line graph and the edges of $S$ can be partitioned into complete subsigraphs in such a way that no vertex lies in more than two of the subsigraphs and each such complete subsigraph is homogeneous.
(2) if each vertex of $S$ belongs to exactly two of these subsigraphs, then the number of all-negative complete sigraphs is even.

Proof. Necessity: We are given a sigraph $S$. Suppose $S$ is a $\bullet$-line sigraph. Then there exists a sigraph $T$, such that $S^{u} \cong L_{\bullet}\left(T^{u}\right)$, so that $S^{u}$ is a line graph. Thus, by Theorem 1, the edges of $S^{u}$ can be partitioned into complete subsigraphs such that no vertex lies in more than two of these. The vertices of any such subsigraph $\mathcal{Q}$ are the $L$-vertices of $L_{\bullet}\left(T^{u}\right)$ corresponding to the edges of $T^{u}$ incident with some vertex $v$ of $T^{u}$, and therefore the edges of $\mathcal{Q}$ are $L$-edges of $L \bullet\left(T^{u}\right)$ signed
with $\mu_{\sigma}(v)$. Thus these complete subsigraphs are homogeneous, and condition (1) is satisfied.

Next, suppose all the vertices in $S$ lie in exactly two such subsigraphs and number of all-negative subsigraphs is odd. By the definition of •-line sigraph, it is clear that there are odd number of negatively marked vertices in the $L_{\bullet}$-root of $S$, a contradiction to the Remark 10. Hence, by contradiction, (2) holds.

Now, suppose there are some vertices in $S$ such that these are not in two such subsigraphs. While making the $L_{\bullet}$-root of $S$ by the help of these subsets, it is clear that there are some pendent vertices in the $L_{\bullet}$-root of $S$. Since the vertices are pendent in $L_{\bullet}$-root of $S$, so there are not any other edges incident with these vertices in the $L_{\bullet}$-root of $S$. By the definition of $L_{\bullet}(S)$, these vertices are not giving any contribution to the signing of any complete subsigraphs in the •-line sigraph. Thus, by including or excluding these vertices the number of negatively marked vertices, in $L_{\bullet}$-root of $S$, is even. So the number of all-negative homogeneous complete subsigraphs is either even or odd according to the number of pendent vertices and their canonical marking in the $L_{\bullet}$-root of $S$.
$S$ ufficiency: Suppose $S$ is a sigraph satisfying the conditions. We shall show that $S$ is the •-line sigraph, that is, there exists a sigraph $T$ such that $S \cong L_{\bullet}(T)$.

Let $T^{u}$ be the graph such that $S^{u}=L\left(T^{u}\right)$. We wish to find a signature $\tau$ on $T^{u}$ such that $\mu_{\tau}$ is the canonical marking of $V\left(T^{u}\right)$ giving the required signature of $S^{u}$. Assume, first that there is an even number of all-negative complete subsigraphs in the partitioning of $E(S)$. Let $U$ be the corresponding set of vertices of $T^{u}$. By Lemma 11, there is a signature $\tau$ on $T^{u}$ such that $\mu_{\tau}$ is negative exactly on $U$. Finally, if $T^{u}$ has a pendent vertex $t$, then by Lemma 12 there is a complete sigraph whose sign may be chosen independently. This completes the proof.

Note: If a sigraph $S$ is disconnected then $S$ is a •-line sigraph if and only if its every component satisfies the conditions of Theorem 13 separately.

Corollary 14. Every homogeneous complete sigraph $K_{n}$ is a •-line sigraph.
Proof. Suppose we are given a homogeneous complete sigraph. It is easy to see that $L_{\bullet}$-root for $K_{n}^{u}$ is star $K_{1, n}^{u}$. Thus, $K_{n}^{u}$ is a line graph. Thus, the conditions of Theorem 13 are satisfied. Hence, every homogeneous complete sigraph $K_{n}$ is a •-line sigraph.

Corollary 15. Every path sigraph $P_{n}$ is $a \bullet$-line sigraph.
Proof. Suppose we are given a path sigraph $P_{n}$. It is easy to see that $L_{\bullet}$-root for $P_{n}^{u}$ is $P_{n+1}^{u}$. Thus, $P_{n}^{u}$ is a line graph. We can partition $P_{n}$ into $(n-1) K_{2}$ complete subsigraphs in such a way that no vertex lies in more than two of the subsigraphs and each such $K_{2}$ is homogeneous. Since there are pendent vertices in $P_{n}$, number of such complete subsigraph may be even or odd. So both the
conditions of Theorem 13 are satisfied. Hence, every path sigraph $P_{n}$ is a $\bullet$-line sigraph.

Corollary 16. A cycle $C_{n}$ is a-line sigraph if it contains even number of negative edges.
Proof. Suppose we are given a cycle $C_{n}$. It is easy to see that the $L_{\boldsymbol{\bullet}}$-root for $C_{n}^{u}$ is $C_{n}^{u}$. Thus, $C_{n}^{u}$ is a line graph. We can partitioned $C_{n}$ into $n, K_{2}$ complete subsigraphs in such a way that no vertex lies in more than two of the subsigraphs and each such $K_{2}$ is homogeneous. Since there are no pendent vertices in $C_{n}$ and there is an even number of negative edges in $C_{n}$, the number of such complete subsigraphs $K_{2}$ is even. So both the conditions of Theorem 13 are satisfied. Hence, every cycle sigraph $C_{n}$ is a $\bullet$-line sigraph.

Corollary 17. Every balanced cycle $C_{n}$ is a $\bullet$-line sigraph.
Proof. The result is trivial by Lemma 3 and Corollary 16.


Figure 3. Showing a $\bullet$-line sigraph and its two $L_{\bullet}$-root sigraphs.

Note: We know that if there are odd (even) number of negative edges incident with any vertex, then due to its canonical marking $\mu_{\sigma}(v)=-\left(\mu_{\sigma}(v)=+\right)$. Therefore, for a given $\bullet$-line sigraph its $L_{\bullet}$-root sigraphs is not unique. The pictorial presentation of this is shown in Figure 3. Hence, the following problem is open.
Problem 18. Characterize •-line sigraphs having exactly one $L_{\bullet}$-root sigraph up to isomorphism.

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