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A NOTE ON THE TOTAL DETECTION NUMBERS OF CYCLES

Henry E. Escuadro[1], Futaba Fujie[2]

and

Chad E. Musick[2]

[1] *Mathematics Department, Juniata College*
Huntingdon, PA 16652, USA

[2] *Graduate School of Mathematics, Nagoya University*
Nagoya, 464-8602, Japan

e-mail: escuadro@juniata.edu
futaba@math.nagoya-u.ac.jp
ce.musick@yahoo.com

Abstract

Let G be a connected graph of size at least 2 and $c : E(G) \to \{0, 1, \ldots, k-1\}$ an edge coloring (or labeling) of G using k labels, where adjacent edges may be assigned the same label. For each vertex v of G, the color code of v with respect to c is the k-vector $\text{code}(v) = (a_0, a_1, \ldots, a_{k-1})$, where a_i is the number of edges incident with v that are labeled i for $0 \leq i \leq k-1$. The labeling c is called a detectable labeling if distinct vertices in G have distinct color codes. The value $\text{val}(c)$ of a detectable labeling c of a graph G is the sum of the labels assigned to the edges in G. The total detection number $\text{td}(G)$ of G is defined by $\text{td}(G) = \min\{\text{val}(c)\}$, where the minimum is taken over all detectable labelings c of G. We investigate the problem of determining the total detection numbers of cycles.

Keywords: vertex-distinguishing coloring, detectable labeling, detection number, total detection number, Hamiltonian graph.

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1. Introduction

We refer to the book [6] for graph-theoretical notation and terminology not described in this paper.

Let G be a connected graph of size at least 2 and $c : E(G) \to \{0, 1, \ldots, k-1\}$ an edge coloring (or labeling) of G for some positive integer k, where adjacent edges may be assigned the same label. If c uses k labels, then c is a k-labeling. The *color code* of a vertex v of G (with respect to c) is the ordered k-vector $\mathrm{code}_c(v) = (a_0, a_1, \ldots, a_{k-1})$ (or simply $\mathrm{code}_c(v) = a_0 a_1 \cdots a_{k-1}$), where a_i is the number of edges incident with v that are labeled i for $0 \le i \le k-1$. Thus,

$$\sum_{i=0}^{k-1} a_i = \deg_G v. \tag{1}$$

If the labeling c is clear, then we use $\mathrm{code}(v)$ to denote the color code of a vertex v. The labeling c is called a *detectable labeling* of G if distinct vertices of G have distinct color codes, that is, for every two vertices of G, there exists a label such that the numbers of incident edges assigned that label are different for these two vertices. Thus, a detectable labeling is a vertex-distinguishing edge labeling. The *detection number* $\det(G)$ of G is the minimum positive integer k for which G has a detectable k-labeling. A detectable labeling of a graph G using $\det(G)$ labels is called a *minimum detectable labeling* of G. Since there is no nontrivial irregular graph (a graph in which no two distinct vertices have the same degree), every detectable labeling of a graph must use at least two labels by (1). Thus, $\det(G) \ge 2$ for every connected graph G of size at least 2. Detectable labelings have been studied in [1, 2, 3, 4, 5], sometimes with different terminology and notation.

For a detectable labeling c of a graph G, define the *value* $\mathrm{val}(c)$ of c by $\mathrm{val}(c) = \sum_{e \in E(G)} c(e)$. The *total detection number* $\mathrm{td}(G)$ of G is then defined by $\mathrm{td}(G) = \min\{\mathrm{val}(c)\}$, where the minimum is taken over all detectable labelings c of G. Thus, in the case of the detection number $\det(G)$ of G, we minimize the number of labels used in a detectable labeling of G; while in the case of the total detection number $\mathrm{td}(G)$ of G, we minimize the sum of labels of the edges of G used in a detectable labeling of G (which may or may not be a minimum detectable labeling). The concept of the total detection numbers of graphs was suggested by Slater and has been studied by Escuadro and Fujie-Okamoto in [8]. Complete bipartite graphs and their total detection numbers were considered in [7].

In general, for a given connected graph, determining the exact value of its detection number is not always a trivial task, although the numbers have been investigated for many well-known graphs. Finding the total detection number of a graph is even more challenging. For example, the detection numbers of complete graphs and cycles have been completely determined while only partial results are known for the total detection numbers for these graphs (see [8], also shown in Table 3).

Theorem 1 [5]**.** *Let $n \geq 3$ be an integer and $p = \left\lceil \sqrt{n/2} \right\rceil$. Then* $\det(K_n) = 3$ *and*

$$\det(C_n) = \begin{cases} 2p-1 & \text{if } 2(p-1)^2 < n \leq p(2p-1), \\ 2p & \text{if } p(2p-1) < n \leq 2p^2. \end{cases}$$

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathrm{td}(K_n)$	3	3	4	6	8	9	11	13	16	18	20	22	25
$\mathrm{td}(C_n)$	3	3	4	6	9	11	12	14	17	21	23	26	29

Table 1. $\mathrm{td}(K_n)$ and $\mathrm{td}(C_n)$ for small values of n.

Note that $\mathrm{td}(K_n) \leq \mathrm{td}(C_n)$ for $3 \leq n \leq 15$. This is true for $n \geq 16$ as well. In fact, the following is among the results obtained in [8].

Theorem 2 [8]**.** *If G is a graph of order at least* 3 *containing a connected regular spanning subgraph H, then* $\mathrm{td}(G) \leq \mathrm{td}(H)$.

Theorem 2 suggests that studying the total detection numbers of cycles may shed some light on the total detection numbers of Hamiltonian graphs in general. For this reason, we discuss the problem of determining the total detection numbers of cycles in this article. In particular, we will give upper and lower bounds for the total detection number of a cycle in terms of its order, namely,

$$\tfrac{2}{3}n^{1.5} - \tfrac{3}{2}n < \mathrm{td}(C_n) \leq \tfrac{2}{3}n^{1.5} - \tfrac{1}{2}n \tag{2}$$

for $n \geq 8$. We also present an infinite set $S \subseteq \{3, 4, 5, \ldots\}$ such that the exact value of $\mathrm{td}(C_n)$ can be calculated for each $n \in S$.

2. A Lower Bound

Let there be given a detectable k-labeling of C_n, where $k \geq \det(C_n)$. Then equation (1) implies that the dot product of the color code of each vertex in the cycle and the k-vector $(1, 1, \ldots, 1)$ equals 2. In [8], a lower bound for $\mathrm{td}(C_n)$ was obtained by considering sets of n distinct vectors $\vec{v}$ such that $\vec{v} \cdot (1, 1, \ldots, 1) = 2$. The lower bound in (2) is then a consequence of this result.

Theorem 3 [8]**.** *For each integer $n \geq 5$, let $p = \lceil \sqrt{n} \rceil$. Then*

$$\mathrm{td}(C_n) \geq \begin{cases} \frac{1}{2}\big(n(2p-3) - \frac{1}{6}p(p-1)(4p-5)\big) & \text{if } (p-1)^2 < n \leq p(p-1) - 2, \\ \frac{1}{2}\big(n(2p-2) - \frac{1}{6}(4p^3 - 3p^2 - p - 6)\big) & \text{if } n \in \{p(p-1)-1, p(p-1)\}, \\ \frac{1}{2}\big(n(2p-2) - \frac{1}{6}p(p-1)(4p+1)\big) & \text{if } p(p-1) < n \leq p^2 - 2, \\ \frac{1}{2}\big(n(2p-1) - \frac{1}{6}(p-1)(4p^2 + 7p + 6)\big) & \text{if } n \in \{p^2 - 1, p^2\}. \end{cases}$$

Theorem 4. *For each integer* $n \geq 3$, $\mathrm{td}(C_n) > \frac{2}{3}n^{1.5} - \frac{3}{2}n$.

Proof. Since the result holds for $3 \leq n \leq 15$, assume that $n \geq 16$. Let $p = \lceil \sqrt{n}\, \rceil$. Therefore, $\sqrt{n} \leq p < \sqrt{n} + 1 \leq \frac{1}{8}n + 3$. We consider the following four cases.

Case 1. $(p-1)^2 + 1 \leq n \leq p(p-1) - 2$. Then $n + p + 2 \leq p^2 \leq n + 2p - 2$. Therefore,

$$\begin{aligned}\mathrm{td}(C_n) &\geq np - \tfrac{3}{2}n - \tfrac{1}{12}(4p^3 - 9p^2 + 5p)\\ &\geq np - \tfrac{3}{2}n - \tfrac{1}{3}p(n + 2p - 2) + \tfrac{3}{4}(n + p + 2) - \tfrac{5}{12}p\\ &\geq \tfrac{2}{3}np - \tfrac{3}{4}n - \tfrac{2}{3}(n + 2p - 2) + p + \tfrac{3}{2}\\ &\geq \tfrac{2}{3}n^{1.5} - \tfrac{17}{12}n - \tfrac{1}{3}\left(\tfrac{1}{8}n + 3\right) + \tfrac{17}{6} = \tfrac{2}{3}n^{1.5} - \tfrac{35}{24}n + \tfrac{11}{6}\\ &> \tfrac{2}{3}n^{1.5} - \tfrac{3}{2}n.\end{aligned}$$

Case 2. $p(p-1) - 1 \leq n \leq p(p-1)$. Then $n + p \leq p^2 \leq n + p + 1$ and so

$$\begin{aligned}\mathrm{td}(C_n) &\geq np - n - \tfrac{1}{12}(4p^3 - 3p^2 - p - 6)\\ &\geq np - n - \tfrac{1}{3}p(n + p + 1) + \tfrac{1}{4}(n + p) + \tfrac{1}{12}p + \tfrac{1}{2}\\ &\geq \tfrac{2}{3}np - \tfrac{3}{4}n - \tfrac{1}{3}(n + p + 1) + \tfrac{1}{2}\\ &\geq \tfrac{2}{3}n^{1.5} - \tfrac{13}{12}n - \tfrac{1}{3}\left(\tfrac{1}{8}n + 3\right) + \tfrac{1}{6} = \tfrac{2}{3}n^{1.5} - \tfrac{9}{8}n - \tfrac{5}{6}\\ &> \tfrac{2}{3}n^{1.5} - \tfrac{3}{2}n.\end{aligned}$$

Case 3. $p(p-1) + 1 \leq n \leq p^2 - 2$. Then $n + 2 \leq p^2 \leq n + p - 1$ and

$$\begin{aligned}\mathrm{td}(C_n) &\geq np - n - \tfrac{1}{12}(4p^3 - 3p^2 - p)\\ &\geq np - n - \tfrac{1}{3}p(n + p - 1) + \tfrac{1}{4}(n + 2) + \tfrac{1}{12}p\\ &\geq \tfrac{2}{3}np - \tfrac{3}{4}n - \tfrac{1}{3}(n + p - 1) + \tfrac{5}{12}p + \tfrac{1}{2} \geq \tfrac{2}{3}n^{1.5} - \tfrac{13}{12}n + \tfrac{1}{12}p + \tfrac{5}{6}\\ &> \tfrac{2}{3}n^{1.5} - \tfrac{3}{2}n.\end{aligned}$$

Case 4. $p^2 - 1 \leq n \leq p^2$. Then $n \leq p^2 \leq n + 1$ and

$$\begin{aligned}\mathrm{td}(C_n) &\geq np - \tfrac{1}{2}n - \tfrac{1}{12}(4p^3 + 3p^2 - p - 6)\\ &\geq np - \tfrac{1}{2}n - \tfrac{1}{3}p(n + 1) - \tfrac{1}{4}(n + 1) + \tfrac{1}{12}p + \tfrac{1}{2}\\ &\geq \tfrac{2}{3}n^{1.5} - \tfrac{3}{4}n - \tfrac{1}{4}\left(\tfrac{1}{8}n + 3\right) + \tfrac{1}{4} = \tfrac{2}{3}n^{1.5} - \tfrac{25}{32}n - \tfrac{1}{2}\\ &> \tfrac{2}{3}n^{1.5} - \tfrac{3}{2}n.\end{aligned}$$

This completes the proof. ■

3. An Upper Bound

For an integer $n \geq 3$, suppose that c is a detectable labeling of $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and consider a corresponding function $c^* : V(C_n) \to \mathbb{Z} \times \mathbb{Z}$ defined by

$$c^*(v_i) = \big(\max\{c(v_{i-1}v_i), c(v_iv_{i+1})\}, \min\{c(v_{i-1}v_i), c(v_iv_{i+1})\}\big)$$

for $1 \leq i \leq n$. Thus, $\mathrm{val}(c) = \frac{1}{2}\sum_{i=1}^{n} c^*(v_i) \cdot (1,1)$.

Now consider the sequence $s : c^*(v_1), c^*(v_2), \ldots, c^*(v_n), c^*(v_1), c^*(v_2)$. Observe that if $(a_{i_1}, b_{i_1}), (a_{i_2}, b_{i_2}), (a_{i_3}, b_{i_3})$ are consecutive terms in s, then either

(i) $a_{i_2} \in \{a_{i_1}, b_{i_1}\}$ and $b_{i_2} \in \{a_{i_3}, b_{i_3}\}$ or

(ii) $a_{i_2} \in \{a_{i_3}, b_{i_3}\}$ and $b_{i_2} \in \{a_{i_1}, b_{i_1}\}$.

As an example, suppose that c is the detectable labeling of C_8 shown in Figure 1. Then we obtain the sequence $s : (0,0), (1,0), (2,1), (2,0), (3,0), (3,1), (4,1), (4,0), (0,0), (1,0)$. Note also that this sequence s can be expressed geometrically in the $\mathbb{R}^2$ plane as shown in Figure 1.

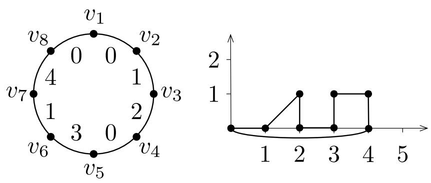

Figure 1. A detectable labeling of C_8 and the corresponding diagram.

Conversely, if there is a sequence $s : c_1, c_2, \ldots, c_{n+2}$ of length $n+2$ where

(a) each term in s is of the form (a,b), where a and b are integers with $0 \leq b \leq a$,

(b) the first n terms in s are distinct and $c_{n+i} = c_i$ for $i = 1, 2$, and

(c) if $(a_{i_1}, b_{i_1}), (a_{i_2}, b_{i_2}), (a_{i_3}, b_{i_3})$ are consecutive terms in s, then either

  (i) $a_{i_2} \in \{a_{i_1}, b_{i_1}\}$ and $b_{i_2} \in \{a_{i_3}, b_{i_3}\}$ or

  (ii) $a_{i_2} \in \{a_{i_3}, b_{i_3}\}$ and $b_{i_2} \in \{a_{i_1}, b_{i_1}\}$,

then one can find a corresponding detectable labeling of C_n.

For example, suppose that $n = (2p)^2 + 1$ for some positive integer p. Then one can construct a diagram inducing a sequence satisfying (a)–(c) from which we obtain a detectable labeling of C_n whose value is $\frac{2}{3}p(8p^2 - 3p + 1)$. See Figure 2 for how such a diagram can be obtained for $n = 5, 17, 37$ and note that this can be easily generalized for all $n = (2p)^2 + 1$. Similarly, one can show that there is a detectable labeling of C_n whose value equals $\frac{1}{6}(n-1)(4\sqrt{n} - 3)$ when $\sqrt{n}$ is an odd integer. (See Figure 3 for $n = 9, 25, 49$.) Hence, we obtain the following.

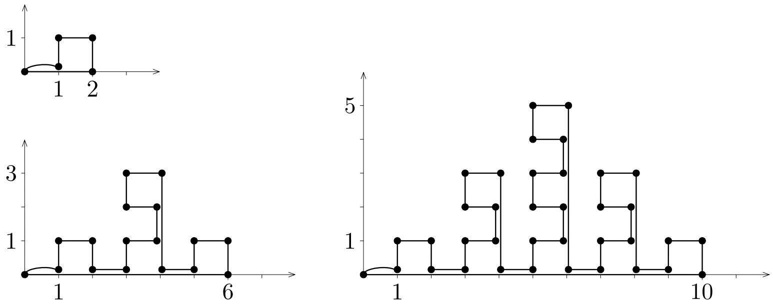

Figure 2. Grid diagrams for $n = 5, 17, 37$.

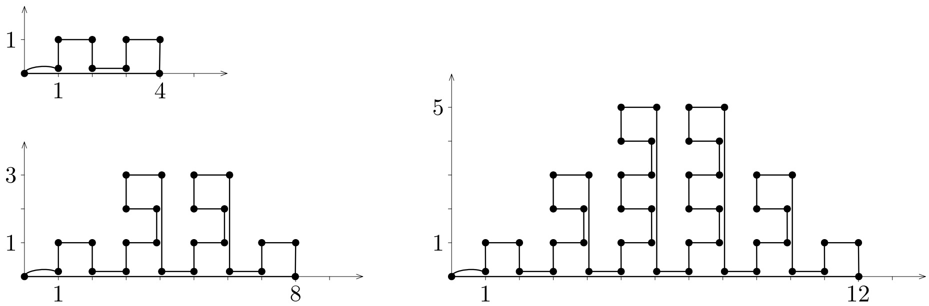

Figure 3. Grid diagrams for $n = 9, 25, 49$.

Proposition 5. *For an integer* $n \geq 5$,

$$\mathrm{td}(C_n) \leq \begin{cases} \frac{1}{6}(n-1)(4\sqrt{n-1}-3) + \frac{1}{3}\sqrt{n-1} & \textit{if } \sqrt{n-1} \textit{ is an even integer,} \\ \\ \frac{1}{6}(n-1)(4\sqrt{n}-3) & \textit{if } \sqrt{n} \textit{ is an odd integer.} \end{cases}$$

For other values of $n \geq 16$ not described in Proposition 5, we can modify these diagrams to obtain detectable labelings of cycles of the desired orders and calculate their values. The diagrams in Figure 4 for $20 \leq n \leq 23$ are obtained from the diagram for $n' = 17$ in Figure 2. We then see that $\mathrm{td}(C_{20}) \leq 49$, $\mathrm{td}(C_{21}) \leq 52$, $\mathrm{td}(C_{22}) \leq 54$, and $\mathrm{td}(C_{23}) \leq 58$.

Let us describe this procedure more precisely. For a given $n \geq 16$, let $p = \lfloor \sqrt{n}/2 \rfloor$. First, if $(2p)^2 \leq n < 4(p^2+p+1)$, then we begin with the diagram D' for $n' = (2p)^2 + 1$. For $0 \leq i \leq 3$, let $n'_i = n' + i - 1$. Since $n'_1 = n'$, we already have a detectable labeling of $C_{n'_1}$ whose value equals $t'_1 = \frac{2}{3}p(8p^2 - 3p + 1) = 2pn'_1 - \frac{2}{3}p(4p^2+3p+2)$. For $n'_0 = n' - 1$, delete the vertex at $(2p-1, 2p-1)$ from D' and join the two vertices at $(2p-1, 2p-2)$ and $(2p, 2p-1)$. The new diagram results in a detectable labeling of $C_{n'_0}$ whose value equals $t'_0 = t'_1 - (2p-1) = 2pn'_0 - \frac{2}{3}p(4p^2+3p+2) + 1$. For $n'_2 = n' + 1$, start again with D' and insert

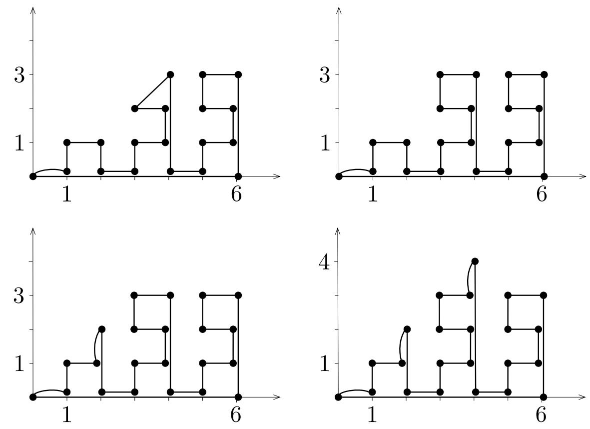

Figure 4. Grid diagrams for $20 \leq n \leq 23$.

a new vertex at $(2,2)$ between the two at $(2,1)$ and $(2,0)$. In addition, insert another vertex at $(4,4)$ between the two at $(4,3)$ and $(4,0)$ for $n'_3 = n' + 2$. Then the corresponding labelings are detectable labelings of $C_{n'_2}$ and $C_{n'_3}$ whose values equal $t'_2 = t'_1 + 2 = 2pn'_2 - \frac{2}{3}p(4p^2 + 3p + 5) + 2$ and $t'_3 = t'_1 + 6 = 2pn'_3 - \frac{2}{3}p(4p^2 + 3p + 8) + 6$, respectively. Now for $(2p)^2 + 4 \leq n < 4(p^2 + p + 1)$, add the appropriate number of sets of four vertices. These are indicated as shaded "squares" in Figure 5, which describes the situation for $p = 3$. While adding any

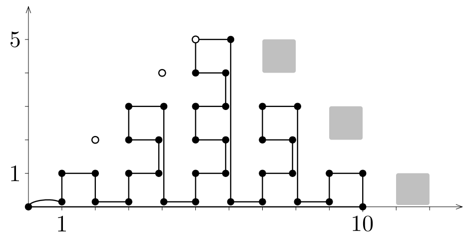

Figure 5. Constructing grid diagrams for $(2p)^2 \leq n < 4(p^2 + p + 1)$ where $p = 3$.

one of these squares increases the order by 4, it also increases the value of the corresponding labeling by $8p$. Thus, the order in which these squares (there are p of them) are added does not matter.

For $4(p^2 + p + 1) \leq n < (2p + 2)^2$, the procedure is quite similar. We first add four vertices at $(2p+1, 2p), (2p+1, 2p+1), (2p+2, 2p), (2p+2, 2p+1)$ to the diagram for $n'' = (2p+1)^2$, as shown in Figure 6, and call this new diagram D''. For $0 \leq i \leq 3$, let $n''_i = n'' + i + 3$. Then for $n''_1 = n'' + 4$, we obtain a detectable labeling of $C_{n''_1}$ whose value equals $t''_1 = \frac{2}{3}p(p+1)(8p+1) + (8p+4) = (2p+1)n''_1 - \frac{2}{3}p(4p^2 + 9p + 8) - 1$. As before, one can obtain a detectable labeling

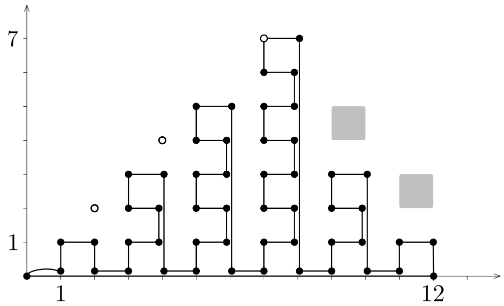

Figure 6. Constructing grid diagrams for $4(p^2+p+1) \leq n < (2p+2)^2$ where $p = 3$.

of $C_{n''_0}$ by deleting the vertex at $(2p+1, 2p+1)$ from D'' and joining the two vertices at $(2p+1, 2p)$ and $(2p+2, 2p+1)$. Similarly, adding the vertices at $(2,2)$ and $(4,4)$ to D'', we obtain detectable labelings of $C_{n''_2}$ and $C_{n''_3}$.

For $4(p^2+p+1)+4 \leq n < (2p+2)^2$, adjust the order by adding the appropriate number of squares, which is also shown in Figure 6. Any one of these squares (there are $p-1$ of them) increases the value of the labeling by $8p+4$.

As a consequence, we obtain the following for general values of $n \geq 16$.

Theorem 6. *For each integer $n \geq 16$, let $p = \lfloor \sqrt{n}/2 \rfloor$ and define $f : [16, \infty) \cap \mathbb{Z} \to \mathbb{Z}$ by*

$$f(n) = \begin{cases} 2pn - \frac{2}{3}p(4p^2+3p+2)+1 & \text{if } n < 4(p^2+p+1) \text{ and } n \equiv 0 \pmod 4, \\ 2pn - \frac{2}{3}p(4p^2+3p+2) & \text{if } n < 4(p^2+p+1) \text{ and } n \equiv 1 \pmod 4, \\ 2pn - \frac{2}{3}p(4p^2+3p+5)+2 & \text{if } n < 4(p^2+p+1) \text{ and } n \equiv 2 \pmod 4, \\ 2pn - \frac{2}{3}p(4p^2+3p+8)+6 & \text{if } n < 4(p^2+p+1) \text{ and } n \equiv 3 \pmod 4, \\ (2p+1)n - \frac{2}{3}p(4p^2+9p+8)-1 & \text{if } n \geq 4(p^2+p+1) \text{ and } n \equiv 0, 1 \pmod 4, \\ (2p+1)n - \frac{2}{3}p(4p^2+9p+11) & \text{if } n \geq 4(p^2+p+1) \text{ and } n \equiv 2 \pmod 4, \\ (2p+1)n - \frac{2}{3}p(4p^2+9p+14)+3 & \text{if } n \geq 4(p^2+p+1) \text{ and } n \equiv 3 \pmod 4. \end{cases}$$

Then there exists a detectable labeling of C_n whose value equals $f(n)$.

In order to verify that $\mathrm{td}(C_n) \leq \frac{2}{3}n^{1.5} - \frac{1}{2}n$ for $n \geq 16$, we present a few lemmas.

Lemma 7. *For an integer $n \geq 16$, let $p = \lfloor \sqrt{n}/2 \rfloor$. If f is the function defined in Theorem 6, then*

$$f(n) \leq \begin{cases} f(4p^2) + 2p(n - 4p^2) & \text{if } n < 4(p^2 + p + 1), \\ f(4(p^2 + p + 1)) + (2p + 1)(n - 4(p^2 + p + 1)) & \text{if } n \geq 4(p^2 + p + 1). \end{cases}$$

Lemma 8. *If f is the function defined in Theorem 6, then $f(n) \leq \frac{2}{3}n^{1.5} - \frac{1}{2}n$ for each integer $n \geq 16$.*

Proof. Let $g : (1, \infty) \to \mathbb{R}$ be given by $g(x) = \frac{2}{3}x^{1.5} - \frac{1}{2}x$. Also, for each positive integer p, define $g_{i,p} : \mathbb{R} \to \mathbb{R}$ by

$$g_{i,p}(x) = (2p + i)\left(x - (2p + \tfrac{1}{2} + i)^2\right) + g\left((2p + \tfrac{1}{2} + i)^2\right)$$

for $i = 0, 1$. Then $g_{i,p} \leq g$ on $(1, \infty)$. Also, observe that

$$\begin{aligned} &g_{0,p}(4p^2) - f(4p^2) = \tfrac{5}{6}\left(p - \tfrac{5}{4}\right), \\ &g_{1,p}(4(p^2 + p + 1)) - f(4(p^2 + p + 1)) = \tfrac{5}{6}\left(p - \tfrac{3}{20}\right). \end{aligned}$$

Hence, $g_{0,p}(4p^2) > f(4p^2)$ and $g_{1,p}(4(p^2 + p + 1)) > f(4(p^2 + p + 1))$ for $p \geq 2$. For $n < 4(p^2 + p + 1)$,

$$f(n) \leq f(4p^2) + 2p(n - 4p^2) < g_{0,p}(4p^2) + 2p(n - 4p^2) = g_{0,p}(n) \leq g(n).$$

Similarly, for $n \geq 4(p^2 + p + 1)$,

$$\begin{aligned} f(n) &\leq f(4(p^2 + p + 1)) + (2p + 1)(n - 4(p^2 + p + 1)) \\ &< g_{1,p}(4(p^2 + p + 1)) + (2p + 1)(n - 4(p^2 + p + 1)) \\ &= g_{1,p}(n) \leq g(n). \end{aligned}$$

This gives us the desired result. ■

The following is therefore a consequence of Table 1, Theorem 6, and Lemma 8.

Corollary 9. *For each integer $n \geq 3$, $\mathrm{td}(C_n) \leq \frac{2}{3}n^{1.5} - \frac{1}{2}n$ except for $n = 3, 7$ ($\mathrm{td}(C_n) < \frac{2}{3}n^{1.5} - \frac{1}{2}n + 1.1$ for every $n \geq 3$).*

In closing, we determine the exact values of the total detection numbers of C_n for each $n \in \{p(4p + 1), p(4p + 1) + 1, p(4p + 5), p(4p + 5) + 1 : p \in \mathbb{N}\}$. For example, see Figure 7 for how we obtain grid diagrams of C_{18} and C_{19} that produce detectable labelings whose values are the minimum possible. Generalizing this construction, we obtain the following result.

Theorem 10. *For each $p \in \mathbb{N}$,*

$$\begin{aligned} \mathrm{td}(C_{p(4p+1)}) &= \tfrac{1}{3}p(16p^2 - 3p - 1) = \mathrm{td}(C_{p(4p+1)+1}) - 2p, \\ \mathrm{td}(C_{p(4p+5)}) &= \tfrac{1}{3}p(16p^2 + 21p - 1) = \mathrm{td}(C_{p(4p+5)+1}) - 2p. \end{aligned}$$

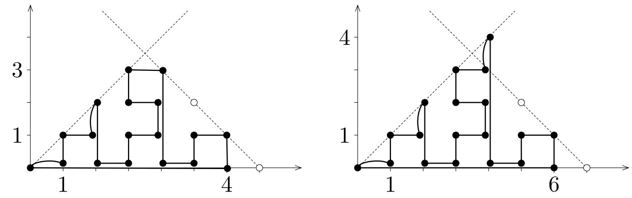

Figure 7. Obtaining detectable labelings of C_{18} and C_{19}.

We have seen that $\frac{2}{3}n^{1.5} - \frac{3}{2}n < \mathrm{td}(C_n) \leq \frac{2}{3}n^{1.5} - \frac{1}{2}n$ for $n \geq 8$. Table 2 suggests that $\frac{2}{3}n^{1.5} - \frac{3}{4}n$ could be an improved lower bound for $\mathrm{td}(C_n)$. Indeed, we see that it is the case for those n that can be written in terms of p as in Theorem 10 by verifying that $\frac{4}{9}n^3 < \left(\mathrm{td}(C_n) + \frac{3}{4}n\right)^2$. Also, Table 3 shows that the same holds for $3 \leq n \leq 20$.

n	5	6	9	10	18	19	26	27	39	40	51	52
$\mathrm{td}(C_n)$	4	6	12	14	38	42	70	74	134	140	206	212
$\lceil \frac{2}{3}n^{1.5} - \frac{3}{4}n \rceil$	4	6	12	14	38	41	69	74	134	139	205	211

n	68	69	84	85	105	106	125	126	150	151	174	175
$\mathrm{td}(C_n)$	324	332	452	460	640	650	840	850	1114	1126	1402	1414
$\lceil \frac{2}{3}n^{1.5} - \frac{3}{4}n \rceil$	323	331	451	459	639	649	838	849	1113	1124	1400	1413

Table 2. $\mathrm{td}(C_n)$ for $n \in \{p(4p+1), p(4p+1)+1, p(4p+5), p(4p+5)+1 : 1 \leq p \leq 6\}$.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\mathrm{td}(C_n)$	3	3	4	6	9	11	12	14	17	21	23	26	29	33	35	38	42	47
$\lceil \frac{2}{3}n^{1.5} - \frac{3}{4}n \rceil$	2	3	4	6	8	10	12	14	17	19	22	25	28	31	34	38	41	45

Table 3. $\mathrm{td}(C_n)$ for $3 \leq n \leq 20$.

Conjecture 11. *For every* $n \geq 8$, $\frac{2}{3}n^{1.5} - \frac{3}{4}n \leq \mathrm{td}(C_n) \leq \frac{2}{3}n^{1.5} - \frac{1}{2}n$.

Problem 12. How close is $\mathrm{td}(C_n)$ to $\frac{2}{3}n^{1.5} - \frac{3}{4}n$?

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