

AN ORIENTED VERSION OF THE 1-2-3 CONJECTURE

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Abstract

The well-known 1-2-3 Conjecture addressed by Karoński, Łuczak and Thomason asks whether the edges of every undirected graph G with no isolated edge can be assigned weights from $\{1, 2, 3\}$ so that the sum of incident weights at each vertex yields a proper vertex-colouring of G . In this work, we consider a similar problem for oriented graphs. We show that the arcs of every oriented graph \vec{G} can be assigned weights from $\{1, 2, 3\}$ so that every two adjacent vertices of \vec{G} receive distinct sums of outgoing weights. This result is tight in the sense that some oriented graphs do not admit such an assignment using the weights from $\{1, 2\}$ only. We finally prove that deciding whether two weights are sufficient for a given oriented graph is an NP-complete problem. These results also hold for product or list versions of this problem.

Keywords: oriented graph, neighbour-sum-distinguishing arc-weighting, complexity, 1-2-3 Conjecture.

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1. INTRODUCTION

Let G be an undirected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. For every vertex v of G , we denote by $N(v)$ the set of vertices neighbouring v . A k -edge-weighting w of G is an assignment $w : E(G) \rightarrow \{1, 2, \dots, k\}$. From w , one naturally deduces a vertex-colouring ϕ_w of G , where $\phi_w(v) = \sum_{u \in N(v)} w(vu)$

for every vertex v . In other words, every vertex v receives the sum of its incident weights by w as its “colour”. If ϕ_w is proper, i.e. we have $\phi_w(u) \neq \phi_w(v)$ for every two adjacent vertices u and v of G , then we say that w is *neighbour-sum-distinguishing* (*nsd* for short).

The study of neighbour-sum-distinguishing edge-weighting of graphs was initiated in 2004, with Karoński, Łuczak and Thomason posing the following conjecture.

1-2-3 Conjecture. *Every graph with no isolated edge admits an nsd 3-edge-weighting.*

Despite many efforts to tackle it, the 1-2-3 Conjecture is still an open question. The best result towards the 1-2-3 Conjecture at the moment is due to Kalkowski, Karoński and Pfender, who proved that every graph with no isolated edge admits an nsd 5-edge-weighting [6].

Many edge-weighting problems inspired by the 1-2-3 Conjecture have been introduced in the literature. As examples, let us mention the notions of *detectable colouring* [12] or *locally irregular edge-colouring* [11] of graphs. We refer the interested reader to [1], where numerous more variants of the original problem are surveyed. Most of these works are devoted to undirected graphs, but one could wonder about an oriented version of the 1-2-3 Conjecture.

We first introduce some terminology related to oriented graphs. Let \vec{G} be an oriented graph, i.e. a loopless directed graph whose every two vertices are joined by at most one arc in either direction, with vertex and arc sets $V(\vec{G})$ and $A(\vec{G})$, respectively. Given a vertex v of \vec{G} , we denote by $N^-(v)$ (resp. $N^+(v)$) the set $\{u \in V(\vec{G}) : \vec{uv} \in A(\vec{G})\}$ (resp. $\{u \in V(\vec{G}) : \vec{vu} \in A(\vec{G})\}$). The *indegree* (resp. *outdegree*) of v , denoted $d^-(v)$ (resp. $d^+(v)$), is $|N^-(v)|$ (resp. $|N^+(v)|$).

To our knowledge, the only link between the 1-2-3 Conjecture and oriented graphs is the following problem. Let w be a k -arc-weighting of \vec{G} , and let $q_w^-(v)$ and $q_w^+(v)$ be $\sum_{u \in N^-(v)} w(\vec{uv})$ and $\sum_{u \in N^+(v)} w(\vec{vu})$, respectively, for every vertex v . The functions q_w^- and q_w^+ naturally yield a vertex-colouring q_w of \vec{G} , where $q_w(v) = q_w^+(v) - q_w^-(v)$ for every vertex v of \vec{G} . It was proved in [8] that every oriented graph admits a 2-arc-weighting w which yields a *proper* vertex-colouring q_w . A list version of the same result was also proved independently in [7] and [9] using different methods.

We here investigate another problem. As for the undirected case, a k -arc-weighting w of \vec{G} yields a vertex-colouring ϕ_w of \vec{G} where $\phi_w(v) = \sum_{u \in N^+(v)} w(\vec{vu})$ for every $v \in V(\vec{G})$. This time, the “colour” of v by ϕ_w , sometimes called its *weighted outdegree* (with respect to w), is the sum of its outgoing weights (one could similarly consider the sum of its ingoing weights). Again, if ϕ_w has the property to be proper, then we say that w is *neighbour-sum-distinguishing* (*nsd* for short).

A quick investigation on small oriented graphs suggests that all oriented graphs should admit an nsd 3-arc-weighting. Besides, there exist oriented graphs, such as the circuit on three vertices, which do not admit an nsd 2-arc-weighting. We hence investigate the following question.

Question 1. *Does every oriented graph admit an nsd 3-arc-weighting?*

Although Question 1 and the 1-2-3 Conjecture are quite similar in essence, the two underlying problems do not seem to share any systematic relationship. The fact that weighting some arc \overrightarrow{uv} only affects the weighted outdegree of u makes Question 1 easier to handle. We hence answer this question in the affirmative in Section 2. We then turn our concern on conditions for some specific classes of oriented graphs to admit an nsd 2-arc-weighting in Section 3. We next prove, in Section 4, that an “easy” characterization of oriented graphs which admit an nsd 2-arc-weighting cannot exist unless $P=NP$. For this purpose, we show that the problem of deciding whether an oriented graph admits an nsd 2-arc-weighting is NP-complete. Concluding remarks can be found in Section 5. In particular, we point out that our results directly apply to product or list versions of the problem.

2. ALL ORIENTED GRAPHS ADMIT AN NSD 3-ARC-WEIGHTING

Our first result states that every oriented graph admits an nsd 3-arc-weighting. This relies on the fact that every oriented graph has a “convenient” vertex, i.e. a vertex which admits a large number of potential weighted outdegrees compared to its number of neighbours. The existence of such a vertex allows the use of an inductive proof scheme. Our proof also yields a polynomial-time algorithm for finding an nsd 3-arc-weighting of every oriented graph.

Theorem 2. *Every oriented graph \vec{G} admits an nsd 3-arc-weighting.*

Proof. The claim is proved by induction on the size of \vec{G} , i.e. its number of arcs. As a base case, the claim is clearly true when \vec{G} has size 0 or 1. Suppose now that the claim is true for every oriented graph with at most $m - 1$ arcs, and assume \vec{G} has size $m \geq 2$.

Note that \vec{G} necessarily has a vertex v such that $d^+(v) > 0$ and $d^+(v) \geq d^-(v)$ since otherwise we would have $\sum_{v \in V(\vec{G})} d^-(v) \neq \sum_{v \in V(\vec{G})} d^+(v)$. A nsd 3-arc-weighting of \vec{G} is then obtained as follows. Start by removing the arcs outgoing from v . According to the induction hypothesis, the remaining oriented graph admits an nsd 3-arc-weighting w . Now put back the arcs outgoing from v to \vec{G} , and extend w to these arcs in such a way that the weighted outdegree of v is different from the weighted outdegrees of the $d^-(v) + d^+(v)$ vertices neighbouring

v . This is possible since there are $2d^+(v) + 1$ potential weighted outdegrees for v , namely those from $\{d^+(v), d^+(v) + 1, \dots, 3d^+(v)\}$, while the number of forbidden weighted outdegrees is at most $d^-(v) + d^+(v) < 2d^+(v) + 1$ by our assumption on $d^-(v)$ and $d^+(v)$. Because assigning a weight to the arcs outgoing from v does not affect the weighted outdegree by w of any vertex neighbouring v , the extension of w to \vec{G} remains neighbour-sum-distinguishing. ■

3. CONDITIONS FOR SOME FAMILIES OF ORIENTED GRAPHS TO ADMIT AN NSD 2-ARC-WEIGHTING

By Theorem 2, we know that every oriented graph admits an nsd 3-arc-weighting. Throughout this section, we focus on some common families of oriented graphs and exhibit conditions for their members to admit an nsd 2-arc-weighting.

3.1. Acyclic oriented graphs

An oriented graph is *acyclic* if it has no directed cycle $\overrightarrow{v_1 v_2 \cdots v_k v_1}$, with $k \geq 3$, as a subgraph. We show that every such oriented graph admits an nsd 2-arc-weighting.

Theorem 3. *Every acyclic oriented graph admits an nsd 2-arc-weighting.*

Proof. We prove the claim by induction on the order, i.e. the number of vertices, of acyclic oriented graphs. As a starting point, note that an oriented graph with only one vertex admits an nsd 2-arc-weighting. Suppose now that the claim is true for every acyclic oriented graph with order at most $n - 1$ for some $n \geq 2$, and let \vec{G} be an acyclic oriented graph on n vertices.

Since \vec{G} is acyclic, there are vertices of \vec{G} with indegree 0. Let v be such a vertex, and consider the graph \vec{G}' obtained by removing v from \vec{G} . Clearly \vec{G}' is acyclic and admits an nsd 2-arc-weighting w according to the induction hypothesis. We now extend w to \vec{G} , i.e. we weight the arcs outgoing from v in such a way that w remains neighbour-sum-distinguishing. There are $d^+(v) + 1$ possible weighted outdegrees for v , namely those from $\{d^+(v), d^+(v) + 1, \dots, 2d^+(v)\}$, while there are at most $d^+(v)$ forbidden weighted outdegrees for v , namely the weighted outdegrees by w of the vertices in $N^+(v)$. Since weighting the arcs outgoing from v does not alter the weighted outdegree of any vertex neighbouring v , we can choose an available weighted outdegree for v and weight the arcs outgoing from v consequently. This completes the proof. ■

3.2. Oriented graphs whose underlying graphs are k -colourable

Given an undirected graph G , a *proper k -vertex-colouring* of G is a partition of $V(G)$ into k parts V_1, \dots, V_k such that V_i is an independent set for every

$i \in \{1, 2, \dots, k\}$. The least number of parts of a proper vertex-colouring of G is referred to as the *chromatic number* of G , denoted $\chi(G)$. Assuming \vec{G} is an orientation of G , i.e. \vec{G} is obtained by orienting every edge of G in either direction, we denote by $\text{und}(\vec{G})$ the *underlying* undirected graph of \vec{G} , that is G .

As pointed out in some references of the literature (see e.g. [12] or [10]), first partitioning a graph into several independent sets before weighting its edges can be a good method for finding a specific edge-weighting. This is also the case regarding neighbour-sum-distinguishing arc-weighting, as shown in the following result.

Theorem 4. *Every oriented graph \vec{G} admits an nsd $\chi(\text{und}(\vec{G}))$ -arc-weighting.*

Proof. Let $k = \chi(\text{und}(\vec{G}))$, and V_0, \dots, V_{k-1} be a proper k -vertex-colouring of $\text{und}(\vec{G})$. Process the vertices of \vec{G} in arbitrary order. If the vertex v belongs to the part V_i , then weight the arcs outgoing from v with weights from $\{1, 2, \dots, k\}$ in such a way that the weighted outdegree of v is congruent to i modulo k , e.g. by assigning i to one arc outgoing to v (or k if $i = 0$), and k to all of its other outgoing arcs. This is possible unless $d^+(v) = 0$ since, in such a situation, the only possible weighted outdegree for v is 0. Once the process is achieved, two adjacent vertices u and v cannot have the same weighted outdegrees since otherwise either u and v would both belong to some part V_i , which is impossible since V_i is an independent set, or we would have $d^+(u) = d^+(v) = 0$, which is impossible since u and v are adjacent. ■

As a corollary of Theorem 4, we get in particular the following result.

Corollary 5. *Every oriented graph \vec{G} whose underlying graph is bipartite admits an nsd 2-arc-weighting.*

3.3. Tournaments

Our strategy for weighting the arcs of a tournament \vec{T} is based on the following lemma, which could be also deduced from result of Landau regarding so-called *score sequences* (see [3], Theorem 29).

Lemma 6. *For every $k \in \{1, 2, \dots, |V(\vec{T})|\}$, let $n_k \geq 0$ denote the number of vertices with outdegree at most k of a tournament \vec{T} . Then $n_k \leq 2k + 1$.*

Proof. Let k be fixed, with $1 \leq k \leq |V(\vec{T})|$. Denote by $X \subseteq V(\vec{T})$ the set of the n_k vertices of \vec{T} whose outdegree is at most k , and by s the sum of outdegrees of the vertices in X . Naturally, we have $s \leq n_k k$. We also have $s \geq \frac{n_k(n_k-1)}{2}$ since X induces a tournament, and there may be arcs of \vec{T} whose tails lie in X ,

and whose heads do not lie in X . We hence get $\frac{n_k(n_k-1)}{2} \leq n_k k$, which implies that $n_k \leq 2k + 1$. ■

We now give an easy sufficient condition for a tournament to admit an nsd 2-arc-weighting.

Theorem 7. *For every $k \in \{1, 2, \dots, |V(\vec{T})|\}$, let $n_k \geq 0$ denote the number of vertices with outdegree at most k of a tournament \vec{T} . If $n_k \leq k + 1$ for every $k \in \{1, 2, \dots, |V(\vec{T})|\}$, then \vec{T} admits an nsd 2-arc-weighting.*

Proof. The proof is based on the following simple weighting scheme for \vec{T} . Process the vertices of \vec{T} in increasing order of their outdegrees. For each vertex v , weight the arcs outgoing from v in such a way that the weighted outdegree of v gets the smallest possible value which does not appear among the weighted outdegrees of the vertices considered in earlier steps of the process.

It has to be noted that this weighting scheme necessarily produces an nsd arc-weighting of \vec{T} when the weights among $\{1, 2, 3\}$ are used. Suppose indeed that, at some point of the process, we are dealing with a vertex v but we cannot weight v satisfyingly. Set $k = d^+(v)$. This situation means that we have attributed all the weighted outdegrees among $\{k, k+1, \dots, 3k\}$ to the vertices considered before v , i.e. that at least $2k + 1$ vertices have been treated before v . Due to how the process is led, these vertices have outdegree at most k . But then it means that $n_k > 2k + 1$, which is impossible according to Lemma 6.

Now assume we are using the weights among $\{1, 2\}$ only. Since the weighted outdegree of v can take any value from $\{k, k+1, \dots, 2k\}$ and at most $n_k - 1 < k + 1$ vertices have been considered in earlier steps of the process, there is necessarily one non-conflicting value which can be chosen as the weighted outdegree of v . We then just have to weight the arcs outgoing from v consequently. ■

It is worth mentioning that a tournament \vec{T} admits an nsd 1-arc-weighting if and only if the vertices of \vec{T} have distinct outdegrees, i.e. \vec{T} is transitive. This improves Theorem 7 for transitive tournaments.

3.4. Cartesian products of oriented graphs

Let \vec{G} and \vec{H} be two oriented graphs. The *Cartesian product* of \vec{G} and \vec{H} , denoted $\vec{G} \square \vec{H}$, is the oriented graph with vertex set $V(\vec{G}) \times V(\vec{H})$, and whose two vertices (u, v) and (u', v') are joined by an arc from (u, v) towards (u', v') if and only if $u = u'$ and $\vec{vv'} \in E(\vec{H})$, or $\vec{uu'} \in E(\vec{G})$ and $v = v'$.

The Cartesian product of graphs is a classic graph operation which has been studied a lot since its introduction [2]. The reason for focusing on the Cartesian product of oriented graphs is that if \vec{G} and \vec{H} both admit an nsd k -arc-weighting

for some value of k , one could expect $\vec{G} \square \vec{H}$ to need k' weights to obtain an nsd arc-weighting, where k' depends on k . In the next result, we show that the existence of nsd k -arc-weightings of \vec{G} and \vec{H} implies the existence of an nsd k -arc-weighting of $\vec{G} \square \vec{H}$.

Theorem 8. *Assume \vec{G} and \vec{H} admit an nsd k - and ℓ -arc-weighting, respectively. Then $\vec{G} \square \vec{H}$ admits an nsd $\max\{k, \ell\}$ -arc-weighting.*

Proof. Let $w_{\vec{G}}$ and $w_{\vec{H}}$ be nsd k - and ℓ -arc-weighting of \vec{G} and \vec{H} , respectively. Let w be a $\max\{k, \ell\}$ -arc-weighting of $\vec{G} \square \vec{H}$ defined as follows:

$$w(\overrightarrow{(u, v)(u', v')}) = \begin{cases} w_{\vec{H}}(\overrightarrow{vv'}) & \text{if } u = u', \\ w_{\vec{G}}(\overrightarrow{uu'}) & \text{otherwise.} \end{cases}$$

Assume $\overrightarrow{(u, v)(u', v')}$ is an arc of $\vec{G} \square \vec{H}$. Then we have $\phi_w((u, v)) = \phi_{w_{\vec{G}}}(u) + \phi_{w_{\vec{H}}}(v)$ and $\phi_w((u', v')) = \phi_{w_{\vec{G}}}(u') + \phi_{w_{\vec{H}}}(v')$. Since (u, v) and (u', v') are adjacent, we have either $u = u'$ or $v = v'$ by construction. Assume $u = u'$ without loss of generality. Then $\phi_{w_{\vec{G}}}(u) = \phi_{w_{\vec{G}}}(u')$. Now, because $w_{\vec{H}}$ is neighbour-sum-distinguishing, we have $\phi_{w_{\vec{H}}}(v) \neq \phi_{w_{\vec{H}}}(v')$. It then follows that $\phi_w((u, v)) \neq \phi_w((u', v'))$. ■

An immediate corollary of Theorem 8 is the following result.

Corollary 9. *Assume \vec{G} and \vec{H} both admit an nsd 2-arc-weighting. Then $\vec{G} \square \vec{H}$ admits an nsd 2-arc-weighting.*

4. ALGORITHMIC COMPLEXITY

In this section, we focus on the complexity of the following decision problem.

NEIGHBOUR-SUM-DISTINGUISHING k -ARC-WEIGHTING— k -NSDAW

Instance: An oriented graph \vec{G} .

Question: Does \vec{G} admit an nsd k -arc-weighting?

An oriented graph \vec{G} admits an nsd 1-arc-weighting if and only if every two adjacent vertices of \vec{G} have distinct outdegrees. Since this property can be checked in polynomial time, the problem 1-NSDAW is in P. Besides, every problem k -NSDAW with $k \geq 3$ is also in P since the answer to every of its instances is *yes*, according to Theorem 2.

We deal with the complexity of the remaining problem, i.e. 2-NSDAW. We show this problem to be NP-complete in Theorem 12 below, by reduction from

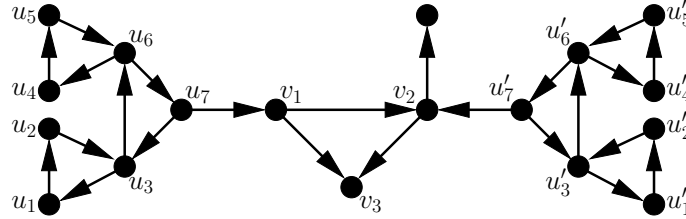


Figure 1. The forbidding gadget $\overrightarrow{F}_{3,4}$.

3-SAT. For this purpose, we first introduce several gadgets to “force” the propagation of an nsd 2-arc-weighting along an oriented graph.

We first introduce two kinds of *forbidding* gadgets. A forbidding gadget \overrightarrow{F} is composed of one *root vertex* with outdegree 0 adjacent to *forcing vertices*. The weighting property of \overrightarrow{F} is that each of its forcing vertices has always the same weighted outdegree by every nsd 2-arc-weighting of \overrightarrow{F} . Assume x_1, x_2, \dots, x_k denote the respective outdegrees of the forcing vertices. Then, after having identified the root of \overrightarrow{F} with a vertex v of some graph \overrightarrow{G} , the vertex v cannot have weighted outdegree x_1, x_2, \dots, x_k by an nsd 2-arc-weighting of \overrightarrow{G} because of the forcing vertices of \overrightarrow{F} neighbouring v .

First, we define a $(2k - 1, 2k)$ -*forbidding gadget*, denoted $\overrightarrow{F}_{2k-1,2k}$, for every integer $k \geq 2$. These gadgets are defined inductively. The gadget $\overrightarrow{F}_{3,4}$ is the one depicted in Figure 1. The root of $\overrightarrow{F}_{3,4}$ is v_3 , while its forcing vertices are v_1 and v_2 . Now, for every value of $k \geq 3$ such that the oriented graphs $\overrightarrow{F}_{2k'-1,2k'}$ have been defined for every $k' < k$, the oriented graph $\overrightarrow{F}_{2k-1,2k}$ is constructed as follows. Let v_1^k, v_2^k and v_3^k be three distinct vertices joined by $\overrightarrow{v_1^k v_2^k}, \overrightarrow{v_1^k v_3^k}$ and $\overrightarrow{v_2^k v_3^k}$. Now, for every $k' \in \{2, 3, \dots, k - 1\}$, identify v_1^k and the root of a copy of $\overrightarrow{F}_{2k'-1,2k'}$. Repeat the same procedure but with v_2^k instead of v_1^k and new copies of the forbidding gadgets. Finally add an arc from v_1^k towards $k - 2$ new vertices with outdegree 0, and similarly from v_2^k towards $k - 1$ new vertices with outdegree 0. The root of $\overrightarrow{F}_{2k-1,2k}$ is v_3^k , while its forcing vertices are v_1^k and v_2^k .

Lemma 10. *Let $k \geq 2$ be fixed. In every nsd 2-arc-weighting of $\overrightarrow{F}_{2k-1,2k}$, one of the forcing vertices has weighted outdegree $2k - 1$, while the other forcing vertex has weighted outdegree $2k$.*

Proof. We prove the claim by induction on k . At each step, let w be an nsd 2-arc-weighting of the considered forbidding gadget. Start with $\overrightarrow{F}_{3,4}$. Since u_1 and u_2 are adjacent and both have outdegree 1, we have $\{\phi_w(u_1), \phi_w(u_2)\} = \{1, 2\}$. By the same argument, we have $\{\phi_w(u_4), \phi_w(u_5)\} = \{1, 2\}$. Since u_3

and u_6 are adjacent, both adjacent to vertices with weighted outdegree 2, and have outdegree 2, we necessarily have $\{\phi_w(u_3), \phi_w(u_6)\} = \{3, 4\}$. Because u_7 is adjacent to u_3 and u_6 and has outdegree 2, we necessarily get $\phi_w(u_7) = 2$. Repeating the same arguments to the oriented subgraph of $\overrightarrow{F_{3,4}}$ induced by the u'_i 's, we also obtain $\phi_w(u'_7) = 2$. Finally, since v_1 and v_2 are adjacent, both adjacent to a vertex with weighted outdegree 2, and have outdegree 2, we have $\{\phi_w(v_1), \phi_w(v_2)\} = \{3, 4\}$ as claimed.

Assume the claim is true for every k up to $i - 1$, and consider $\overrightarrow{F_{2k-1,2k}}$. Because v_1^k and v_2^k have outdegree k by construction, their weighted outdegree by w can only take value from $\{k, k + 1, \dots, 2k\}$. However, since these two vertices are both identified with the roots of forbidding gadgets $\overrightarrow{F_{3,4}}, \overrightarrow{F_{5,6}}, \dots, \overrightarrow{F_{2k-3,2k-2}}$, their weighted outdegree cannot take value from $\{3, 4, \dots, 2k-3, 2k-2\}$ according to the induction hypothesis. Therefore, we have $\{\phi_w(v_1^k), \phi_w(v_2^k)\} = \{2k - 1, 2k\}$ since v_1^k and v_2^k are adjacent. ■

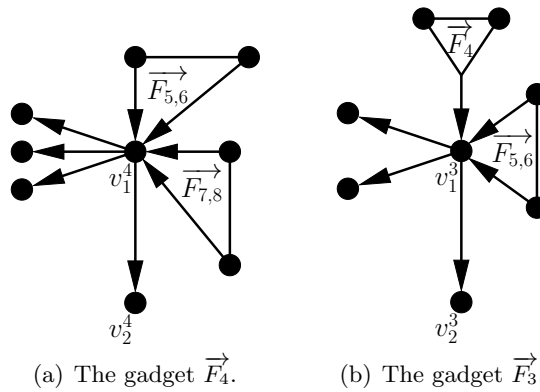


Figure 2. Two examples of forbidding gadgets. A triangle represents a forbidding gadget.

We now define a k -forbidding gadget, denoted $\overrightarrow{F_k}$, for every integer $k \geq 3$. The oriented graph $\overrightarrow{F_k}$ originally consists in an arc $v_1^k v_2^k$. We call v_2^k and v_1^k the root and the forcing vertex of $\overrightarrow{F_k}$, respectively. Next add an arc from v_1^k towards $k - 1$ new vertices with outdegree 0. The end of the construction depends on the parity of k . If k is even, then identify v_1^k and the root of each of the forbidding gadgets $\overrightarrow{F_{k+1,k+2}}, \overrightarrow{F_{k+3,k+4}}, \dots, \overrightarrow{F_{2k-1,2k}}$. Otherwise, i.e. if k is odd, then identify v_1^k and the roots of $\overrightarrow{F_{k+1}}$, and $\overrightarrow{F_{k+2,k+3}}, \overrightarrow{F_{k+4,k+5}}, \dots, \overrightarrow{F_{2k-1,2k}}$. The gadgets $\overrightarrow{F_3}$ and $\overrightarrow{F_4}$ are depicted in Figure 2.

Lemma 11. *Let $k \geq 3$ be fixed. In every nsd 2-arc-weighting of $\overrightarrow{F_k}$, the forcing vertex has weighted outdegree k .*

Proof. Let w be an nsd 2-arc-weighting of $\overrightarrow{F_k}$. Assume k is even. Since v_1^k has outdegree k , its weighted outdegree by w can only take value from $\{k, k +$

$1, \dots, 2k\}$. But, because v_1^k is the root of forbidding gadgets $\overrightarrow{F_{k+1, k+2}}, \overrightarrow{F_{k+3, k+4}}, \dots, \overrightarrow{F_{2k-1, 2k}}$, it is adjacent to vertices with weighted outdegrees $k+1, k+2, \dots, 2k$ according to Lemma 10. Hence, the only remaining weighted outdegree for v_1^k by w is k . The claim follows similarly when k is odd, the value $k+1$ being forbidden as the weighted outdegree of v_1^k because it was identified with the root of a forbidding gadget $\overrightarrow{F_{k+1}}$ with $k+1$ being even. ■

Thanks to the two kinds of forbidding gadgets introduced above, we can now “force” a vertex of some oriented graph to have a specific weighted outdegree by an nsd 2-arc-weighting. Let v be a vertex of some oriented graph \overrightarrow{G} , and $k \geq d^+(v)$ be some integer. Assume we are given a set $D \subseteq \{k, k+1, \dots, 2k\}$ of “allowed” weighted outdegrees for v by an nsd 2-arc-weighting of \overrightarrow{G} . Then, by “turning v into a D -vertex”, we refer to the following operations:

- first add arcs from v towards $k - d^+(v)$ new vertices with outdegree 0 so that v has outdegree k ,
- then identify v and the respective root of each of the forbidding gadgets $\overrightarrow{F_i}$ with $i \in \{k, k+1, \dots, 2k\} - D$.

Clearly, because of the forcing vertices neighbouring v , the weighted outdegree of v by an nsd 2-arc-colouring of \overrightarrow{G} necessarily takes value among D .

We are now ready to introduce our hardness reduction.

Theorem 12. *The problem 2-NSDAW is NP-complete.*

Proof. Given a 2-arc-weighting w of \overrightarrow{G} , one can first compute the vertex-colouring ϕ_w of \overrightarrow{G} from w , and then check whether it is proper. Since this procedure can be achieved in polynomial time, 2-NSDAW is in NP.

We now prove that 2-NSDAW is NP-hard by reduction from the following classical NP-complete problem [5].

3-SAT

Instance: A 3CNF formula F over clauses C_1, \dots, C_m and variables x_1, \dots, x_n .

Question: Does F admit a satisfying truth assignment?

Note that we can assume that every possible literal appears in F . Indeed, if ℓ_i does not appear in any clause of F , then the 3CNF formula $F \wedge (\ell_i \vee \ell_i \vee \overline{\ell_i})$ is satisfiable if and only if F is satisfiable. By repeating this procedure for all literals which do not appear in F , we obtain a formula equivalent to F but involving all possible literals over its variables. This procedure is achieved in polynomial time.

We introduce some more terminology regarding an instance of 3-SAT. The $2n$ literals of F over its n variables are denoted by $\ell_1, \ell_2, \dots, \ell_{2n}$, the ordering being arbitrary. By $n_i \geq 1$, we refer to the number of distinct clauses of F that contain

the literal ℓ_i for every $i \in \{1, 2, \dots, 2n\}$. By $c_j \in \{1, 2, 3\}$, we denote the number of distinct literals which appear in the clause C_j for every $j \in \{1, 2, \dots, m\}$. In the case where $c_j = 1$, i.e. C_j is of the form $(\ell_i \vee \ell_i \vee \ell_i)$, note that ℓ_i is set to true by every satisfying truth assignment of F . In such a situation, we say that ℓ_i is *forced* to true by C_j .

Our hardness reduction is described below. From a 3CNF formula F , we construct an oriented graph \overrightarrow{G}_F such that F is satisfiable if and only if \overrightarrow{G}_F admits an nsd 2-arc-weighting w_F .

Let t and f be two injective mappings from $\{x_1, x_2, \dots, x_n\}$ to $\{2n, 2n + 1, \dots, 3n - 1\}$ and $\{3n, 3n + 1, \dots, 4n\}$, respectively. Assuming $\ell_j = x_i$ and $\ell_{j'} = \bar{x}_i$, i.e. ℓ_j and $\ell_{j'}$ are the literals associated with the variable x_i , we set $t(\ell_j) = f(\ell_{j'}) = t(x_i)$ and $f(\ell_j) = t(\ell_{j'}) = f(x_i)$.

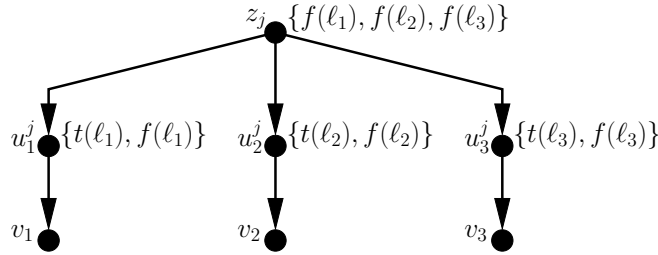


Figure 3. Partial resulting clause gadget for a clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$. The integer sets represent the allowed weighted outdegrees at each vertex by an nsd 2-arc-weighting of \overrightarrow{G}_F .

First, for every literal ℓ_i of F , add a vertex v_i in \overrightarrow{G}_F . Now consider every clause C_j of F . We associate a *clause gadget* in \overrightarrow{G}_F with C_j , its structure depending on the value of c_j . Denote by $\ell_{j_1}, \dots, \ell_{j_{c_j}}$ the distinct literals of C_j . Let $\overrightarrow{u_{j_1}^j v_{j_1}}, \dots, \overrightarrow{u_{j_{c_j}}^j v_{j_{c_j}}}$ be c_j arcs of \overrightarrow{G}_F , where $u_{j_1}^j, \dots, u_{j_{c_j}}^j$ are new vertices. If $c_j = 1$, i.e. ℓ_{j_1} is forced to true by C_j , then turn $u_{j_1}^j$ into a $\{t(\ell_{j_1})\}$ -vertex. Otherwise, i.e. $c_j \in \{2, 3\}$, turn each vertex $u_{j_i}^j$ into a $\{t(\ell_{j_i}), f(\ell_{j_i})\}$ -vertex, add a vertex z_j to \overrightarrow{G}_F , add arcs from z_j towards $u_{j_1}^j, \dots, u_{j_{c_j}}^j$, and turn z_j into a $\{f(\ell_{j_1}), \dots, f(\ell_{j_{c_j}})\}$ -vertex. This construction is depicted in Figure 3.

Claim 13. *Let $C_j = (\ell_{j_1} \vee \ell_{j_2} \vee \ell_{j_3})$ be a clause of F . Then at least one of $t(\ell_{j_1})$, $t(\ell_{j_2})$ and $t(\ell_{j_3})$ belongs to $\{\phi_{w_F}(u_{j_1}^j), \phi_{w_F}(u_{j_2}^j), \phi_{w_F}(u_{j_3}^j)\}$.*

Proof. The claim is true when $c_j = 1$ since $u_{j_1}^j$ is a $\{t(\ell_{j_1})\}$ -vertex. When $c_j \in \{2, 3\}$, note that we cannot have $\phi_{w_F}(u_{j_1}^j) = f(\ell_{j_1}), \dots, \phi_{w_F}(u_{j_{c_j}}^j) = f(\ell_{j_{c_j}})$ since z_j is a $\{f(\ell_{j_1}), \dots, f(\ell_{j_{c_j}})\}$ -vertex. On the contrary, note that if there is

an $i \in \{1, 2, \dots, c_j\}$ such that $\phi_{w_F}(u_{j_i}^j) = t(\ell_{j_i})$, then we can weight the arcs outgoing from z_j in such a way that the weighted outdegree of z_j by w_F is $f(\ell_{j_i})$. □

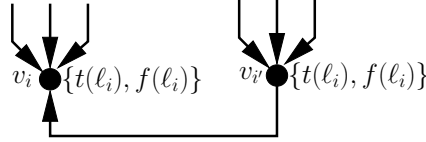


Figure 4. Partial subgraph of $\overrightarrow{G_F}$ for two literals ℓ_i and $\ell_{i'}$ such that $\ell_{i'} = \bar{\ell}_i$. The integer sets represent the allowed weighted outdegrees at each vertex by an nsd 2-arc-weighting of $\overrightarrow{G_F}$.

Let $i \in \{1, 2, \dots, 2n\}$. Note that, so far, the vertex v_i has indegree n_i . Consider $i' \in \{1, 2, \dots, 2n\}$ such that $\ell_{i'} = \bar{\ell}_i$. To finish the construction of $\overrightarrow{G_F}$, add the arc $\overrightarrow{v_{i'}v_i}$, and turn v_i and $v_{i'}$ into $\{t(\ell_i), f(\ell_i)\}$ -vertices. This step of the construction is illustrated in Figure 4.

Claim 14. *Let $i \in \{1, 2, \dots, 2n\}$, and i_1, i_2, \dots, i_{n_i} be the indexes of the distinct clauses of F that contain ℓ_i . Then $\phi_{w_F}(u_i^{i_1}) = \phi_{w_F}(u_i^{i_2}) = \dots = \phi_{w_F}(u_i^{i_{n_i}})$.*

Proof. Recall that the $u_i^{i_j}$'s can only have weighted outdegree $t(\ell_i)$ or $f(\ell_i)$ by w_F . Now note that if one of the $u_i^{i_j}$'s has weighted outdegree $t(\ell_i)$ by w_F while another such vertex has weighted outdegree $f(\ell_i)$, then w_F cannot be extended to the arcs outgoing from v_i since v_i is a $\{t(\ell_i), f(\ell_i)\}$ -vertex. On the contrary, if all the $u_i^{i_j}$'s neighbouring v_i have the same weighted outdegree, say $t(\ell_i)$, then the arcs outgoing from v_i can be weighted in such a way that $\phi_{w_F}(v_i) = f(\ell_i)$. □

Claim 15. *Let $i, i' \in \{1, 2, \dots, 2n\}$ be two integers such that $\ell_{i'} = \bar{\ell}_i$. Then $\phi_{w_F}(v_i) \neq \phi_{w_F}(v_{i'})$.*

Proof. The claim follows from the fact that v_i and $v_{i'}$ are adjacent. □

We now claim that F has a satisfying truth assignment if and only if $\overrightarrow{G_F}$ admits its nsd 2-arc-weighting w_F . Assume $C_j = (\ell_{j_1} \vee \ell_{j_2} \vee \ell_{j_3})$ is a clause of F , and that having $\phi_{w_F}(u_{j_i}^j) = t(\ell_{j_i})$ (resp. $f(\ell_{j_i})$) simulates the assignment of ℓ_{j_i} to true (resp. false) in C_j by a truth assignment of F . Then, by Claim 13, every clause gadget of $\overrightarrow{G_F}$ must have a vertex $u_{j_i}^j$ whose weighted outdegree by w_F is $t(\ell_{j_i})$. This simulates the fact that every clause of F must have one true literal by a satisfying truth assignment of F . Claim 14 depicts the fact that, by a truth assignment of F , every literal of F provides the similar truth value to every clause it appears in. Finally, Claim 15 represents the fact that two opposite literals cannot be assigned the same truth value by a truth assignment of F .

With these arguments, we can deduce a satisfying truth assignment of F from w_F , and vice-versa. ■

5. DISCUSSION

Recall that the proof of Theorem 2 mainly relies on the fact that the number of possible weighted outdegrees by an arc-weighting for a vertex with outdegree d is sufficiently large, i.e. at least $2d + 1$, when the weights from $\{1, 2, 3\}$ are allowed for each arc. By showing this property to hold for every triple $\{a, b, c\}$ of weights, we can strengthen Theorem 2.

Lemma 16. *Let v be a vertex with outdegree d of some oriented graph \vec{G} , and $\{a, b, c\}$ be a set of three real numbers. Then there are at least $2d + 1$ possible weighted outdegrees for v by an arc-weighting of \vec{G} assigning value among $\{a, b, c\}$ to the arcs outgoing from v .*

Proof. We prove this claim by induction on d . If $d = 1$, then the arc outgoing from v can be weighted either a , b , or c by an arc-weighting of \vec{G} . Since a , b and c are distinct, there are exactly three weighted outdegrees for v , namely a , b and c , respectively.

Assume the claim is true for every value of d up to $i - 1$, and assume $d = i$. Let \vec{G}' be the oriented graph obtained by removing exactly one arc \vec{vu} outgoing from v . Then there are at least $2(d - 1) + 1$ possible weighted outdegrees for v by an arc-weighting of \vec{G}' taking value among $\{a, b, c\}$ according to the induction hypothesis. Let D' be the set of these possible weighted outdegrees, and denote inf and sup the minimum and maximum elements of D' , respectively, and w'_{inf} and w'_{sup} two arc-weighting of \vec{G}' such that $\phi_{w'_{inf}}(v) = inf$ and $\phi_{w'_{sup}}(v) = sup$, respectively.

Assume $a < b < c$. Note that if the result holds for $\{a, b, c\}$, then it also holds for $\{-a, -b, -c\}$. Hence, we only have two cases to consider, namely

1. $0 \leq a < b < c$, and
2. $a < 0 \leq b < c$.

In the first case, by extending every arc-weighting of \vec{G}' to \vec{G} by weighting the arc \vec{vu} with weight a , we directly obtain that the set $D = \{x + a : x \in D'\}$ is a set of at least $2(d - 1) + 1$ possible weighted outdegrees for v . The two remaining weighted outdegrees for v are obtained by extending w'_{sup} by weighting b or c the arc \vec{vu} . We then obtain that $sup + b$ and $sup + c$ are two other possible weighted outdegrees for v since none of these two values can appear in D because $a < b < c$. There are thus at least $2d + 1$ possible weighted outdegrees for v .

In the second case, by extending every arc-weighting of \vec{G}' to \vec{G} by weighting b the arc \vec{vu} , we get that $D = \{x + b : x \in D'\}$ is a set of at least $2(d - 1) + 1$ weighted outdegrees for v . The two remaining weighted outdegrees for v are obtained by extending w'_{inf} and w'_{sup} to \vec{G} by weighting a and c , respectively, the arc \vec{vu} . From these two extensions, we get that v can also have weighted outdegree $inf + a$ and $sup + c$, which do not appear in D by our assumptions on a , b and c . This completes the proof. ■

As a corollary of Lemma 16, we directly get that the proof of Theorem 2 is applicable no matter what are the three weights allowed to weight the arcs outgoing from every vertex. This implies the following list version of our main result.

Corollary 17. *For every vertex v of some oriented graph \vec{G} , let $L(v)$ be an arbitrary list of three distinct real weights allowed at v . Then \vec{G} admits an nsd arc-weighting such that the arcs outgoing from every vertex v are weighted with values among $L(v)$.*

As for the undirected case, one can also consider a variant of the problem investigated in this work where the weighted outdegree of a vertex is the *product* of its outgoing weights rather than their sum (see e.g. [4]). Formally, from a k -arc-weighting w of some oriented graph \vec{G} one obtains a vertex-colouring ρ_w defined as $\rho_w(v) = \prod_{u \in N^+(v)} w(\vec{vu})$ for every $v \in V(\vec{G})$. If ρ_w is proper, then we say that w is *neighbour-product-distinguishing* (*npd* for short).

Regarding npd-arc-weightings, note that the range of possible weighted outdegrees for a vertex is as wider as in the product version than in the sum version when the weights from $\{1, 2, 3\}$ are allowed (this can be proved in a similar manner as Lemma 16). Hence, our proof of Theorem 2 is also a proof that every oriented graph admits an npd-3-arc-weighting.

Theorem 18. *Every oriented graph \vec{G} admits an npd-3-arc-weighting.*

Note that there are $k + 1$ possible weighted outdegrees for a vertex with outdegree k by an npd-2-arc-weighting of some oriented graph, namely those from $\{1, 2, 4, \dots, 2^k\}$. Since there are as many possible weighted outdegrees for a vertex by an nsd 2-arc-weighting and an npd-2-arc-weighting, our results from Section 3 also hold regarding npd-2-arc-weightings.

Finally, we can adapt the reduction scheme from Section 4 to prove that it is NP-complete to decide whether a given oriented graph admits an npd-2-arc-weighting. The forbidding gadgets can be obtained, for instance, as follows. Start from the circuit $\overrightarrow{u_1 u_2 u_3 u_1}$ on three vertices, and add an arc $\overrightarrow{u_1 u_4}$ where u_4 is a new vertex. This resulting oriented graph \vec{F}_4 is a 4-forbidding gadget since u_1 necessarily gets weighted outdegree 4 by every npd-2-arc-weighting. The root of

\vec{F}_4 is u_4 . Now consider another oriented graph $\vec{F}_{1,2}$ with vertices v_1, v_2, v_3 and v_4 such that $\vec{v_1v_2}, \vec{v_1v_3}, \vec{v_2v_3}$ and $\vec{v_2v_4}$ are arcs, and v_1 and v_2 are each identified with the root of one copy of \vec{F}_4 . Clearly, since v_1 and v_2 are adjacent vertices with outdegree 2, and they are both identified with the root of a gadget \vec{F}_4 , their weighted outdegree can only be 1 and 2 without loss of generality, and $\vec{F}_{1,2}$ is thus a (1, 2)-forbidding gadget with root v_3 . Now to obtain a 2^k -forbidding gadget \vec{F}_{2^k} assuming that a $2^{k'}$ -forbidding gadget has been defined for every $k' < k$ (with the exception that there is a (1, 2)-forbidding gadget rather than a 1-forbidding gadget and a 2-forbidding gadget), start from the arc $\vec{w_1w_2}$, then add arcs from w_1 towards $k - 1$ new vertices so that w_1 has outdegree k , and finally identify w_1 and the roots of all the forbidding gadgets constructed in previous steps. Clearly, w_1 can only have weighted outdegree 2^k by every npd-2-arc-weighting of \vec{F}_{2^k} . Thus, \vec{F}_{2^k} is a 2^k -forbidding gadget with root w_2 . With these forbidding gadgets, our reduction scheme can then be directly adapted for the product version of the problem.

Another direction for extending our problem could be to consider undirected graphs.

Question 19. *What is the least $k \in \{1, 2, 3\}$ such that every undirected graph admits an orientation which admits an nsd k -arc-weighting?*

Recall that an oriented graph admits an nsd 1-arc-weighting if and only if every two of its adjacent vertices have distinct outdegrees. According to a result from [8], the answer to Question 19 is 1. We give a reformulated proof of this statement using our terminology.

Lemma 20. *Every undirected graph G admits an orientation in which every two adjacent vertices have distinct outdegrees.*

Proof. We prove this result by induction on the order n of G . Since the result is true for $n = 1$, we assume the claim is true for every n up to $i - 1$, and now consider $n = i$. Let v be a vertex whose degree is maximum in G . According to the induction hypothesis, the graph $G' = G - v$ admits an orientation \vec{G}' in which every two adjacent vertices have distinct outdegrees. Note that in \vec{G}' , the outdegree of every vertex in $N(v)$ is at most $d(v) - 1$ since v has maximum degree in G . Now start from \vec{G}' , and let \vec{G} be the orientation of G obtained by orienting all edges incident with v from v towards its neighbours. Since the outdegree of v in \vec{G} is then $d(v)$, and the outdegrees of all vertices neighbouring v are not altered, the orientation still satisfies the claim. ■

Corollary 21. *Every undirected graph admits an orientation which admits an nsd 1-arc-weighting.*

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