# EIGENVALUE CONDITIONS FOR INDUCED SUBGRAPHS 

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#### Abstract

Necessary conditions for an undirected graph $G$ to contain a graph $H$ as induced subgraph involving the smallest ordinary or the largest normalized Laplacian eigenvalue of $G$ are presented.


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## 1. InTRODUCTION

We consider two fixed finite, undirected, and simple graphs: Let $G=(V, E)$ be a graph without isolated vertices, where $V=\{1, \ldots, n\}$ and $E$ (with $|E|=m$ ) denote the vertex set and the edge set of $G$, respectively. Let $\delta \geq 1$ denote the minimum degree of $G$. Furthermore, let $d_{H}=\frac{2 e}{h}$ be the average degree of a graph $H=(V(H), E(H))$, where $|V(H)|=h$ and $|E(H)|=e$.

The eigenvalues $\lambda_{1} \leq \cdots \leq \lambda_{n}$ of the adjacency matrix $A$ of $G$ are the ordinary eigenvalues (or shortly the eigenvalues) of $G$. Note that $-r \leq \lambda \leq \lambda_{n}=r$ for all eigenvalues $\lambda$ of an $r$-regular graph $G$, and if $G$ is connected, then $\lambda_{1}=-\lambda_{n}$ if and only if $G$ is bipartite $[4,7]$.

Let $D$ be the degree matrix of $G$, that is an $(n \times n)$ diagonal matrix, where the degree $d_{i}$ of vertex $i \in V$ is the $i$-th entry at the main diagonal. Moreover, let $0=\eta_{1} \leq \cdots \leq \eta_{n}$ be the eigenvalues of the Laplacian $L=D-A$ of $G[1,13]$. If $G$ is $r$-regular, then $\eta$ is an eigenvalue of the Laplacian if and only if $r-\eta$ is an eigenvalue of $A$.

For $G$ without isolated vertices, the normalized Laplacian is the $(n \times n)$ matrix $\mathcal{L}=\left(l_{i j}\right)$ with $l_{i j}=1$ if $i=j, l_{i j}=-\frac{1}{\sqrt{d_{i} d_{j}}}$ if $i j \in E$, and $l_{i j}=0$ otherwise. The eigenvalues $0=\sigma_{1} \leq \cdots \leq \sigma_{n}$ of $\mathcal{L}$ are the normalized Laplacian eigenvalues of $G$ [5,6,13]. It is known that $1<\sigma_{n} \leq 2$ and that $G$ is bipartite if and only if $\sigma_{n}=2$ [10, 12, 13]. For an $r$-regular graph $G, \sigma$ is a normalized Laplacian eigenvalue if and only if $r(1-\sigma)$ is an eigenvalue of $A$.

For further notation and terminology we refer to [8].
In the present paper, we are interested in necessary conditions in terms of eigenvalues for the fact that $G$ contains a copy of $H$ as an induced subgraph. If all eigenvalues of $G$ and all eigenvalues $\phi_{1} \leq \cdots \leq \phi_{h}$ of the adjacency matrix $A_{H}$ of $H$ are taken into consideration, then Theorem 1 is a typical result of this kind.

Theorem 1 (Cauchy's Inequalities, Interlacing Theorem [4, 7]). If $H$ is an induced subgraph of $G$ with eigenvalues $\phi_{1} \leq \cdots \leq \phi_{h}$, then $\lambda_{i} \leq \phi_{i} \leq \lambda_{n-h+i}$ for $i=1, \ldots, h$.

In general, it is difficult to determine the spectra of large graphs $G$ and $H$, however, the largest and the smallest eigenvalues of the matrices $A, L$, and $\mathcal{L}$ of a graph are well investigated ( $[1,4,5,6]$ ). Hence, we focus on simpler necessary conditions for $H$ being an induced subgraph of $G$ just involving smallest or largest eigenvalues. The inequalities (1) obtained from Theorem 1 are possible results of this type.

$$
\begin{equation*}
\lambda_{1} \leq \phi_{1} \quad \text { and } \quad \lambda_{n} \geq \phi_{h} . \tag{1}
\end{equation*}
$$

If the largest Laplacian eigenvalue $\eta_{n}$ of $G$ and the degrees of the vertices of $H$ in $G$ are taken into account, then the assertion of Theorem 2 holds.

Theorem 2 (Bollobás, Nikiforov [3]). If $H$ is an induced subgraph of $G$, then $\left(\sum_{i \in V(H)} d_{i}-2 e\right) n \leq \eta_{n} h(n-h)$.
In general, it is not easy to determine the value $\sum_{i \in V(H)} d_{i}$ exactly. If the degrees of $G$ do not differ too much, then the inequality $\sum_{i \in V(H)} d_{i} \geq \delta h$ is reasonable and it follows

Corollary 3. If $H$ is an induced subgraph of $G$, then $\eta_{n} h \leq\left(d_{H}+\eta_{n}-\delta\right) n$.

Note that Corollary 3 only makes sense if $\delta>d_{H}$. If $G$ is $r$-regular, then $\delta=r$, $\eta_{n}=r-\lambda_{1}$, and $\sum_{i \in V(H)} d_{i}=r h$, hence, Theorem 2, Corollary 3, and the following Corollary 4, proved by Haemers already in [9], coincide in this case.

Corollary 4 (Haemers [9]). If $H$ is an induced subgraph of the r-regular graph $G$, then $\left(r-\lambda_{1}\right) h \leq\left(d_{H}-\lambda_{1}\right) n$.

The identity matrix is the $(n \times n)$ square matrix with ones on the main diagonal and zeros elsewhere. It is denoted simply by $I$ if the size is immaterial or can be trivially determined by the context. In the sequel, $\underline{x}$ denotes a vector, where $\underline{1}=(1,1, \ldots, 1)^{T}$ and $\underline{0}=(0,0, \ldots, 0)^{T}$, and we write $\underline{x} \geq \underline{0}$ if $x_{i} \geq 0$ for each entry $x_{i}$ of $\underline{x}$.

Our first result is Theorem 5 concerning the case that $G$ is regular and involving the smallest eigenvalue $\lambda_{1}$ of $G$.

Theorem 5. Let $G$ be $r$-regular. If $H$ is an induced subgraph of $G$, then $\left(A_{H}-\right.$ $\left.\lambda_{1} I\right) \underline{x}=\underline{1}$ is solvable, and, for any solution $\underline{x}$ of this equation,

$$
\frac{r-\lambda_{1}}{n} \leq \min \left\{\underline{z}^{T}\left(A_{H}-\lambda_{1} I\right) \underline{z} \mid \underline{z} \in R^{|V(H)|}, \underline{1}^{T} \underline{z}=1\right\}=\frac{1}{\underline{1}^{T} \underline{x}} .
$$

Moreover, if $\lambda_{1}<\phi_{1}$, then $A_{H}-\lambda_{1} I$ is regular and $\underline{1}^{T} \underline{x}$ equals the sum of all entries of $\left(A_{H}-\lambda_{1} I\right)^{-1}$.
If $\underline{z}=\left(\frac{1}{h}, \ldots, \frac{1}{h}\right)^{T} \in R^{h}$, then $\underline{1}^{T} \underline{z}=1$ and $\underline{z}^{T}\left(A_{H}-\lambda_{1} I\right) \underline{z}=\frac{2 e-\lambda_{1} h}{h^{2}}$. Thus, Theorem 5 is an extension of Corollary 4. If in Theorem 5, additionally, $H$ is assumed to be $\rho$-regular, then $\underline{x}=\left(\frac{1}{\rho-\lambda_{1}}, \ldots, \frac{1}{\rho-\lambda_{1}}\right)^{T}$ is a solution of $\left(A_{H}-\right.$ $\left.\lambda_{1} I\right) \underline{x}=\underline{1}$, thus, $\frac{1}{1^{T} \underline{x}}=\frac{\left(\rho-\lambda_{1}\right)}{h}=\frac{\left(d_{H}-\lambda_{1}\right)}{h}$, hence, Corollary 4 and Theorem 5 coincide in this case.

Now consider the following example, where the assertion of Theorem 5 is stronger than that one of Corollary 4 and inequalities (1) only lead to trivial statements. We ask for a necessary condition that the $r$-regular graph $G$ contains $k \geq 1$ disjoint and independent copies of the path $P_{3}$ on 3 vertices, that is, $H$ consists of $k$ components each of them is isomorphic to $P_{3}$. The eigenvalues of $P_{3}$ are $-\sqrt{2}, 0, \sqrt{2}([4])$, hence, with Theorem 1 we may assume $\lambda_{1} \leq-\sqrt{2}<-\frac{4}{3}$. With $h=3 k$ and $d_{H}=\frac{4}{3}$, Corollary 4 leads to $k \leq \frac{4-3 \lambda_{1}}{9\left(r-\lambda_{1}\right)} n$.

If we consider the system $\left(A_{H}-\lambda_{1} I\right) \underline{x}=\underline{1}$, then, by Theorem 5 , it is solvable and it follows $\underline{1}^{T} \underline{x}=k \underline{1}^{T} \underline{y}$, where $\underline{y}$ is a solution of $\left(A_{P_{3}}-\lambda_{1} I\right) \underline{y}=\underline{1}$. It is easy to see that $\underline{1}^{T} \underline{y}=\frac{4+3 \lambda_{1}}{2-\lambda_{1}^{2}}$, thus, again by Theorem $5, k \leq \frac{2-\lambda_{1}^{2}}{\left(4+3 \lambda_{1}\right)\left(r-\lambda_{1}\right)} n$, which is stronger than $k \leq \frac{4-3 \lambda_{1}}{9\left(r-\lambda_{1}\right)} n$.

If, additionally, $G$ is assumed to be bipartite, then $\lambda_{1}=-r$ and $\lambda_{n}=r$. The inequalities (1) just imply $\sqrt{2} \leq r$ in this case.

Next we consider again the case that $G$ is not necessarily regular and try to establish a result similar to Theorem 5. Therefore, let $M(G, H)$ be the set of non-empty induced subgraphs $H^{*}$ of $H$ such that $B \underline{y}=\underline{1}$ has a solution $\underline{y}=$ $\left(y_{1}, \ldots, y_{t}\right)^{T}$ with $y_{s}>0$ for $s=1, \ldots, t=\left|V\left(H^{*}\right)\right|^{-}$, where $A_{H^{*}}$ denotes the adjacency matrix of $H^{*}$ and $B=A_{H^{*}}+\left(\sigma_{n}-1\right) \delta I$. In this case $\underline{y}$ is called a positive solution of $B \underline{y}=\underline{1}$. With $H^{*}=K_{1}$ and $y_{1}=\frac{1}{\left(\sigma_{n}-1\right) \delta}>\overline{0}$ (note that $\left.\sigma_{n}>1\right)$, it follows $K_{1} \in M(G, H) \neq \emptyset$.

If $H^{*} \in M(G, H)$ and $\underline{y_{1}}$ and $\underline{y_{2}}$ are positive solutions of $B \underline{y}=\underline{1}$, then, since $B$ is symmetric, $\underline{1}^{T} y_{1}=y_{2}{ }^{T} B y_{1}=y_{2}{ }^{T} \underline{1}=\underline{1}^{T} y_{2}$, hence, the value $\underline{1}^{T} y$ is independent on the choice of the positive solution $\underline{y}$. We define $g\left(G, H^{*}\right)=\underline{1}^{T} \underline{y}$, where $\underline{y}$ is an arbitrary positive solution of $B \underline{y}=\underline{1}$.

If the induced subgraph $H^{*}$ of $H$ is $\rho$-regular, then it is easy to see that $\left(A_{H^{*}}+\left(\sigma_{n}-1\right) \delta I\right) \underline{y}=\underline{1}$ has a positive solution $\underline{y}=\left(\frac{1}{\rho+\left(\sigma_{n}-1\right) \delta}, \ldots, \frac{1}{\rho+\left(\sigma_{n}-1\right) \delta}\right)^{T}$, hence, $H^{*} \in M(G, H)$.

If $H_{1}^{*}$ and $H_{2}^{*}$ are independent induced subgraphs of $H$ and $H_{1}^{*}, H_{2}^{*} \in M(G, H)$, then the disjoint union $H_{1}^{*} \cup H_{2}^{*}$ of $H_{1}^{*}$ and $H_{2}^{*}$ also belongs to $M(G, H)$ and $g\left(G, H_{1}^{*} \cup H_{2}^{*}\right)=g\left(G, H_{1}^{*}\right)+g\left(G, H_{2}^{*}\right)$.

Eventually, let $f(G, H)=\min _{H^{*} \in M(G, H)} \frac{1}{g\left(G, H^{*}\right)}$. Our second result is Theorem 6 involving the largest normalized Laplacian eigenvalue $\sigma_{n}$ of $G$.

Theorem 6. If $H$ is an induced subgraph of $G$, then

$$
\frac{\sigma_{n} \delta^{2}}{2 m} \leq \min \left\{\underline{z}^{T}\left(A_{H}+\left(\sigma_{n}-1\right) \delta I\right) \underline{z} \mid \underline{z} \in R^{|V(H)|}, \underline{1}^{T} \underline{z}=1, \underline{z} \geq \underline{0}\right\}=f(G, H)
$$

If $G$ is $r$-regular, then the assertion of Theorem 6 is weaker than that one of Theorem 5 because $\lambda_{1}=r\left(1-\sigma_{n}\right), \frac{2 m}{\sigma_{n} \delta^{2}}=\frac{n}{r-\lambda_{1}}$, and $\min \left\{\underline{z}^{T}\left(A_{H}-\lambda_{1} I\right) \underline{z} \mid \underline{z} \in\right.$ $\left.R^{|V(H)|}, \underline{1}^{T} \underline{z}=1\right\} \leq \min \left\{\underline{z}^{T}\left(A_{H}-\lambda_{1} I\right) \underline{z} \mid \underline{z} \in R^{|V(H)|}, \underline{1}^{T} \underline{z}=1, \underline{z} \geq \underline{0}\right\}$ in this case.

In general, it is not easy to calculate $\min \left\{\underline{z}^{T}\left(A_{H}+\left(\sigma_{n}-1\right) \delta I\right) \underline{z} \mid \underline{1}^{T} \underline{z}=1\right.$, $\underline{z} \geq \underline{0}\}$, however, in special cases it can be done efficiently.

Therefore, we consider an example, where the graph $G$ is non-regular (i.e., Corollary 4 and Theorem 5 are not applicable), $f(G, H)$ can be determined easily, and the necessary condition of Theorem 6 for the graph $H$ to be an induced subgraph of $G$ is stronger than that one of Theorem 2 .

For positive integers $p$ and $q$, where $p$ is even, let $G=C_{p} \square P_{3}$ be the Cartesian product ${ }^{1}$ of the cycle $C_{p}$ and the path $P_{3}$ on 3 vertices (for $p=20, G$ is shown in the figure) and let $H$ consist of $q$ copies of $K_{1,4}$.

[^0]

We have $n=3 p, m=5 p, \delta=3$, and, since $G$ is bipartite, $\sigma_{n}=2$. The Laplacian eigenvalues of $C_{p}$ and of $P_{3}$ are $2-2 \cos \left(\frac{2 \pi j}{p}\right)$ for $j=0, \ldots, p-1$ and $0,1,3$, respectively ([4]). Moreover, if $\eta^{\prime}$ and $\eta^{\prime \prime}$ are Laplacian eigenvalues of $G^{\prime}$ and $G^{\prime \prime}$, respectively, then $\eta^{\prime}+\eta^{\prime \prime}$ is a Laplacian eigenvalue of $G^{\prime} \square G^{\prime \prime}([4])$. Because $p$ is even, it follows $\eta_{n}=2-2 \cos (\pi)+3=7$.

It is easy to see that $\sum_{i \in V(H)} d_{i}-2 e=10 q$ and, using $h=5 q$, Theorem 2 implies $q \leq \frac{3}{7} p$ in this case.

If $H^{*}$ is an induced subgraph of $K_{1,4}$, then $H^{*}=K_{1, s}$ or $H^{*}=\bar{K}_{s}$ (the edgeless graph on $s$ vertices) for suitable $s \in\{1,2,3,4\}$.

Let $H^{*}=K_{1, s}$ and consider the system $\left(A_{H^{*}}+\left(\sigma_{n}-1\right) \delta I\right) \underline{y}=\left(A_{H^{*}}+\right.$ $3 I) \underline{y}=1$. It is easy to see that $K_{1,4}, K_{1,3} \notin M(G, H), K_{1,2}, K_{1,1} \in M(G, H)$, $g\left(\bar{G}, K_{1,2}\right)=\frac{5}{7}$, and $g\left(G, K_{1,1}\right)=\frac{1}{2}$.

If $H^{*}=\bar{K}_{s}$, then $H^{*} \in M(G, H)$ and $\left(A_{H^{*}}+3 I\right) \underline{y}=\underline{1}$ lead to $g\left(G, H^{*}\right)=\frac{s}{3}$, hence, $f(G, H)=\frac{3}{4}$. By Theorem 6, it follows $q \leq \frac{5}{12} p<\frac{3}{7} p$.

If $H^{*}$ with $\left|V\left(H^{*}\right)\right| \geq 1$ is an arbitrary induced subgraph of $H$ and $\underline{z}=$ $\left(z_{1}, \ldots, z_{h}\right)^{T}$ with $z_{i}=\frac{1}{\left|V\left(H^{*}\right)\right|}$ if $i \in V\left(H^{*}\right)$ and $z_{i}=0$ otherwise, then $\underline{1}^{T} \underline{z}=1$ and $\underline{z}^{T}\left(A_{H}+\left(\sigma_{n}-1\right) \delta I\right) \underline{z}=\frac{d_{H^{*}}+\left(\sigma_{n}-1\right) \delta}{\left|V\left(H^{*}\right)\right|}$, where $d_{H^{*}}$ denotes the average degree of $H^{*}$. Thus, Corollary 7 is a consequence of Theorem 6 .
Corollary 7. If $H$ is an induced subgraph of $G$, then $\frac{\sigma_{n} \delta^{2}}{2 m} \leq \frac{d_{H^{*}}+\left(\sigma_{n}-1\right) \delta}{\left|V\left(H^{*}\right)\right|}$, where $H^{*}$ is an arbitrary induced subgraph of $H$ with $\left|V\left(H^{*}\right)\right| \geq 1$.

Obviously, Corollary 7 is an extension of Corollary 4 if $G$ is regular. We conclude with an example, where Corollary 3 is weaker than Corollary 7 for not necessarily regular $G$. Therefore, let $V(H)$ be an independent set of $G$, i.e. $d_{H}=0$. By Corollary 3 and Corollary 7, it follows that $h \leq \frac{\eta_{n}-\delta}{\eta_{n}} n$ and $h \leq \frac{2\left(\sigma_{n}-1\right)}{\sigma_{n} \delta} m$ if $G$ contains $h$ independent vertices, respectively. In [11], it is shown that there are infinitely many graphs $G$ such that $\frac{2\left(\sigma_{n}-1\right)}{\sigma_{n} \delta} m<\frac{\eta_{n}-\delta}{\eta_{n}} n$.

## 2. Proofs

In [11], the following Lemma 8 is proved. For completeness we give a proof here.

Lemma 8. If $x_{1}, \ldots, x_{n}$ are real numbers, then

$$
\begin{equation*}
\sigma_{n}\left(\sum_{i=1}^{n} d_{i} x_{i}\right)^{2}-2\left(\sigma_{n}-1\right) m \sum_{i=1}^{n} d_{i} x_{i}^{2} \leq 4 m \sum_{i j \in E} x_{i} x_{j} \tag{2}
\end{equation*}
$$

Proof. It is easy to see that $\sigma$ is an eigenvalue of $\mathcal{L}$ if and only if $\mu=1-\sigma$ fulfills $\operatorname{det}(A-\mu D)=0$, see $[10,12,14]$. Let $\mu_{i}=1-\sigma_{n-i+1}$ for $i=1, \ldots, n$.
Note that $D$ is positive definite since $\delta \geq 1$. Define $\underline{x}^{T} D \underline{y}$ as the inner product for vectors $\underline{x}, \underline{y} \in R^{n}$ and let $\underline{x}$ and $\underline{y}$ be called $D$-orthogonal if $\underline{x}^{T} D \underline{y}=0$. If $\underline{x}^{T} D \underline{x}=1$ then $\underline{x}$ is called $D$-normal. A set of $D$-normal vectors being pairwise $D$-orthogonal is a $D$-orthonormal set.

We consider the generalized eigenvalue problem $A \underline{x}=\mu D \underline{x}$ for $\mu \in R$ and $\underline{x} \in R^{n}$ with $\underline{x} \neq \underline{0}$. If the pair $(\mu, \underline{x})$ is a solution of this equation, then $\underline{x}$ is a $D$-eigenvector of $G$ and $\mu$ is the corresponding $D$-eigenvalue of $G$.

We use the well known fact (e.g. see [14]) that there is a $D$-orthonormal basis of $R^{n}$ consisting of $D$-eigenvectors of $G$. Next we will show the following assertion. If $\left\{\underline{u_{1}}, \ldots, \underline{u_{n}}\right\}$ is a $D$-orthonormal basis of $R^{n}$ such that $\underline{u_{i}}$ is a $D$-eigenvector with corresponding $D$-eigenvalue $\mu_{i}$ for $i=1, \ldots, n$, then, for any vector $\underline{x} \in R^{n}$,

$$
\begin{equation*}
\left(\mu_{2}-\mu_{1}\right)\left(\underline{x}^{T} D \underline{u}_{2}\right)^{2}+\cdots+\left(\mu_{n}-\mu_{1}\right)\left(\underline{x}^{T} D \underline{u}_{n}\right)^{2}+\mu_{1} \underline{x}^{T} D \underline{x}=\underline{x}^{T} A \underline{x} . \tag{3}
\end{equation*}
$$

To see this, let $\underline{x}$ be given. There are real numbers $a_{1}, \ldots, a_{n}$ such that $\underline{x}=$ $a_{1} \underline{u_{1}}+\cdots+a_{n} \underline{u_{n}}$.

Then $\underline{x}^{T} A \underline{x}=\mu_{1} a_{1}^{2}+\cdots+\mu_{n} a_{n}^{2}, \underline{x}^{T} D \underline{x}=a_{1}^{2}+\cdots+a_{n}^{2}$, and $\underline{x}^{T} D \underline{u_{i}}=a_{i}$ for $i=1, \ldots, n$. The desired equality (3) is equivalent to $\left(\mu_{2}-\mu_{1}\right) a_{2}^{2}+\cdots+\left(\mu_{n}-\right.$ $\left.\mu_{1}\right) a_{n}^{2}+\mu_{1}\left(a_{1}^{2}+\cdots+a_{n}^{2}\right)=\mu_{1} a_{1}^{2}+\ldots+\mu_{n} a_{n}^{2}$.

As a consequence,

$$
\begin{equation*}
\left(\mu_{n}-\mu_{1}\right)\left(\underline{x}^{T} D{\underline{u_{n}}}^{2}\right)^{2}+\mu_{1} \underline{x}^{T} D \underline{x} \leq \underline{x}^{T} A \underline{x} . \tag{4}
\end{equation*}
$$

The vector $\frac{1}{\sqrt{2 m}} \underline{1}$ is a $D$-normal $D$-eigenvector of $G$ with corresponding $D$-eigenvalue $\mu_{n}=1$, thus, inequality (4) and $\sigma_{n}=1-\mu_{1}$ imply the lemma.

Proof of Theorem 5. Inequality (2) and $\lambda_{1}=r\left(1-\sigma_{n}\right)$, if $G$ is $r$-regular, imply the fact that if $G$ is $r$-regular and $x_{1}, \ldots, x_{n}$ are real numbers, then

$$
\begin{equation*}
\left(r-\lambda_{1}\right)\left(\sum_{i=1}^{n} x_{i}\right)^{2}+\lambda_{1} n \sum_{i=1}^{n} x_{i}^{2} \leq 2 n \sum_{i j \in E} x_{i} x_{j} . \tag{5}
\end{equation*}
$$

Let $U$ be an induced subgraph of $G$ isomorphic to $H$ and $\phi: V(H) \rightarrow V(U)$ be a graph isomorphism from $H$ to $U$.

For real numbers $z_{1}, \ldots, z_{h}$ with $\sum_{q=1}^{h} z_{q}=1$, let $x_{1}, \ldots, x_{n}$ be defined as follows: If $i \in V(U)$, then there is a suitable $q \in\{1, \ldots, h\}$ such that $i=\phi\left(v_{q}\right)$. Set $x_{i}=z_{q}$ in this case. If $i \in V \backslash V(U)$, then let $x_{i}=0$.

With $\underline{z}=\left(z_{1}, \ldots, z_{h}\right)^{T}$, we obtain $\sum_{i \in V} x_{i}=\sum_{q=1}^{h} z_{q}=1, \sum_{i \in V} x_{i}^{2}=$ $\sum_{q=1}^{h} z_{q}^{2}$, and $2 \sum_{i j \in E} x_{i} x_{j}=2 \sum_{v_{q} v_{q^{\prime}} \in E(H)} z_{q} z_{q^{\prime}}=\underline{z}^{T} A_{H} \underline{z}$.

Inequality (5) implies $\left(r-\lambda_{1}\right)+\lambda_{1}\left(\sum_{q=1}^{h} z_{q}^{2}\right) n \leq \underline{z}^{T} A_{H} \underline{z} n$, hence, with $B=$ $\left(A_{H}-\lambda_{1} I\right), 1 \leq \frac{n}{\left(r-\lambda_{1}\right)} \min \underline{z}^{T} B \underline{z}=\frac{n}{\left(r-\lambda_{1}\right)} M I N$, where the minimum is taken over all vectors $\underline{z}=\left(z_{1}, \ldots, z_{h}\right)^{T}$ with $\sum_{q=1}^{h} z_{q}=1$.

Note that this minimum exists, because $\lambda_{1} \leq \phi_{1}$ follows from Theorem 1, hence, all eigenvalues $\phi_{1}-\lambda_{1}, \phi_{2}-\lambda_{1}, \ldots, \phi_{h}-\lambda_{1}$ of $B$ are non-negative. It follows that $B$ is positive semidefinite.

To investigate this value MIN, we consider the Lagrange function $L(\underline{z}, \kappa)=$ $\underline{z}^{T} B \underline{z}-2 \kappa\left(\sum_{q=1}^{h} z_{q}-1\right)$ with Lagrange multiplier $2 \kappa$ and the necessary optimality conditions $L_{z_{q}}=0$ for $q=1, \ldots, h$ (for more details an Lagrange Theory see [2]).

We obtain that the equations $B \underline{z}=\kappa \underline{1}$ and $\underline{1}^{T} \underline{z}=1$ are simultaneously solvable.

Next we will show that $\kappa$ is unique. If $B \underline{z_{1}}=\kappa_{1} \underline{1}, \underline{1}^{T} \underline{z_{1}}=1, B \underline{z_{2}}=\kappa_{2} \underline{1}$, and $\underline{1}^{T} \underline{z}_{2}=1$, then $\kappa_{1}=\kappa_{1} \underline{1}^{T} \underline{z}_{2}=\underline{z}^{T} B \underline{z}_{2}=\kappa_{2} \underline{z}_{1}^{T} \underline{1}=\kappa_{2}$.

With $1 \leq \frac{n}{\left(r-\lambda_{1}\right)} M I N$, it follows $M I N=\underline{z}^{T} B \underline{z}=\kappa>0$.
If $\underline{x}=\frac{1}{\kappa} \underline{z}$, then $B \underline{x}=\underline{1}$ and $\underline{1}^{T} \underline{x}=\frac{1}{\kappa}$.
If $\lambda_{1}<\phi_{1}$, then $B$ is regular and $1=\underline{1}^{T} \underline{z}=\kappa \underline{1}^{T} B^{-1} \underline{1}$, hence, $\underline{1}^{T} \underline{x}=\underline{1}^{T} B^{-1} \underline{1}$.

Proof of Theorem 6. The proof of Theorem 6 is similar to that one of Theorem 5.

Let $x_{i} \geq 0$ for $i=1, \ldots, n$ and, since $\sigma_{n}>1$, inequality (2) implies

$$
\sigma_{n}\left(\sum_{i=1}^{n} d_{i} x_{i}\right)^{2}-\frac{2\left(\sigma_{n}-1\right) m}{\delta} \sum_{i=1}^{n}\left(d_{i} x_{i}\right)^{2} \leq \frac{4 m}{\delta^{2}} \sum_{i j \in E}\left(d_{i} x_{i}\right)\left(d_{j} x_{j}\right) .
$$

Substituting $w_{i}=d_{i} x_{i}$ for $i=1, \ldots, n$, it follows

$$
\begin{equation*}
\sigma_{n} \delta^{2}-2\left(\sigma_{n}-1\right) m \delta \sum_{i=1}^{n} w_{i}^{2} \leq 4 m \sum_{i j \in E} w_{i} w_{j}, \tag{6}
\end{equation*}
$$

for arbitrary $w_{i} \geq 0$ for $i=1, \ldots, n$ with $\sum_{i=1}^{n} w_{i}=1$.
Again, let $U$ be an induced subgraph of $G$ isomorphic to $H$ and $\phi: V(H) \rightarrow$ $V(U)$ be a graph isomorphism from $H$ to $U$, and, for real numbers $z_{1}, \ldots, z_{h} \geq 0$ with $\sum_{q=1}^{h} z_{q}=1$, let $w_{1}, \ldots, w_{n}$ be defined as follows: If $i \in V(U)$, then there is a suitable $q \in\{1, \ldots, h\}$ such that $i=\phi\left(v_{q}\right)$. Set $w_{i}=z_{q}$ in this case. If $i \in V \backslash V(U)$, then let $w_{i}=0$.

Inequality (6) implies $\frac{\sigma_{n} \delta^{2}}{2 m} \leq \min \left(\underline{z}^{T} A_{H} \underline{z}+\left(\sigma_{n}-1\right) \delta \underline{z}^{T} \underline{z}\right)=M I N$, where the minimum is taken over $\mathcal{S}_{h}=\left\{\underline{z}=\left(z_{1}, \ldots, z_{h}\right)^{T} \mid z_{q} \geq 0\right.$ for $q=1, \ldots, h, \sum_{q=1}^{h} z_{q}$ $=1\}$. Note that this minimum exists because $\underline{z}^{T} A_{H} \underline{z}+\left(\sigma_{n}-1\right) \delta \underline{z}^{T} \underline{z}$ is a continuous function and $\mathcal{S}_{h}$ is a compact set.

Let $\underline{z}=\left(z_{1}, \ldots, z_{h}\right)^{T} \in \mathcal{S}_{h}$ with $\underline{z}^{T} A_{H} \underline{z}+\left(\sigma_{n}-1\right) \delta \underline{z}^{T} \underline{z}=M I N$. Furthermore, let $H^{\prime}$ be the induced subgraph of $H$ with vertex set $V\left(H^{\prime}\right)=\left\{q \in V(H) \mid z_{q}>\right.$ $0\} \neq \emptyset$.

If $t=\left|V\left(H^{\prime}\right)\right|=1$, then $H^{\prime}=K_{1} \in M(G, H)$ with $V\left(H^{\prime}\right)=\{q\}, z_{q}=1$, and MIN $=\left(\sigma_{n}-1\right) \delta>0$. Hence, $\underline{y}=\left(\frac{1}{\left(\sigma_{n}-1\right) \delta}\right)$ is a positive solution of $\left(A_{H^{\prime}}+\left(\sigma_{n}-1\right) \delta I\right) \underline{y}=\underline{1}$ and it follows $g\left(G, H^{\prime}\right)=\underline{1}^{T} \underline{y}=\frac{1}{\left(\sigma_{n}-1\right) \delta}=\frac{1}{M I N}$ and $1 \leq \frac{2 m}{\sigma_{n} \delta^{2} g\left(G, H^{\prime}\right)}$.

If $t \geq 2$, then $0<z_{q}<1$ for all $q \in V\left(H^{\prime}\right)$. Thus, $M I N=\min \left(\underline{u}^{T} A_{H^{\prime}} \underline{u}+\right.$ $\left.\left(\sigma_{n}-1\right) \delta \underline{u}^{T} \underline{u}\right)$, where the minimum is taken over the relative interior $\operatorname{rint}\left(\mathcal{S}_{t}\right)=$ $\left\{\underline{u}=\left(u_{1}, \ldots, u_{t}\right)^{T} \mid u_{s}>0\right.$ for $\left.s=1, \ldots, t, \sum_{s=1}^{t} u_{s}=1\right\}$ of $\mathcal{S}_{t}$, consequently, this minimum is a local minimum at the hyperplane $\mathcal{H}_{t}=\left\{\underline{u}=\left(u_{1}, \ldots, u_{t}\right)^{T} \mid \sum_{s=1}^{t} u_{s}\right.$ $=1\}$.

To investigate this value MIN, we consider the Lagrange function $L(\underline{u}, \kappa)=$ $\underline{u}^{T} A_{H^{\prime}} \underline{u}+\left(\sigma_{n}-1\right) \delta \underline{\delta}^{T} \underline{u}-2 \kappa\left(\sum_{s=1}^{t} u_{s}-1\right)$ with Lagrange multiplier $2 \kappa$ and the necessary optimality conditions $L_{u_{s}}=0$ for $s=1, \ldots, t$.

With $B=A_{H^{\prime}}+\left(\sigma_{n}-1\right) \delta I$, we obtain that the system $B \underline{u}=\kappa \underline{1}, \underline{1}^{T} \underline{u}=1$ has a positive solution $\underline{u}$.

Next we will show that $\kappa$ is unique. If $B \underline{u}_{1}=\kappa_{1} \underline{1}, \underline{1}^{T} \underline{u}_{1}=1, B \underline{u_{2}}=\kappa_{2} \underline{1}$, and $\underline{1}^{T} \underline{u_{2}}=1$, then $\kappa_{1}=\kappa_{1} \underline{1}^{T} \underline{u_{2}}=\underline{u}_{1}^{T} B \underline{u_{2}}=\kappa_{2} \underline{u}_{1}^{T} \underline{1}=\kappa_{2}$.

With $1 \leq \frac{2 m}{\sigma_{n} \delta^{2}} M I N$, it follows $\overline{M I N}=\underline{u^{T}} B \underline{u}=\kappa>0$.
If $\underline{y}=\frac{1}{\kappa} \underline{u}$, then $B \underline{y}=\underline{1}$ has a positive solution $\underline{y}$, consequently, $H^{\prime} \in$ $M(G, \bar{H})$. Moreover, $g\left(\overline{G,} H^{\prime}\right)=\underline{1}^{T} \underline{y}=\frac{1}{\kappa}=\frac{1}{M I N}$ and we obtain $1 \leq \frac{2 m}{\sigma_{n} \delta^{2} g\left(G, H^{\prime}\right)}$.

To see that $f(G, H)=\frac{1}{g\left(G, H^{\prime}\right)}$, assume there is $H^{\prime \prime} \in M(G, H)$ with $g\left(G, H^{\prime \prime}\right)$ $>g\left(G, H^{\prime}\right)$. Then there exists $\underline{u} \in \operatorname{rint}\left(\mathcal{S}_{t}\right)$ with $t=\left|V\left(H^{\prime \prime}\right)\right|$ such that $\underline{u}^{T} A_{H^{\prime \prime}} \underline{u}+$ $\left(\sigma_{n}-1\right) \delta \underline{u}^{T} \underline{u}<M I N$.

Let $x_{i}=u_{i}$ if $i \in V\left(H^{\prime \prime}\right)$ and $x_{i}=0$ for $i \in V(H) \backslash V\left(H^{\prime \prime}\right)$.
It follows $\underline{x}=\left(x_{1}, \ldots, x_{h}\right)^{T} \in \mathcal{S}_{|V(H)|}$ and $\underline{x}^{T} A_{H} \underline{x}+\left(\sigma_{n}-1\right) \delta \underline{x}^{T} \underline{x}<M I N$, contradicting the definition of MIN.

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[^0]:    ${ }^{1}$ Given graphs $G_{1}$ and $G_{2}$ with vertex set $V_{1}$ and $V_{2}$, respectively, their Cartesian product $G_{1} \square G_{2}$ is the graph with vertex set $V_{1} \times V_{2}$, where $\left(v_{1}, v_{2}\right)\left(w_{1}, w_{2}\right) \in E\left(G_{1} \square G_{2}\right)$ when either $v_{1}=w_{1}$ and $v_{2} w_{2} \in E\left(G_{2}\right)$ or $v_{2}=w_{2}$ and $v_{1} w_{1} \in E\left(G_{1}\right)$.

