Discussiones Mathematicae
Graph Theory 35 (2015) 133–140
doi:10.7151/dmgt.1788

REMARKS ON DYNAMIC MONOPOLIES WITH GIVEN AVERAGE THRESHOLDS

Carmen C. Centeno

Pontifícia Universidade Católica de Goiás
Goiás, Brazil

e-mail: cecilia@pucgoias.edu.br

and

Dieter Rautenbach

Institute of Optimization and Operations Research
Ulm University, Ulm, Germany

e-mail: dieter.rautenbach@uni-ulm.de

Abstract

Dynamic monopolies in graphs have been studied as a model for spreading processes within networks. Together with their dual notion, the generalized degenerate sets, they form the immediate generalization of the classical notions of vertex covers and independent sets in a graph. We present results concerning dynamic monopolies in graphs of given average threshold values extending and generalizing previous results of Khoshkhah *et al.* [*On dynamic monopolies of graphs: The average and strict majority thresholds*, Discrete Optimization **9** (2012) 77–83] and Zaker [*Generalized degeneracy, dynamic monopolies and maximum degenerate subgraphs*, Discrete Appl. Math. **161** (2013) 2716–2723].

Keywords: dynamic monopoly, degenerate set, vertex cover, independent set.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

We consider finite, simple, and undirected graphs and use standard terminology. For a graph G, a vertex u of G, and a subset U of the vertex set $V(G)$ of G, we denote by $N_U(u)$ the set of neighbors of u in U, that is, $N_U(u) = \{v \in U :$

$uv \in E(G)\}$. Furthermore, let $d_U(u)$ denote the cardinality of $N_U(u)$. With this notation, the neighborhood $N_G(u)$ and the degree $d_G(u)$ of u in G are $N_{V(G)}(u)$ and $d_{V(G)}(u)$.

For a graph G and an integer-valued threshold function $\tau : V(G) \to \mathbb{Z}$, a *$\tau$-dynamic monopoly of* G is a set M of vertices of G such that every non-empty subset N of $V(G) \setminus M$ contains a vertex u with $d_{V(G)\setminus N}(u) \geq \tau(u)$. Equivalently, the set M is a τ-dynamic monopoly of G if starting with the set M and iteratively adding to the current set further vertices u that have at least $\tau(u)$ neighbors in it, results in the entire vertex set of G. The notion of a dynamic monopoly has been proposed in various contexts as a graph-theoretical model for disease opinion fault spreading within a network. In view of the vast amount of literature concerning this notion and its close variants, we restrict our references to a minimum only citing papers that we really refer to in a non-superficial way.

From the above definition one readily sees that the notion of a τ-dynamic monopoly is dual to a generalized notion of *degeneracy* as observed by Zaker in [8] where, for an integer-valued function $\kappa : V(G) \to \mathbb{Z}$, a set D of vertices of G is *κ-degenerate* if every subset N of D contains a vertex u with $d_N(u) \leq \kappa(u)$. Clearly, M is a τ-dynamic monopoly if and only if $V(G) \setminus M$ is κ-degenerate for $\kappa = d_G - \tau$. For constant κ, this notion has been studied by Alon, Kahn, and Seymour [1]. Clearly, every superset of a dynamic monopoly (subset of a degenerate set) is again a dynamic monopoly (a degenerate set). Therefore, as is standard procedure for such dual notions, one is interested in the smallest cardinality $\mathrm{dyn}_\tau(G)$ of a τ-dynamic monopoly of G as well as the largest cardinality $\alpha_\kappa(G)$ of a κ-degenerate set of G. The duality immediately implies

$$\mathrm{dyn}_\tau(G) + \alpha_\kappa(G) = n(G)$$

where $n(G)$ denote the order $|V(G)|$ of G. For simplicity, we denote a constant function by its unique value. It has been observed that dynamic monopolies and degenerate sets are immediate generalizations of *vertex covers* and *independent sets*. More precisely, a vertex cover of G coincides with a d_G-dynamic monopoly and an independent set of G coincides with a 0-degenerate set.

In the present note we exploit some well-known arguments that were used to prove bounds on the independence or vertex cover number. We present results on dynamic monopolies in graphs of given average threshold values extending and generalizing previous results of Khoshkhah, Soltani, and Zaker [6], and Zaker [8].

In [6] and [8], Khoshkhah, Soltani, and Zaker study the smallest and largest values of $\mathrm{dyn}_\tau(G)$ for a given graph G and a given average $\overline{\tau} = \frac{1}{n(G)} \sum_{u \in V(G)} \tau(u)$ of the threshold function τ. Clearly, such a study only makes sense under reasonable restrictions on the threshold function: If G is a graph, the rational number

t is such that $t \cdot n(G)$ is an integer, and u^* is a vertex of G, then

$$\tau_1(u) = \begin{cases} t \cdot n(G), & u = u^* \text{ and} \\ 0, & u \in V(G) \setminus \{u^*\} \end{cases}$$

and

$$\tau_2(u) = \begin{cases} t \cdot n(G) - \sum_{v \in V(G)\setminus\{u^*\}} (d_G(v) + 1), & u = u^* \text{ and} \\ d_G(u) + 1, & u \in V(G) \setminus \{u^*\}, \end{cases}$$

satisfy $\mathrm{dyn}_{\tau_1}(G) \leq 1$ and $\mathrm{dyn}_{\tau_2}(G) \geq n(G) - 1$ while both average values $\overline{\tau}_1$ and $\overline{\tau}_2$ are exactly t. That is, without some restriction on τ, the smallest value of $\mathrm{dyn}_\tau(G)$ for given average threshold is 1 or 0 and the largest value is $n(G) - 1$ or $n(G)$, which does not really capture any property of the underlying graph.

In [6] as well as in [8], τ is assumed to be at most the degree function d_G. Furthermore, while in [6] τ is assumed to be non-negative, in [8] τ may assume arbitrary negative integer values[1]. The following assumptions on the threshold function seem reasonable.

- $\tau \geq 0$.

Allowing negative values for τ does not make graph-theoretical sense. In the irreversible conversion process modeled by the dynamic monopoly, vertices u with $\tau(u) \leq 0$ behave exactly the same regardless of the specific value of $\tau(u)$. Hence changing the non-positive values does not change the graph-theoretical interpretation but allows to manipulate the average threshold value in a meaningless way.

- $\tau \leq d_G + 1$.

The reason for this restriction is essentially the same as for $\tau \geq 0$; values of τ above $d_G + 1$ make no graph-theoretical sense.

- $\tau \leq d_G$

Not allowing the value $d_G(u) + 1$ for $\tau(u)$ implies that the vertex u does not lie in every dynamic monopoly. For algorithmic purposes, such an assumption may make sense, because one can simply remove a vertex u with $\tau(u) = d_G(u) + 1$ from the graph and reduce the threshold value of its neighbors by 1. Considering the smallest largest value of $\mathrm{dyn}_\tau(G)$ for given average threshold, allowing the value $d_G(u) + 1$ for $\tau(u)$ makes sense; it is possible to 'vaccinate' the vertex u such that it cannot be 'infected' by its neighbors.

[1]In view of τ_2, the remark in [8] just before Theorem 13 stating that Theorem 13 in [8] is still true for non-positive threshold values is not true.

2. Results

Our first result is based on adapting the folklore proof (cf. [3]) for Caro [5] and Wei's [7] lower bound on the independence number. For two integers d and τ, let

$$p(d,\tau)=\begin{cases}0, & \tau<0,\\ \frac{\tau}{d+1}, & 0\le\tau\le d+1, \text{ and}\\ 1, & \tau>d+1.\end{cases}$$

Theorem 1. *For a graph G and a function $\tau : V(G)\to\mathbb{Z}$,*

$$\mathrm{dyn}_\tau(G)\le\sum_{u\in V(G)}p(d_G(u),\tau(u)).$$

Proof. If $u_1,\ldots,u_n$ is a linear order of the vertices of G that is chosen uniformly at random, then $M=\left\{u_i\in V(G): d_{\{u_j:1\le j\le i-1\}}(u_i)<\tau(u_i)\right\}$ is a τ-dynamic monopoly. Since the probability that $d_{\{u_j:1\le j\le i-1\}}(u_i)<\tau(u_i)$ holds is exactly $p(d_G(u),\tau(u))$, the expected cardinality of M is at most $\sum_{u\in V(G)}p(d_G(u),\tau(u))$, which implies the given bound by the first-moment method. ■

The following result generalizes and improves Theorem 5 from [6].

Theorem 2. *Let G be a graph of order n, size m, and vertex degrees $d_1\le\cdots\le d_n$. Let t be a rational number such that $t\cdot n$ is an integer.*

(i) *If $0\le t\cdot n\le 2m+n$, then $\max\{\mathrm{dyn}_\tau(G): 0\le\tau\le d_G+1$ and $\overline{\tau}=t\}$ equals*

$$\max\left\{k\in[n]:\sum_{i=1}^{k}(d_i+1)\le t\cdot n\right\}.$$

(ii) *If $0\le t\cdot n<2m$, then $\max\{\mathrm{dyn}_\tau(G): 0\le\tau\le d_G$ and $\overline{\tau}=t\}$ is at most*

$$\sum_{i=1}^{k^*}\frac{d_i}{d_i+1}+\frac{1}{d_{i+1}+1}\left(t\cdot n-\sum_{i=1}^{k^*}d_i\right)$$

where

$$k^*=\max\left\{k\in[n]:\sum_{i=1}^{k}d_i\le t\cdot n\right\}.$$

For G, n, m, d_i, and t as in Theorem 2 such that $0\le t\cdot n\le 2m$, Theorem 5 in [6] states

$$\max\{\mathrm{dyn}_\tau(G):0\le\tau\le d_G \text{ and } \overline{\tau}=t\}\le\max\left\{k\in[n]:\sum_{i=1}^{k}(d_i+1)\le t\cdot n\right\},$$

that is, (i) of Theorem 2 shows that this bound is actually the correct value for the more general threshold functions τ with $0\le\tau\le d_G+1$ and (ii) of

Theorem 2 improves the upper bound for threshold functions τ with $0 \le \tau \le d_G$ as considered in [6].

Note that in (ii) the case $t \cdot n = 2m$ is not allowed. If $t \cdot n = 2m$ and $\tau \le d_G$, then $\tau = d_G$ and $\mathrm{dyn}_\tau(G)$ coincides with the vertex cover number of G.

***Proof of Theorem* 2.** (i) Let $\tilde{k} = \max\left\{k \in [n] : \sum_{i=1}^{k}(d_i + 1) \le t \cdot n\right\}$. The integer linear program

$$\begin{array}{lrcll} \max & \sum_{i=1}^n \frac{1}{d_i+1}\tau_i & & & \\ \text{s.th.} & \sum_{i=1}^n \tau_i & = & t \cdot n & \\ & \tau_i & \le & d_i + 1 & \text{for each } i \in [n] \\ & \tau_i & \ge & 0 & \text{for each } i \in [n]. \end{array}$$

has the optimum solution

$$\tilde{\tau} = (\tilde{\tau}_1, \ldots, \tilde{\tau}_n) = \left(d_1 + 1, \ldots, d_{\tilde{k}} + 1, t \cdot n - \sum_{i=1}^{\tilde{k}}(d_i + 1), 0, \ldots, 0\right).$$

Since $t \cdot n - \sum_{i=1}^{\tilde{k}}(d_i + 1) < d_{\tilde{k}+1} + 1$, Theorem 1 implies

$\max\left\{\mathrm{dyn}_\tau(G) : 0 \le \tau \le d_G + 1 \text{ and } \overline{\tau} = t\right\} \le \sum_{i=1}^n \frac{1}{d_i+1}\tilde{\tau}_i$

$\le \left\lfloor \tilde{k} + \frac{1}{d_{\tilde{k}+1}+1}\left(t \cdot n - \sum_{i=1}^{\tilde{k}}(d_i + 1)\right)\right\rfloor = \tilde{k}.$

Since every $\tilde{\tau}$-dynamic monopoly contains the $\tilde{k}$ vertices with degrees $d_1, \ldots, d_{\tilde{k}}$, the equality stated in (i) follows.

(ii) Let k^* be as in the statement of (ii). The integer linear program

$$\begin{array}{lrcll} \max & \sum_{i=1}^n \frac{1}{d_i+1}\tau_i & & & \\ \text{s.th.} & \sum_{i=1}^n \tau_i & = & t \cdot n & \\ & \tau_i & \le & d_i & \text{for each } i \in [n] \\ & \tau_i & \ge & 0 & \text{for each } i \in [n]. \end{array}$$

has the optimum solution

$$\tau^* = (\tau_1^*, \ldots, \tau_n^*) = \left(d_1, \ldots, d_{k^*}, t \cdot n - \sum_{i=1}^{k^*} d_i, 0, \ldots, 0\right)$$

and Theorem 1 implies the inequality stated in (ii). ■

The following result strengthens Theorem 4 in [8]. If H is an induced subgraph of a graph G, we write $H \subseteq_{\mathrm{ind}} G$.

Theorem 3. *Let G be a graph of order n and size m. Let t be a rational number such that $t \cdot n$ is an integer.*

(i) *If* $0 \leq t \cdot n \leq 2m + n$, *then* $\min\{\mathrm{dyn}_\tau(G) : 0 \leq \tau \leq d_G + 1 \text{ and } \overline{\tau} = t\}$ *equals*

$$n - \max\{n(H) : H \subseteq_{\mathrm{ind}} G \text{ and } n(H) + m(H) \leq 2m - (t-1)n\}.$$

(ii) *If* $0 \leq t \cdot n \leq 2m$, *then* $\min\{\mathrm{dyn}_\tau(G) : 0 \leq \tau \leq d_G \text{ and } \overline{\tau} = t\}$ *equals*

$$n - \max\{n(H) : H \subseteq_{\mathrm{ind}} G \text{ and } m(H) \leq 2m - t \cdot n\}.$$

For G, n, m, and t as in Theorem 3, Theorem 4 in [8] states that

$$\max\{\mathrm{dyn}_\tau(G) : \tau \leq d_G \text{ and } \overline{\tau} = t\}$$

equals

$$n - \max\{n(H) : H \subseteq_{\mathrm{ind}} G \text{ and } m(H) \leq 2m - t \cdot n\},$$

that is, (ii) shows that the restriction to non-negative threshold functions τ does not change the minimum possible value. The following proof relies on the proof given in [8].

Proof of Theorem 3. (i) Let the threshold function τ minimize $\mathrm{dyn}_\tau(G)$ subject to the conditions $0 \leq \tau \leq d_G + 1$ and $\overline{\tau} = t$. Let M be a τ-dynamic monopoly of G with $|M| = \mathrm{dyn}_\tau(G)$. Since $H = G - M$ is $(d_G - \tau)$-degenerate and $d_G(u) - \tau(u) \geq -1$ for every $u \in V(G)$ we have $m(H) \leq \sum_{u \in V(H)}(d_G(u) - \tau(u)) \leq \sum_{u \in V(G)}(d_G(u) - \tau(u)) + |M| = 2m - t \cdot n + |M| = 2m - t \cdot n + n - n(H)$, which implies $n(H) + m(H) \leq 2m - (t-1)n$. Hence $\min\{\mathrm{dyn}_\tau(G) : 0 \leq \tau \leq d_G + 1 \text{ and } \overline{\tau} = t\}$ is at least the given value.

For the converse inequality, let H be an induced subgraph of G with $n(H) + m(H) \leq 2m - (t-1)n$. Let $v_1, \ldots, v_h$ be a linear ordering of the vertices of H and let $\tau_0(v_i) = d_G(v_i) - d_{\{v_j : 1 \leq j \leq i-1\}}(v_i)$ for $i \in [h]$. Let $M = V(G) \setminus V(H)$. For $u \in M$, let $\tau_0(u) = d_G(u) + 1$. We obtain

$$\begin{aligned} \sum\nolimits_{u \in V(G)} \tau_0(u) &= \sum\nolimits_{u \in M} (d_G(u) + 1) + \sum\nolimits_{i=1}^{n} \left(d_G(v_i) - d_{\{v_j : 1 \leq j \leq i-1\}}(v_i)\right) \\ &= 2m + n - n(H) - m(H) \\ &\geq t \cdot n. \end{aligned}$$

Since $t \cdot n \geq 0$ and $0 \leq \tau_0 \leq d_G + 1$, it is possible to reduce some values of τ_0 to obtain a threshold function τ with $0 \leq \tau \leq d_G + 1$ and $\overline{\tau} = t$. Since M is a τ_0-dynamic monopoly, it is also a τ-dynamic monopoly. Hence $\min\{\mathrm{dyn}_\tau(G) : 0 \leq \tau \leq d_G + 1 \text{ and } \overline{\tau} = t\}$ is at most the given value.

(ii) The proof is analogous to (i). ∎

Our final result extends Theorem 1.7 from Alon *et al.* [2]

Theorem 4. *Let r and s be positive integers. If G is a graph of order n and maximum degree Δ at least 2, then*

$$\alpha_{r+s}(G) \geq \alpha_r(G) + \frac{s}{\Delta(\Delta-1)}\left(n - \alpha_r(G)\right).$$

Proof. Let I be an r-degenerate set of G of maximum cardinality $\alpha_r(G)$. Clearly, we may assume that $\alpha_r(G) < n$. Let R be the set $V(G)\setminus I$. Let H be the subgraph of the square of G induced by R. By the choice of I, every vertex in R has at least $r+1$ neighbors in I. Hence, if $u \in R$, then

$$\begin{aligned} d_H(u) &\leq \sum\nolimits_{v\in N_I(u)} (d_R(v)-1) + \sum\nolimits_{v\in N_R(u)} d_R(v) \\ &\leq \sum\nolimits_{v\in N_I(u)} (\Delta-1) + \sum\nolimits_{v\in N_R(u)} (\Delta-r-1) \\ &\leq d_I(u)(\Delta-1) + (\Delta - d_I(u))(\Delta-r-1) \\ &\leq \Delta(\Delta-1). \end{aligned}$$

If $d_H(u) = \Delta(\Delta-1)$, then u has Δ neighbors in G that all lie in I, every neighbor of u has Δ neighbors in G that all lie in R, and u is the only common neighbor in G of neighbors of u in G. This implies that, if some component of H is $\Delta(\Delta-1)$-regular, then there is a component K of G such that K intersects I and R and $V(K)\cap I$ is an independent set. Since adding a vertex from $V(K)\cap R$ to I yields an r-degenerate set of G, this implies a contradiction to the choice of I, that is, no component of H is $\Delta(\Delta-1)$-regular. By Brooks' theorem [4], the graph H has a $\Delta(\Delta-1)$-coloring $R = V_1 \cup \cdots \cup V_{\Delta(\Delta-1)}$. Note that every vertex of G has at most one neighbor in each of the color classes V_i. Therefore, adding to I the s largest color classes V_i yields an $(r+s)$-degenerate set of G, the desired bound follows. ■

References

[1] N. Alon, J. Kahn and P.D. Seymour, *Large induced degenerate subgraphs*, Graphs Combin. **3** (1987) 203–211.
doi:10.1007/BF01788542

[2] N. Alon, D. Mubayi and R. Thomas, *Large induced forests in sparse graphs*, J. Graph Theory **38** (2001) 113–123.
doi:10.1002/jgt.1028

[3] P. Borowiecki, F. Göring, J. Harant and D. Rautenbach, *The potential of greed for independence*, J. Graph Theory **71** (2012) 245–259.
doi:10.1002/jgt.20644

[4] R.L. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Philos. Soc. **37** (1941) 194–197.
doi:10.1017/S030500410002168X

[5] Y. Caro, *New results on the independence number*, Technical Report, Tel-Aviv University, 1979.

[6] K. Khoshkhah, H. Soltani and M. Zaker, *On dynamic monopolies of graphs: The average and strict majority thresholds*, Discrete Optimization **9** (2012) 77–83. doi:10.1016/j.disopt.2012.02.001

[7] V.K. Wei, A lower bound on the stability number of a simple graph, Technical Memorandum, TM 81-11217-9, Bell Laboratories, 1981.

[8] M. Zaker, *Generalized degeneracy, dynamic monopolies and maximum degenerate subgraphs*, Discrete Appl. Math. **161** (2013) 2716–2723. doi:10.1016/j.dam.2013.04.012

Received 5 September 2013
Revised 18 March 2014
Accepted 18 March 2014