Discussiones Mathematicae
Graph Theory 35 (2015) 229–235
doi:10.7151/dmgt.1787

$(k-1)$-KERNELS IN STRONG k-TRANSITIVE DIGRAPHS

Ruixia Wang

School of Mathematical Sciences
Shanxi University
Taiyuan Shanxi,030006, PR China

e-mail: wangrx@sxu.edu.cn

Abstract

Let $D = (V(D), A(D))$ be a digraph and $k \geq 2$ be an integer. A subset N of $V(D)$ is k-independent if for every pair of vertices $u, v \in N$, we have $d(u, v) \geq k$; it is l-absorbent if for every $u \in V(D) - N$, there exists $v \in N$ such that $d(u, v) \leq l$. A (k, l)-kernel of D is a k-independent and l-absorbent subset of $V(D)$. A k-kernel is a $(k, k-1)$-kernel.

A digraph D is k-transitive if for any path $x_0x_1 \cdots x_k$ of length k, x_0 dominates x_k. Hernández-Cruz [3-*transitive digraphs*, Discuss. Math. Graph Theory **32** (2012) 205–219] proved that a 3-transitive digraph has a 2-kernel if and only if it has no terminal strong component isomorphic to a 3-cycle. In this paper, we generalize the result to strong k-transitive digraphs and prove that a strong k-transitive digraph with $k \geq 4$ has a $(k-1)$-kernel if and only if it is not isomorphic to a k-cycle.

Keywords: digraph, transitive digraph, k-transitive digraph, k-kernel.

2010 Mathematics Subject Classification: 05C20.

1. Terminology and Introduction

We assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1] for terminology not defined here. We only consider finite digraphs without loops or multiple arcs. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For a vertex x in D, its *out-neighborhood* $N^+(x) = \{y \in V(D) : xy \in A(D)\}$ and its *in-neighborhood* $N^-(x) = \{y \in V(D) : yx \in A(D)\}$. For a pair X, Y of vertex sets of D, define $(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$.

Let x and y be two vertices of $V(D)$. The *distance* from x to y in D, denoted $d(x, y)$, is the minimum length of an (x, y)-path, if y is reachable from x, and otherwise $d(x, y) = \infty$. The distance from a set X to a set Y of vertices in D is $d(X, Y) = \max\{d(x, y) : x \in X, y \in Y\}$. The *diameter* of D is $\mathrm{diam}(D) = d(V(D), V(D))$. A digraph D is said to be *strongly connected* or just *strong*, if for every pair x, y of vertices of D, there is an (x, y)-path. Clearly, D has finite diameter if and only if D is strong.

A *cycle* is a finite sequence of distinct vertices $C = x_0x_1 \cdots x_nx_0$ such that $x_{i-1} \to x_i$ for every $1 \leq i \leq n$ and $x_n \to x_0$, whose length is $n + 1$. We denote the subpath of C from x_i to x_j by $C[x_i, x_j] = x_ix_{i+1} \cdots x_j$. Let C be a cycle of length $k \geq 2$ and let $V_1, V_2, \ldots, V_k$ be pairwise disjoint vertex sets. *The extended k-cycle* $C[V_1, V_2, \ldots, V_k]$ is the digraph with vertex set $V_1 \cup V_2 \cup \cdots \cup V_k$ and arc set $\bigcup_{i=1}^{k}\{v_iv_{i+1} : v_i \in V_i, v_{i+1} \in V_{i+1}\}$, where subscripts are taken modulo k.

A *biorientation* of the graph G is a digraph D obtained from G by replacing each edge $\{x, y\} \in E(G)$ by either the arc xy or the arc yx or the pair of arcs xy and yx. A *complete digraph* is a biorientation of a complete graph obtained by replacing each edge $\{x, y\}$ by the arcs xy and yx. A *complete bipartite digraph* is a biorientation of a complete bipartite graph obtained by replacing each edge $\{x, y\}$ by the arcs xy and yx. A digraph is *k-transitive* if for any path $x_0x_1 \cdots x_k$ of length k, x_0 dominates x_k. A 2-transitive digraph is called a *transitive digraph.* The family of k-transitive digraphs have been studied in [4, 5, 6, 7].

A subset N of $V(D)$ is *k-independent* if for every pair of vertices $u, v \in N$, we have $d(u, v) \geq k$; it is *l-absorbent* if for every $u \in V(D) - N$ there exists $v \in N$ such that $d(u, v) \leq l$. A *(k, l)-kernel* of D is a k-independent and l-absorbent subset of $V(D)$. A *k-kernel* is a $(k, k-1)$-kernel. A 2-kernel is called *kernel.* Kernels have been widely studied. A nice survey on the subject is [2]. Chvátal proved in [3] that recognizing digraphs that have a kernel is an NP-complete problem, so finding sufficient conditions for a digraph to have a kernel or finding large families of digraphs with kernels have been a very prosperous line of investigation explored by many authors. In 1980, Kwaśnik [9] introduced the concept of (k, l)-kernels generalizing the notion of kernels. The existence of (k, l)-kernels in some digraphs has been studied. In [8], Galeana-Sánchez et al. showed that every k-transitive digraph has a k-kernel. In [5], Hernández-Cruz proved that a 3-transitive digraph has a 2-kernel if and only if it has no terminal strong component isomorphic to a 3-cycle. In this paper, we generalize the result to strong k-transitive digraphs and prove that a strong k-transitive digraph with $k \geq 4$ has a $(k-1)$-kernel if and only if it is not isomorphic to a k-cycle.

2. $(k-1)$-Kernels in Strong k-Transitive Digraphs

We begin with a useful lemma.

Lemma 1 [4]**.** *Let D be a strong k-transitive digraph with $k \geq 2$. Then $\mathrm{diam}(D) \leq k - 1$.*

When a strong k-transitive digraph D contains a cycle of length at least k, Hernández-Cruz and Montellano-Ballesteros characterized the structure of D as follows.

Theorem 2 [7]**.** *Given an integer k with $k \geq 2$, let D be a strong k-transitive digraph. Suppose that D contains a cycle of length n such that $(n, k-1) = d$ and $n \geq k+1$. Then each of the following holds:*

(1) *If $d = 1$, then D is a complete digraph.*

(2) *If $d \geq 2$, then D is either a complete digraph, a complete bipartite digraph or an extended d-cycle.*

Theorem 3 [7]**.** *Given an integer k with $k \geq 2$, let D be a strong k-transitive digraph of order at least $k+1$. If D contains a cycle of length k, then D is a complete digraph.*

When a strong k-transitive digraph D contains a cycle of length $k-1$, the following lemma holds.

Lemma 4. *Let D be a strong k-transitive digraph with $k \geq 4$ and let $C = x_0x_1 \cdots x_{k-2}x_0$ be a cycle of length $k-1$. Then for every $x \in V(D) \setminus V(C)$, $(x, V(C)) \neq \emptyset$ and $(V(C), x) \neq \emptyset$.*

Proof. Since the converse of a k-transitive digraph is still a k-transitive digraph, we only need to show $(x, V(C)) \neq \emptyset$. Again, since D is strong, there exists a path from x to C. Let $P = y_0y_1 \cdots y_s$ be a shortest path from x to C, where $s \geq 1$, $y_0 = x$ and $y_s \in V(C)$. Without loss of generality, assume that $y_s = x_0$. We prove that y_0 dominates some vertex of $V(C)$ by induction on the length s of P. It clearly holds for $s = 1$. Thus, we assume that $s \geq 2$. Note that $y_1 \cdots y_s$ is a path of length $s-1$. By the induction hypothesis, we know that there exists a vertex $x_i \in V(C)$ such that $y_1 \to x_i$. Then $y_0y_1C[x_i, x_{i-1}]$ is a path of length k, which implies that $y_0 \to x_{i-1}$. ∎

The following theorem is our main result.

Theorem 5. *Let D be a strong k-transitive digraph with $k \geq 4$. Then D has a $(k-1)$-kernel if and only if it is not isomorphic to a k-cycle.*

Proof. The necessity is obvious. Now we show the sufficiency. Since every strong digraph contains a cycle, we consider the following four cases.

Case 1. D contains a cycle of length at least $k+1$. By Theorem 2, D is either a complete digraph, a complete bipartite digraph or an extended d-cycle where $d = (n, k-1) \geq 2$. By the definition of k-kernels, a t-kernel consisting of a single vertex must be a k-kernel where $k \geq t$. Clearly, if D is a complete digraph or a complete bipartite digraph, then every vertex in $V(D)$ is a 3-kernel.

Hence, D has a $(k-1)$-kernel. Now assume that D is an extended d-cycle, denote $D = C[E_1, \ldots, E_d]$, where C is a cycle of length d and every E_i is an independent set. Note that $2 \leq d \leq k-1$. It is easy to check that if $d = k-1$, then every E_i is a $(k-1)$-kernel of D; if $2 \leq d < k-1$, then every vertex in $V(D)$ is a $(k-1)$-kernel.

Case 2. D contains a cycle C of length k. Let $C = x_0x_1 \cdots x_{k-1}x_0$. If $V(D) \setminus V(C) \neq \emptyset$, then Theorem 3 implies that D is a complete digraph. Recall that every complete digraph has a 2-kernel consisting of a single vertex. Hence, D has a $(k-1)$-kernel. If $V(D) \setminus V(C) = \emptyset$, then since D is not isomorphic to a k-cycle, C contains a chord x_ix_j, where the length of $C[x_i, x_j]$ is more than or equal to 2. For any $x_l \in V(C)$, if $l \neq j+1$, then $C[x_l, x_j]$ is a path of length at most $k-2$; if $l = j+1$, then $C[x_{j+1}, x_i]x_j$ is a path of length at most $k-2$. This shows $d(x_l, x_j) \leq k-2$ and so x_j is a $(k-1)$-kernel.

Case 3. D contains a cycle C of length $k-1$. Let $C = x_0x_1 \cdots x_{k-2}x_0$. In this case, subscripts are taken modulo $k-1$. If $V(D) \setminus V(C) = \emptyset$, then every vertex of D is a $(k-1)$-kernel. Now assume that $V(D) \setminus V(C) \neq \emptyset$. By Lemma 4, for any $x \in V(D) \setminus V(C)$, $(x, V(C)) \neq \emptyset$ and $(V(C), x) \neq \emptyset$. Define $S_i = \{y \in V(D) \setminus V(C) : y \to x_i\}$, for every $i \in \{0, 1, \ldots, k-2\}$. Clearly $\bigcup_{i=0}^{k-2} S_i = V(D) \setminus V(C)$.

Suppose that C is an induced cycle. We first claim that if there exist $x \in V(D) \setminus V(C)$ and $x_i, x_j \in V(C)$ such that $x_i \to x \to x_j$, where $i = j$ or $C[x_i, x_j]$ is a path of length at least three, then x_j is a $(k-1)$-kernel. Assume, without loss of generality, that $i = 0$. Then $j = 0$ or $3 \leq j \leq k-2$. It is obvious that for any $y \in V(D) - S_{j+1}$, $d(y, x_j) \leq k-2$. Let $z \in S_{j+1}$ be a vertex different from x. If $j = 0$, then $zC[x_1, x_0]x$ is a path of length k, which implies that $z \to x$ and so $d(z, x_0) = 2 \leq k-2$ as $k \geq 4$. Hence, x_0 is a $(k-1)$-kernel. Now assume that $3 \leq j \leq k-2$. Then $zC[x_{j+1}, x_0]xx_j$ is a path of length at most $k-2$. Hence, x_j is a $(k-1)$-kernel and the claim holds.

For any $x \in S_i$, by Lemma 4, there exists $x_l \in V(C)$ such that $x_l \to x$. If $l \in \{0, 1, \ldots, k-2\} \setminus \{i-1, i-2\}$, then by the above claim, D has a $(k-1)$-kernel. If $l = i-1$, then $x_{i-1}xC[x_i, x_{i-1}]$ is a cycle of length k. By Case 2, D has a $(k-1)$-kernel. Hence, we may assume $l = i-2$. Indeed, we may assume that for any vertex $z \in V(D) \setminus V(C)$, there exists $i \in \{0, 1, \ldots, k-2\}$ such that $x_{i-1} \to z \to x_{i+1}$ and z has no other neighbor on C.

Now we show that D is an extended $(k-1)$-cycle. Let $E_i = \{x \in V(D) \setminus V(C) : x_{i-1} \to x \to x_{i+1}\}$, for every $i \in \{0, 1, \ldots, k-2\}$. Clearly, $\bigcup_{i=0}^{k-2} E_i = V(D) \setminus V(C)$. We first prove that every vertex of E_i dominates every vertex of E_{i+1}. For any $x \in E_i$ and $y \in E_{i+1}$, we have $xC[x_{i+1}, x_i]y$ is a path of length k. Hence $x \to y$. Let $x, x' \in V(D) \setminus V(C)$ such that $x' \to x$. Then there exist $i, j \in \{0, 1, \ldots, k-2\}$ such that $x_{j-1} \to x \to x_{j+1}$ and $x_{i-1} \to x' \to x_{i+1}$. Then $x'xC[x_{j+1}, x_j]$ is a path of length k, which implies that $x' \to x_j$. Hence, we have

$j = i+1$. It follows that $D = C[E_0, E_1, \dots, E_{k-2}]$. It is easy to check that every $E_i, i \in \{0, 1, \dots, k-2\}$ is a $(k-1)$-kernel.

Suppose that C is not an induced cycle. Then there exists a chord $x_j x_i$ in C. Now we will show that x_i is a $(k-1)$-kernel. It is obvious that for any $y \in V(D) - S_{i+1}$, $d(y, x_i) \le k-2$. Let $z \in S_{i+1}$. Note that $zC[x_{i+1}, x_j]x_i$ is a path of length at most $k-2$. Hence, $d(z, x_i) \le k-2$ and x_i is a $(k-1)$-kernel.

Case 4. There exists no cycle of length more than or equal to $k-1$ in D. Let x be a vertex of maximum out-degree in D. If $d^+(x) = 1$, then the out-degree of every vertex in D is one. Since D is strong and there exists no cycle of length more than or equal to $k-1$, D is a cycle of length at most $k-2$. Hence, every vertex of D is a $(k-1)$-kernel. Now assume that $d^+(x) \ge 2$. If x is a $(k-1)$-kernel, then we are done; if not, there exists $z \in V(D) \setminus \{x\}$ such that $d(z, x) \ge k-1$. Combining this with Lemma 1, we have $d(z, x) = k-1$. Denote $W_s = \{y : d(y, x) = s\}$, for $s \in \{0, 1, \dots, k-1\}$. Observe that $(W_j, W_1 \cup \dots \cup W_i) = \emptyset$ when $j \ge i+2$.

Claim 1. $N^+(x) \subseteq W_1 \cup \dots \cup W_{k-3}$.

Proof. Since D has no loops, $N^+(x) \cap W_0 = \emptyset$. By the definition of W_s, every vertex of W_s can reach x in s steps. If $N^+(x) \cap (W_{k-1} \cup W_{k-2}) \neq \emptyset$, say $v \in N^+(x) \cap (W_{k-1} \cup W_{k-2})$, then let P be the shortest path from v to x. Then Pv is a cycle of length k or $k-1$, a contradiction to the hypothesis of Case 4. Hence, $N^+(x) \subseteq W_1 \cup \dots \cup W_{k-3}$. □

Claim 2. *Every vertex of $N^+(x)$ is contained in the shortest path from any vertex of W_{k-1} to x.*

Proof. Let $z' \in W_{k-1}$ and P' be a shortest path from z' to x. If there exists $v \in N^+(x)$ such that $v \notin V(P')$, then $P'v$ is a path of length k, which implies that $z' \to v$, a contradiction to $v \in W_1 \cup \dots \cup W_{k-3}$ and $(W_{k-1}, W_1 \cup \dots \cup W_{k-3}) = \emptyset$. Hence, $v \in V(P')$ and furthermore $N^+(x) \subseteq V(P')$. The proof of the claim is complete. □

By Claim 2, $|N^+(x) \cap W_s| \le 1$, for every $s \in \{1, 2, \dots, k-3\}$. Let $r = \min\{j : W_j \cap N^+(x) \neq \emptyset\}$. Denote $N^+(x) \cap W_r = \{w\}$. Now we show that w is a $(k-1)$-kernel. For any $u \in W_{k-1}$, by Claim 2, we can conclude that $d(u, w) \le k-2$. For any $u \in W_0 \cup W_1 \cup \dots \cup W_{k-3}$, since $d(u, x) \le k-3$, we have $d(u, w) \le k-2$. If $d(u, w) \le k-2$ for any $u \in W_{k-2}$, then w is a $(k-1)$-kernel. Since $\operatorname{diam}(D) \le k-1$, we assume that $d(u, w) = k-1$. Let $R = u_{k-2}u_{k-3} \cdots u_1 u_0$ be a shortest path from u to x, where $u_{k-2} = u, u_0 = x$ and $u_j \in W_j$ for $j = 1, 2, \dots, k-3$, and let $Q = x_{k-1}x_{k-2} \cdots x_0$ be a shortest path from z to x, where $x_{k-1} = z$, $x_0 = x$ and $x_j \in W_j$ for $j = 1, 2, \dots, k-2$. By Claim 2 and the choice of r, we have $N^+(x) \subseteq \{x_r, \dots, x_{k-3}\}$ and $w = x_r$. Again since $d^+(x) \ge 2$, there exists $x_p \in N^+(x) \cap \{x_{r+1}, \dots, x_{k-3}\}$. It is obvious that $u \notin V(Q)$ and $w \notin V(R)$, otherwise $d(u, w) \le k-2$, a contradiction. If there exists a vertex $x_i \in \{x_{r+1}, \ldots, x_{k-3}\} \cap V(R)$, then $x_i = u_i$ and $R[u_{k-2}, u_i]P[x_{i-1}, x_r]$ is a path of length $k-2-r$. Recalling $1 \leq r \leq k-3$, we have $1 \leq k-2-r \leq k-3$. So, $R[u_{k-2}, u_i]P[x_{i-1}, x_r]$ is a path of length at most $k-3$, a contradiction to $d(u, w) = k-1$. Hence, $\{x_{r+1}, \ldots, x_{k-3}\} \cap V(R) = \emptyset$. Combining this with $x_p \in N^+(x) \cap \{x_{r+1}, \ldots, x_{k-3}\}$, we have Rx_px_{p-1} is a path of length k. Hence, $u \to x_{p-1}$, a contradiction to $(W_{k-2}, W_1 \cup \cdots \cup W_{k-4}) = \emptyset$, as $p-1 \leq k-4$. So, we have shown that w is a $(k-1)$-kernel. ■

For 3-transitive digraphs, Hernández-Cruz [5] proved the following theorem.

Theorem 6 [5]. *Let D be a 3-transitive digraph. Then D has a 2-kernel if and only if it has no terminal strong component isomorphic to a 3-cycle.*

In view of this result and Theorem 5, we propose the following conjecture.

Conjecture 7. *Let D be a k-transitive digraph. Then D has a $(k-1)$-kernel if and only if it has no terminal strong component isomorphic to a k-cycle.*

Acknowledgement

This work is supported by the National Natural Science Foundation for Young Scientists of China (11201273)(61202017)(11401354)(11401353), the Natural Science Foundation for Young Scientists of Shanxi Province, China (2013021001-5) and Shanxi Scholarship Council of China (2013-017).

References

[1] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications (Springer, London, 2000).

[2] E. Boros and V. Gurvich, *Perfect graphs, kernels, and cores of cooperative games*, Discrete Math. **306** (2006) 2336–2354.
doi:10.1016/j.disc.2005.12.031

[3] V. Chvátal, On the computational complexity of finding a kernel, Report No. CRM-300, Centre de Recherches Mathematiques, Universite de Montreal (1973).

[4] C. Hernández-Cruz and H. Galeana-Sánchez, *k-kernels in k-transitive and k-quasi-transitive digraphs*, Discrete Math. **312** (2012) 2522–2530.
doi:10.1016/j.disc.2012.05.005

[5] C. Hernández-Cruz, 3*-transitive digraphs*, Discuss. Math. Graph Theory **32** (2012) 205–219.
doi:10.7151/dmgt.1613

[6] C. Hernández-Cruz, 4*-transitive digraphs I: the structure of strong transitive digraphs*, Discuss. Math. Graph Theory **33** (2013) 247–260.
doi:10.7151/dmgt.1645

[7] C. Hernández-Cruz and J.J. Montellano-Ballesteros, *Some remarks on the structure of strong k-transitive digraphs*, Discuss. Math. Graph Theory **34** (2014) 651–671. doi:10.7151/dmgt.1765

[8] H. Galeana-Sánchez, C. Hernández-Cruz and M.A. Juárez-Camacho, *On the existence and number of* $(k+1)$*-kings in k-quasi-transitive digraphs*, Discrete Math. **313** (2013) 2582–2591.
doi:10.1016/j.disc.2013.08.007

[9] M. Kwaśnik, On (k,l)-kernels on graphs and their products, Doctoral Dissertation, Technical University of Wrocław, Wrocław, 1980.

Received 7 January 2014
Revised 15 May 2014
Accepted 19 May 2014