# MAXIMUM CYCLE PACKING IN EULERIAN GRAPHS USING LOCAL TRACES 

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#### Abstract

For a graph $G=(V, E)$ and a vertex $v \in V$, let $T(v)$ be a local trace at $v$, i.e. $T(v)$ is an Eulerian subgraph of $G$ such that every walk $W(v)$, with start vertex $v$ can be extended to an Eulerian tour in $T(v)$.

We prove that every maximum edge-disjoint cycle packing $\mathcal{Z}^{*}$ of $G$ induces a maximum trace $T(v)$ at $v$ for every $v \in V$. Moreover, if $G$ is Eulerian then sufficient conditions are given that guarantee that the sets of cycles inducing maximum local traces of $G$ also induce a maximum cycle packing of $G$.


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## 1. INTRODUCTION

We consider a finite and undirected graph $G$ with vertex set $V(G)$ and edge set $E(G)$ that contains no loops. For a finite sequence $v_{i_{1}}, e_{1}, v_{i_{2}}, e_{2}, \ldots, e_{r-1}, v_{i_{r}}$ of vertices $v_{i_{j}}$ and pairwise distinct edges $e_{j}=\left(v_{i_{j}}, v_{i_{j+1}}\right)$ of $G$, the subgraph $W$ of $G$ with vertices $V(W)=\left\{v_{i_{1}}, v_{i_{2}}, \ldots, v_{i_{r}}\right\}$ and edges $E(W)=\left\{e_{1}, e_{2}, \ldots, e_{r-1}\right\}$ is called a walk with start vertex $v_{i_{1}}$ and end vertex $v_{i_{r}}$. If $W$ is closed (i.e, $v_{i_{1}}=v_{i_{r}}$ ) we call it a circuit in $G$. A path is a walk in which all vertices $v$ have degree $d_{W}(v) \leq 2$. A closed path will be called a cycle. A connected graph in which all vertices $v$ have even degree is called Eulerian. For an Eulerian graph $G$, a circuit $W$ with $E(W)=E(G)$ is called an Eulerian tour.

For $1 \leq i \leq k$, let $G_{i} \subset G$ be subgraphs of $G$. We say that $G$ is induced by $\left\{G_{1}, G_{2}, \ldots, G_{k}\right\}$ if $V(G)=V\left(G_{1}\right) \cup V\left(G_{2}\right) \cup \cdots \cup V\left(G_{k}\right)$ and $E(G)=E\left(G_{1}\right) \cup$ $E\left(G_{2}\right) \cup \cdots \cup E\left(G_{k}\right)$. Two subgraphs $G^{\prime}=\left(V^{\prime}, E^{\prime}\right), G^{\prime \prime}=\left(V^{\prime \prime}, E^{\prime \prime}\right)$ of $G$ are called edge-disjoint if $E^{\prime} \cap E^{\prime \prime}=\emptyset$. For $E^{\prime} \subseteq E$ we define $G \backslash E^{\prime}=\left(V, E \backslash E^{\prime}\right)$. For $V^{\prime} \subset V$ we define $G \backslash V^{\prime}=\left.G\right|_{V \backslash V^{\prime}}$, where $V\left(\left.G\right|_{V \backslash V^{\prime}}\right)=V \backslash V^{\prime}$ and $E\left(\left.G\right|_{V \backslash V^{\prime}}\right)=$ $\{e \in E(G) \mid$ both endvertices of $e$ belong to $V\}$.

A packing $\mathcal{Z}(G)=\left\{G_{1}, \ldots, G_{q}\right\}$ of $G$ is a collection of subgraphs $G_{i}$ of $G$ $(i=1, \ldots, q)$ such that all $G_{i}$ are mutually edge-disjoint and $G$ is induced by $\left\{G_{1}, \ldots, G_{q}\right\}$. If exactly $s$ of the $G_{i}$ are cycles, $\mathcal{Z}(G)$ is called a cycle packing of cardinality $s$. The family of cycle-packings of $G$ is denoted by $\mathcal{C}(G)$. If the cardinality of a cycle packing $\mathcal{Z}(G)$ is maximum, it is called a maximum cycle packing. Its cardinality is denoted by $\nu(G)$. If no confusion is possible we will write $\mathcal{Z}$ instead of $\mathcal{Z}(G)$ and $\mathcal{C}$ instead of $\mathcal{C}_{s}(G)$, respectively.

Packing edge-disjoint cycles in graphs is a classical graph-theoretical problem. There is a large amount of literature concerning conditions that are sufficient for the existence of some number of disjoint cycles which may satisfy some further restrictions. A selection of related references is given in [8]. The algorithmic problems concerning edge-disjoint cycle packings are typically hard (e.g. see $[4,5,10])$. There are papers in which practical applications of such packings are mentioned $[1,3,6,9]$.

Starting point of the paper is the attempt to obtain a maximum cycle packing of a graph $G$ by the determination of such packings for specific subgraphs of $G$. In [8] such an approach was studied when the subgraphs were induced by vertex cuts.

In the present paper we study the behaviour of such packings if $G$ is Eulerian and the subgraphs are (local) traces.

In Section 2, local traces are introduced and relations between local traces and maximum cycle packings are given. It turns out in Section 3 that under special conditions a maximum cycle packing can be constructed from maximum cycle packings of maximum local traces.

In Section 4, a mini-max theorem gives a condition whether given maximum local traces are induced by a maximum cycle packing $\mathcal{Z}^{*}$ of $G$. For this the square-length of the cycles is essential.

## 2. Relation Between Maximum Cycles Packings and Local Traces

In this section we will show, how to built up maximum cycle packings iteratively from maximum cycle packings of special subgraphs, if $G$ is Eulerian. This subgraphs will be (local) traces. For special cases Theorem 10 guarantees that the so constructed cycle packing is maximum.

Let $G=(V, E)$ be an Eulerian graph. A vertex $v \in V$ is called proper, if every walk $W$, starting at $v$ can be extended to an Euler-tour in $H$. An Eulerian graph that contains a proper vertex is called a trace. Traces were first considered by Ore in [11] and [2]. Such type of graphs can be characterized in the following way.

Proposition 1. Let $G=(V, E)$ be an Eulerian graph. Let $v \in V$. The following statements are equivalent:
i. $v$ is proper.
ii. If $C$ is an arbitrary cycle in $G$, then $v \in V(C)$.
iii. The number $k$ of components of $G \backslash\{v\}$ is determined by $k=d_{G}(v)-\gamma(G)$, where $\gamma(G)$ denotes the cyclomatic number of $G$.

Proof. See [11].
If $v$ is a proper vertex of degree $d_{G}(v)$, then $G$ is induced by $r=\frac{d(v)}{2}$ edgedisjoint cycles $\left\{C_{1}, \ldots, C_{r}\right\}$, where all $C_{i}$ are passing $v$. Any two of these cycles $C_{i}, C_{j}, i \neq j$ have at most one other vertex in common, and there exists at most one further proper vertex $w \neq v$ in $V$. This is the case if and only if $d(v)=d(w)$ (see [2]).

The following simple characterization relates traces to cycle packings. In [12] it is proved

Proposition 2. If $G=(V, E)$ is Eulerian and $d_{G}(v)=\Delta=\max \left\{d_{G}(u) \mid u \in V\right\}$, then $\nu(G)=\frac{1}{2} \Delta=\frac{1}{2} d_{G}(v)$ if and only if $G$ is a trace with proper vertex $v$.
Proof. Note that $\nu(G) \geq \frac{1}{2} \Delta$ holds since $G$ is Eulerian.
" $\Rightarrow$ ": Let $\nu(G)=\frac{1}{2} \Delta=\frac{1}{2} d_{G}(v)$. Assume that there is a cycle $C \subseteq G$ with $v \notin$ $V(C)$. Obviously, each of the components $G_{1}^{\prime}, G_{2}^{\prime}, \ldots, G_{k}^{\prime}$ of $G \backslash E(C)$ is Eulerian. Let $G_{i}^{\prime}$ be that component that contains $v$. Then $\left.d\right|_{G_{i}^{\prime}}(v)=d(v)=\Delta$. But then, $\nu(G) \geq 1+\sum_{j=1}^{k} \nu\left(G_{j}^{\prime}\right)>\nu\left(G_{i}^{\prime}\right) \geq \frac{1}{2} d_{G_{i}^{\prime}}(v)=\frac{1}{2} d_{G}(v)=\frac{1}{2} \Delta$, contradicting $\nu(G)=\frac{1}{2} \Delta$. Therefore, each cycle $C \subset G$ passes $v$, hence by Proposition $1, v$ is a proper vertex.
" $\Leftarrow$ ": Let $v$ be a proper vertex of $G$. If $\mathcal{Z}^{*}=\left\{C_{1}, C_{2}, \ldots, C_{\nu(G)}\right\}$ is a maximum cycle packing of $G$, then all cycles in $\mathcal{Z}^{*}$ have to pass $v$, i.e, $d_{G}(v)=$ $2 \nu(G) \geq \Delta$. Since $d_{G}(v) \leq \Delta, \nu(G)=\frac{1}{2} \Delta=\frac{1}{2} d_{G}(v)$ follows.

Remark 3. i. For a graph $G$, let $\gamma(G)$ denote the cyclomatic number of $G$. If $G$ is a trace with proper vertex $v$, then the graph $G \backslash\{v\}$ consists of $k=d_{G}(v)-\gamma(G) \geq 1$ components $\left\{B_{1}, B_{k}^{\prime}, \ldots, B_{k}^{\prime}\right\}$ that are all trees. Let $B_{i}$ be such a component and $W_{i}:=\left\{w \in B_{i}^{\prime} \mid d_{B_{i}}(w)\right.$ is odd $\}, r_{i}:=\# W_{i}$. Then the graph $G_{i}=\left(V\left(G_{i}\right), E\left(G_{i}\right)\right)$ with $V\left(G_{i}\right)=V\left(B_{i}\right) \cup\{v\}$ and $E\left(G_{i}\right)=$
$E\left(B_{i}\right) \cup\left\{(w, v) \mid w \in W_{i}\right\}$ is also a trace with proper vertex $v$. Obviously, $\nu\left(G_{i}\right)=\frac{1}{2} r_{i}$ and $\nu(G)=\sum_{i=1}^{k} \nu\left(G_{i}\right)$.
ii. If $G$ is 2 -connected and $k^{\prime}:=\gamma(G)-\nu(G)$, then there is a finite set $\mathcal{P}\left(k^{\prime}\right)$ of graphs (depending only on $k^{\prime}$ not on $G$ ) such that $G$ arises by applying a simple extension rule to a graph in $\mathcal{P}\left(k^{\prime}\right)$ (see [7]). If $G$ is a trace, then this situation is even simpler: since for each of the subgraphs $G_{i}$ it holds $\gamma\left(G_{i}\right)-\nu\left(G_{i}\right)=\frac{1}{2} r_{i}-1=\gamma\left(K_{2}^{r_{i}}\right)-\nu\left(K_{2}^{r_{i}}\right)$ and all edges $E\left(G_{i}\right)$ belong to a maximum cycle packing of $G_{i}, G_{i}$ arises by an extension of $K_{2}^{r_{i}}$. Here $K_{2}^{r_{i}}$ is the multi-graph consisting of two vertices and $r_{i}$ parallel edges.

Now, we will transfer the concept of a trace to an arbitrary graph $G=(V, E)$.
For $v \in V$, an Eulerian subgraph $T(v)=(V(T(v)), E(T(v))) \neq \emptyset$ of $G$ is called a local trace (at v), if $v \in V(T(v))$ and $v$ is proper with respect to $T(v)$. The number $|E(T(v))|$ is called the size of the trace (at $v$ ).

A local trace $T(v)$ is called saturated (at $v$ ), if there is no Eulerian subgraph $H \subset G$ such that $T(v) \subsetneq H$ and $v$ is proper with respect to $H$. It is called maximum, if $T(v)$ is induced by $k(v)$ edge-disjoint cycles $\left\{C_{1}, C_{2}, \ldots, C_{k(v)}\right\} \subset G$ and $k(v)$ is maximum.

$G$


Figure 1. $G$ together with maximum traces $T(u)$ (green colored edges), $T(v)$ (red), $T(w)=T(s)$ (blue).

Being a trace $T(v)$ at $v$ is a local property of the graph $G$. Obviously, each single cycle $C \in G$ that passes $v$ is a local trace at $v$. In general, local traces are not uniquely determined, even maximum local traces are not.

For Eulerian graphs we have

Lemma 4. Let $G=(V, E)$ be Eulerian and $\mathcal{Z}^{*}$ a maximum cycle packing of $G$. For $v \in V$, let $\mathcal{Z}^{*}(v):=\left\{C_{i} \in \mathcal{Z}^{*} \mid v \in V\left(C_{i}\right)\right\}$. Then $\mathcal{Z}^{*}(v)$ induces a maximum trace $T(v)$ at $v$.

Proof. Let $T(v)$ be the subgraph of $G$ induced by the $\frac{d_{G}(v)}{2}$ cycles of $\mathcal{Z}^{*}(v)$. Obviously, $T(v)$ is Eulerian, $v \in V(T(v))$ and $d_{T}(v) \geq d_{T}(u)$ for all $u \in T(v)$. Because $\mathcal{Z}^{*}$ is maximum, $\mathcal{Z}^{*}(v)$ is also a maximum cycle packing of $T(v)$, i.e, $\nu(T(v))=\frac{d_{G}(v)}{2}=\frac{d_{T}(v)}{2}$. Then, by Proposition $2, v$ is a proper vertex of $T(v)$, i.e, $T(v)$ is a maximum trace.

Note, that the fact that $G$ is Eulerian is crucial, i.e, in a general situation a maximum cycle packing must not induce a maximum trace at $v$, even it must not induce a saturated trace

## 3. Getting Maximum Packings of $G$ from Cycle Packings of Maximum Traces

An immediate question that arises is under which conditions the inverse of Lemma 4 is true. In this section such a condition is given, that allows a construction of a maximum cycle packing $\mathcal{Z}^{*}$ of $G$. The construction will use local traces of special subgraphs of $G$.

First, we give construction scheme to obtain a local trace at $v$ from an arbitrary set $C(v)$ of edge-disjoint cycles that all pass $v$.

Lemma 5. Let $G=(V, G), v \in V$. For $r \geq 1$ let $C(v)=\left\{C_{1}, C_{2}, \ldots, C_{r}\right\}$, be a set of edge-disjoint cycles in $G$ that all pass $v$. Then there is a trace $T(v)$, induced by $r$ cycles $\left\{\bar{C}_{1}, \bar{C}_{2}, \ldots, \bar{C}_{r}\right\}$ such that $E(T(v)) \subset E(C(v))$.

Proof. Let $G^{\prime}$ be the graph induced by $C(v)$. If all cycles in $G^{\prime}$ pass $v$, then by Proposition $1 T(v):=G^{\prime}$ is a trace.

Assume that $G^{\prime}$ contains a cycle $C$, that does not pass $v$. The cycle $C$ consists of segments $\left(S_{1}, S_{2}, \ldots, S_{t}\right)$, where a segment $S_{i}$ is a sequence of edges such that $S_{i}$ belongs to one of the cycles $C_{j}$. We can assume that the segments are organized in such a way that different subsequent segments $S_{i}, S_{i+1}$ (modulo $t)$ belong to different cycles. Note, that it may happen, that two different, nonadjacent segments share the same cycle. Let $u_{i}$ and $w_{i}$ be the starting vertex and end-vertex, respectively, of $S_{i}$. Now, consider any of the points $u_{i}=w_{i-1}$. Such a point is the endpoint of two edge-disjoint paths, namely $W_{C_{k}}\left(v, u_{i}\right)$ and $W_{C_{k^{\prime}}}\left(w_{i-1}, v\right)$ for some $k \neq k^{\prime}$. There are exactly two edges $e_{i}(1)$ and $e_{i}(2)$ that are incident with $u_{i}$ such $e_{i}(1) \in W_{C_{k}}\left(v, u_{i}\right)$ and $e_{i}(2) \in W_{C_{k}^{\prime}}\left(v, u_{i}\right)$. Now, $r$ new edge-disjoint cycles $\left\{C_{1}^{\prime}, C_{2}^{\prime}, \ldots, C_{r}^{\prime}\right\}$ are generated in $G^{\prime}$ as follows:
i. If $V\left(C_{k}\right) \cap V(C)=\emptyset$, set $C_{k}^{\prime}=C_{k}$.
ii. If $V\left(C_{k}\right) \cap V(C) \neq \emptyset$, then a new circuit $C_{k}^{\prime}$ is constructed as follows: Start from $v$ along the path $W_{C_{k}}\left(v_{1}, u_{i}\right)$ (we can assume that $u_{i}$ is the first vertex on $W_{C_{k}}\left(v_{1}, u_{i}\right)$ in $C$ ). Then $u_{i}$ is reached on the edge $e_{i}(1) \in W_{C_{k}}\left(v, u_{i}\right)$. Instead of following segment $S_{i} \in C$ we follow along $e_{i}(2) \in W_{C_{k^{\prime}}}\left(v, u_{i}\right)$. If we reach $v$ on $W_{C_{k^{\prime}}}\left(v_{1}, u_{i}\right)$ without visiting another $u_{j} \in C$, the new cycle $C_{k}^{\prime}$ is defined by $C_{k}^{\prime}=W_{C_{k}}\left(v_{1}, u_{i}\right) \cup W_{C_{k^{\prime}}}\left(v_{1}, u_{i}\right)$.
If we reach another vertex, say $u_{j} \in V(C)$, when passing along $W_{C_{k^{\prime}}}\left(v, u_{i}\right)$ from $u_{i}$ we will reach $u_{j}$ on some edge $e_{j}(1)$ before arriving at $v$, we leave $u_{j}$ on edge $e_{j}(2) \in W_{C_{k^{\prime \prime}}}\left(v, u_{i}\right)$, and so on. A new circuit $C_{k}^{\prime}$ is constructed if $v$ is reached for the first time. As a circuit passing $v, C_{k}^{\prime}$ contains a cycle that passes $v$, here also denoted by $C_{k}^{\prime}$.

It is obvious that in this way a set of $r$ cycles $C(v)^{\prime}=\left\{C_{1}^{\prime}, C_{2}^{\prime}, \ldots, C_{r}^{\prime}\right\}$ is determined such that they only use edges in $E(C(v))$. They are mutually edge-disjoint, all pass $v$, but none of them will use any edge in $C$. Hence, $E\left(C(v)^{\prime}\right) \subset E(C(v))$. Now, we consider the graph $G^{\prime \prime}$ induced by $C(v)^{\prime}$. If it contains a cycle $C^{\prime}$, that does not pass $v$, we proceed in the same manner. After a finite number of steps a set $\bar{C}(v)=\left\{\bar{C}_{1}, \bar{C}_{2}, \ldots, \bar{C}_{r}\right\}$ of edge-disjoint cycles is all passing $v$, is constructed, such that in the induced graph $\bar{G}$ every cycle passes $v$. Hence $T(v):=\bar{G}$ is a trace. Obviously, $E(T(v)) \subset E(C(v))$.

The next lemma gives a relation between maximum and saturated traces.
Lemma 6. Let $G=(V, E)$ and $T(v) \neq \emptyset$ be a maximum trace at $v$. Then $T(v)$ is saturated.

Proof. Assume, that this is not the case. Then there is an Eulerian graph $H \subset G$ such that $T(v) \subsetneq H$ and $v$ is proper with respect to $H$.

Let $T(v)$ be induced by $\left\{C_{1}, \ldots, C_{k^{*}}\right\}$ and $H$ be induced by $\left\{C_{1}^{\prime}, \ldots, C_{\bar{k}^{*}}\right\}$, respectively. Note, that $k^{*}=\bar{k}^{*} \leq\left\lfloor\frac{d(v)}{2}\right\rfloor$, otherwise $T(v)$ would not be maximal. Let $\bar{E}=\{e \mid e$ is incident with $v\} \cap E(T(v))$. Without loss of generality, we can assume that the representations of $T$ and $H$, respectively, have no common cycle $C$. Otherwise, if there is such a cycle $C$, then we consider $T(v) \backslash C$ and $H \backslash C$, respectively.

We will show that $H$ must contain a cycle $\tilde{C}$ that does not pass $v$, which is impossible. For this, take a cycle $C_{i_{1}}$ and the two edges $e_{i_{1}}, e_{i_{2}} \in E\left(C_{i_{1}}\right) \cap \bar{E}$. Since $\bar{E} \subset E(H)$, there is a cycle $C_{j_{1}}^{\prime}$ with $e_{i_{2}} \in E\left(C_{j_{1}}^{\prime}\right)$. The cycle $C_{j_{1}}^{\prime}$ also contains an edge $e_{i_{3}} \in \bar{E}$. The edge $e_{i_{3}}$ is then again contained in a cycle $C_{i_{2}}$, which also contains an edge $e_{i_{4}} \in \bar{E}$ and so on. In such a way, we get a sequence $C_{i_{1}}, C_{j_{1}}^{\prime}, C_{i_{2}}, C_{j_{2}}^{\prime}, \ldots$ of cycles that alternately belong to the representations of
$T(v)$ and $H$, respectively. Within this sequence, there must be one cycle $C_{j_{k}}^{\prime}$ that contains the edge $e_{i_{1}}$.

Now, let $P(v)$ be a path along $C_{i_{1}}$ starting at $v$ and using the edge $e_{i_{2}}$. Let $w_{j_{1}}$ be the last vertex in $P(v)$ that belongs to $C_{i_{1}} \cap C_{j_{1}}^{\prime}$. Such a vertex must exist and, obviously, $w_{j_{1}} \neq v$. We now construct the cycle $\tilde{C}$ : starting from $w_{j_{1}}$ we pass along the cycle $C_{j_{1}}^{\prime}$ until to the first vertex $w_{i_{2}} \neq v$ in $C_{i_{2}}$. From there we pass along $C_{i_{2}}$ until to the first vertex $w_{j_{2}} \neq v$ in $C_{j_{2}}^{\prime}$ and so on. We proceed until we reach the vertex $w_{j_{k}} \neq v$ in $C_{j_{k}}^{\prime}$. From there we pass along $C_{j_{k}}^{\prime}$ until we reach $w_{i_{1}} \neq v$ in $C_{i_{1}}$. From there it is possible to pass along $C_{i_{1}}$ to the vertex $w_{j_{1}}$, not using $v$. In such a way we have constructed a cycle $\tilde{C} \subset H$ that does not pass through $v$, contradicting that $v$ is proper with respect to $H$.

Note that the converse is not true in general, even if $G$ is Eulerian. In the following figure a saturated trace $T(w)$ is drawn which is not maximum.


Figure 2. Saturated local trace $T(w)$ (red) in $G$ that is not maximum.
Using a similar construction scheme as in Lemma 5 we now can give a characterization for a a maximum trace to be unique. For $v \in V$, let $\mathcal{C}(v)$ the family of sets of edge-disjoint cycles that induce a maximum trace $T(v)$ at $v$.

Lemma 7. Let $G=(V, E)$ be Eulerian, $v \in V$ and $T(v) \neq \emptyset$ be a maximum trace at $v$. Then the following is equivalent:
i. $T(v)$ is unique.
ii. For all $C(v) \in \mathcal{C}(v)$ it holds: a cycle in $G \backslash\{v\}$ and a cycle in $C(v)$ has no common edge.

Proof. "i. $\Rightarrow$ ii.": Let $T(v)$ be uniquely induced by the edge-disjoint cycles $C(v)=\left\{C_{1}, C_{2}, \ldots, C_{r}\right\} \in \mathcal{C}(v)$. Assume there is a cycle $C \subset G \backslash\{v\}$, such that $E(C) \cap E(C(v)) \neq \emptyset$. Then $C$ contains segments ( $S_{0}, S_{1}, S_{2}, \ldots, S_{t}$ ), where a segment $S_{i}$ is a sequence of edges such that $S_{i}$ belongs to one of the cycles $C_{j}$ or $S_{i}$ does not belong to $T(v)$. At least one such segment, say $S_{0}$, cannot belong to $T(v)$ since otherwise $T(v)$ would not be a trace. $S_{0}$ is now used to construct
a set $C^{\prime}(v) \in \mathcal{C}(v)$ that induces a maximum trace, different from $T(v)$. This will give the contradiction.

Let $u$ and $u^{\prime}$ be the endpoints of $S_{0}$ in $C$. Then there are $C_{i}$ and $C_{j}$ such that $C_{i}=W_{1}^{(i)}(v, u) \cup W_{2}^{(i)}(u, v)$ and $C_{j}=W_{1}^{(j)}\left(v, u^{\prime}\right) \cup W_{2}^{(j)}\left(u^{\prime}, v\right)$. If $C_{i}=C_{j}$ then $W_{1}^{(i)}\left(u, u^{\prime}\right) \subset W_{1}^{(i)}\left(v, u^{\prime}\right)$. Then set

$$
\tilde{C}_{i}=C_{i} \backslash W_{1}^{(i)}\left(u, u^{\prime}\right) \cup S_{0}
$$

The cycles $\left\{C_{1}, C_{2}, \ldots, C_{r}\right\} \backslash C_{i} \cup \tilde{C}_{i}$ then induce a maximum trace at $v$ not containing $W_{1}^{(i)}\left(u, u^{\prime}\right)$.

For the case that $C_{i} \neq C_{j}$, we distinguish two situations.
Case a. There is a vertex $w$ different from $v$ such that $w \in V\left(C_{i}\right) \cap V\left(C_{j}\right)$. Note that at most one such vertex can exist. If $w \in\left\{u, u^{\prime}\right\}$, say $w=u^{\prime}$, then set

$$
\tilde{C}_{i}=C_{i} \backslash W_{1}^{(i)}\left(u, u^{\prime}\right) \cup S_{0}
$$

Again, the cycles $\left\{C_{1}, C_{2}, \ldots, C_{r}\right\} \backslash C_{i} \cup \tilde{C}_{i}$ then induce a maximum trace at $v$ not containing $W_{1}^{(i)}\left(u, u^{\prime}\right)$.

If $w \notin\left\{u, u^{\prime}\right\}$, then assume $w \in W_{1}^{(i)}(v, u)$ and $w \in W_{1}^{(j)}\left(v, u^{\prime}\right)$. Now, set

$$
\tilde{C}_{i}:=W_{1}^{(i)}(v, w) \cup W_{1}^{(j)}(v, w) \tilde{C}_{j}:=W_{2}^{(i)}(v, u) \cup S_{0} \cup W_{2}^{(j)}\left(v, u^{\prime}\right)
$$

Then the cycles $\left\{C_{1}, C_{2}, \ldots, C_{r}\right\} \backslash\left\{C_{i}, C_{j}\right\} \cup\left\{\tilde{C}_{i}, \tilde{C}_{j}\right\}$ induce a maximum trace at $v$ not containing $W_{1}^{(i)}(w, u)$ and $W_{1}^{(j)}\left(w, u^{\prime}\right)$.

Case b. The only common vertex of $C_{i}$ and $C_{j}$ is $v$. In this case we use a similar construction as in Lemma 5. We start from $v$ along the path $W_{1}^{(i)}(v, u)$. Then $u$ is reached on the edge $e_{i}(1) \in W_{1}^{(i)}(v, u)$. Instead of following $W_{i}^{(2)}(v, u)$ we follow along $S_{0} \in C$ until reaching $u^{\prime}$ and follow the path $W_{j}^{(2)}\left(v, u^{\prime}\right)$.

If we reach $v$ on $W_{j}^{(2)}\left(v, u^{\prime}\right)$ without visiting another $u^{\prime \prime} \in C$, then the new cycle $\tilde{C}_{i}$ is defined by $\tilde{C}_{j}:=W_{2}^{(i)}(v, u) \cup S_{0} \cup W_{2}^{(j)}\left(v, u^{\prime}\right)$.

If we reach another vertex, say $u^{\prime \prime} \in V(C)$, when passing along $W_{j}^{(2)}\left(v, u^{\prime}\right)$, then from $u^{\prime}$ we will reach $u^{\prime \prime}$ using a segment $S_{k}$ before arriving at $v$; we then leave $u^{\prime \prime}$ on the segment $S_{k+1}$ using $W_{s}^{(1)}\left(v, u^{\prime \prime}\right)$, and so on.

In such a way $r$ circuits $\left\{\tilde{C}_{1}, \tilde{C}_{2}, \ldots, \tilde{C} r\right\}$ are constructed (all passing $v$ ), that do not contain $W_{i}^{(2)}(v, u)$ and $W_{j}^{(2)}\left(v, u^{\prime}\right)$.
"ii. $\Rightarrow i$.": First note that the components $B_{1}, B_{2}, \ldots, B_{s}$ of $G \backslash\{v\}$ are uniquely determined and that a subset of cycles in $C(v) \in \mathcal{C}(v)$ induce a maximum trace $T_{i}(v)$ for the (Eulerian) graph $G_{i}$ induced by $B_{i} \cup\{v\}$. And vice versa.

Let $T(v)$ and $T^{\prime}(v)$ be two maximum traces at $v$. Let $C(v), C^{\prime}(v) \in \mathcal{C}$ be the sets of cycles that induce $T(v)$ and $T^{\prime}(v)$, respectively.

If $G \backslash\{v\}$ contains no cycle, then non of the $B_{i}$ contain a cycle, i.e, $B_{i}$ is a tree for all $i$. The subgraphs $T_{i}(v), T_{i}^{\prime}(v) \subset G_{i}$ are two maximum traces for $G_{i}$ that, by Lemma 6 , are saturated. But $v$ is a proper vertex with respect to the graphs $G_{i}$. Hence $G_{i}=T_{i}(v)=T_{i}^{\prime}(v)$, i.e, $T(v)=T^{\prime}(v)$.

If $G \backslash\{v\}$ contains a cycle $C$, then by assumption, $E(C) \cap E(C(v))=E(C) \cap$ $E\left(C^{\prime}(v)\right)=\emptyset$. We then consider the Eulerian graph $G^{\prime}=G \backslash E(C)$. For $G^{\prime}, T(v)$ and $T^{\prime}(v)$ are maximum traces at $v$ and we can perform the same considerations as before. In the case that $G^{\prime} \backslash\{v\}$ contains no cycle, we again get $T(v)=T^{\prime}(v)$, otherwise we remove the cycle from $G^{\prime}$. Proceeding in this way we will terminate with a Eulerian graph $\bar{G}$ in which $T(v)$ and $T^{\prime}(v)$ are maximum traces at $v$ and $\bar{G} \backslash\{v\}$ contains no cycle, concluding then $T(v)=T^{\prime}(v)$.

By Lemma 7 we have proved
Proposition 8. Let $G=(V, E)$ be Eulerian. If there is $v \in V$ such that the maximum local trace $T(v) \neq \emptyset$ is unique, then

$$
\nu(G)=\frac{d_{G}(v)}{2}+\nu(G \backslash\{v\})
$$

and

$$
\mathcal{Z}^{*}(G)=C(v) \cup \mathcal{Z}^{*}(G \backslash\{v\}) .
$$

In the following section, we will give a more general sufficient condition that makes the cycle packings $C(v)$ corresponding to maximum traces $T(v), v \in V$, to build up a maximum cycle packing in $G$.

## 4. A Mini-max Theorem

We start with the observation that there are Eulerian graphs $G$ with corresponding cycle packing $\mathcal{Z}_{1}=\left\{C_{1}, C_{2}, \ldots, C_{s}\right\}$ of cardinality $s<\nu(G)$ such that $G$ is induced by $\mathcal{Z}_{1}$ and for every $v \in V$ the subgraph $T(v)$ of $G$, induced by the cycles in $\mathcal{Z}_{1}(v)$, is a maximum trace.

It follows there are cases that maximum traces of $G$ can be induced by cycle packings of $G$ that are not maximum. In Figure 3 such an example is illustrated.

The question arises what are conditions that guarantee that a set $\{T(v) \mid v \in$ $V\}$ of maximum local traces of $G$ is induced by a maximum cycle packing $\mathcal{Z}^{*}$ of $G$.

We now investigate such a situation more generally. For $1 \leq s \leq \nu(G)$, we consider the family of cycle packings $\mathcal{C}_{s}^{*} \subset \mathcal{C}_{s}$ of $G$. A packing $\mathcal{Z}$ belongs to $\mathcal{C}_{s}^{*}$ if it is a cycle packing of cardinality $s$ and for all $v \in V$ the subgraph $T(v)$ of $G$


Figure 3. $\left|\mathcal{Z}_{1}\right|=5, F\left(\mathcal{Z}_{1}\right)=66$ whereas $\left|\mathcal{Z}_{2}\right|=6=\nu(G), F\left(\mathcal{Z}_{2}\right)=54$.
induced by the cycles in $\mathcal{Z}(v)$ is a maximum trace at $v$. A first (simple) condition can be derived as an immediate consequence of Lemma 7 .

Corollary 9. Let $\mathcal{Z} \in \mathcal{C}_{s}^{*}$ and let $\left(v_{0}, v_{1}, \ldots, v_{k}\right)$ be a sequence of vertices in $G$. With $G_{0}:=G$ denote by $G_{i+1}:=G_{i} \backslash E\left(\mathcal{Z}\left(v_{i}\right)\right), i=0,1, \ldots, k$, the sequence of subgraphs of $G$ recursively induced by maximum traces $T_{G_{i}}\left(v_{i}\right)$ at $v_{i}$ in $G_{i}$. If maximum traces $T_{G_{i}}\left(v_{i}\right) \subset G_{i}$ are unique (with respect to $G_{i}$ ) and $G_{k+1}=\emptyset$, then $s=\nu(G)$, i.e, $\mathcal{Z}$ is maximum.

For a more general condition we first prove a theorem, which is true not only for Eulerian graphs.

For this let $G$ be a graph with $\nu(G) \geq 1$. For $0 \leq s \leq \nu(G)$, let $\mathcal{C}_{s}(G)$ be the set of cycle packings of cardinality $s$. Then $\mathcal{C}(G)=\bigcup_{s}^{\nu(G)} \mathcal{C}_{s}(G)$ describes the set of all cycle-packings of $G$. Note, that $\mathcal{C}_{0}(G)=\emptyset$ if and only if $G$ is a cycle. If this is not the case, then $\mathcal{C}_{s}(G) \neq \emptyset$ implies $\mathcal{C}_{s-1}(G) \neq \emptyset, s \geq 1$. For $s \geq 1$ a packing $\mathcal{Z}=\left\{C_{1}, C_{2}, \ldots, C_{s}, \tilde{G}_{s}\right\} \in \mathcal{C}_{s}(G)$ consists of $s$ cycles $C_{i}$ and a "reminder" $\tilde{G}_{s}$. Let $l_{i}=\left|E\left(C_{i}\right)\right|$. For $\mathcal{Z} \in \mathcal{C}_{s}, s \geq 1$, define

$$
\bar{L}(\mathcal{Z})=\sum_{i=1}^{s} l_{i}^{2}+\left|E\left(\tilde{G}_{s}\right)\right|^{2}
$$

For $\mathcal{Z} \in \mathcal{C}_{0}$, set $\bar{L}(\mathcal{Z}):=|E(G)|^{2}$. We get
Theorem 10. Let $\nu(G) \geq 1$. Every cycle packing $\mathcal{Z}^{*}$ that minimizes $\bar{L}$ on $\mathcal{C}(G)$ is maximum, i.e, $\mathcal{Z}^{*} \in \mathcal{C}_{\nu(G)}$.
Proof. Obviously, the theorem is true if $G$ is a cycle. Therefore, assume $G$ is not a cycle. For $s \in\{0,1,2, \ldots, \nu(G)\}$ let $\bar{m}_{s}(G):=\min \left\{\bar{L}\left(\mathcal{Z} \mid \mathcal{Z} \in \mathcal{C}_{s}(G)\right\}\right.$. We
will show that

$$
\bar{m}_{s-1}(G)>\bar{m}_{s}(G), \quad s=1,2, \ldots, \nu(G) .
$$

To prove the inequality we will use the induction on $r \leq \nu(G)$. Obviously, $\bar{m}_{0}(G)=|E(G)|^{2}$.

Let $r=1, \mathcal{C}_{1}(G) \neq \emptyset$. Let $\mathcal{Z}_{1} \in \mathcal{C}_{1}(G)$, i.e, $\mathcal{Z}_{1}=\left\{C_{1}, \tilde{G}_{1}\right\}$ and $l_{1}=\left|E\left(C_{1}\right)\right|$.
Since $G$ is not a cycle, $l_{1}<|E(G)|$ and we immediately get $\bar{L}\left(\mathcal{Z}_{1}\right):=l_{1}^{2}+$ $\left(|E(G)|-l_{1}\right)^{2}=2 l_{1}^{2}+|E(G)|^{2}-2 l_{1}^{2}|E(G)|<|E(G)|^{2}$, i.e, $\bar{m}_{0}(G)>\bar{m}_{1}(G)$. Now, let $r \geq 1$ such that $\mathcal{C}_{r}(G) \neq \emptyset$ and let us assume that for all graphs $G$ such that $\nu(G) \leq r$ and all $r^{\prime} \leq r$ the relations $\bar{m}_{r^{\prime}-1}(G)>\bar{m}_{r^{\prime}}(G)$ hold.

Let $G$ be a graph such that $\mathcal{C}_{r+1}(G) \neq \emptyset$. Hence $\mathcal{C}_{r}(G) \neq \emptyset$. Since $\mathcal{C}_{r}(G) \neq \emptyset$ there exists $\mathcal{Z}_{r}(G) \in \mathcal{C}_{r}(G)$ such that $\bar{L}\left(\mathcal{Z}_{r}(G)\right)=\bar{m}_{r}(G)$. Take the cycle $C_{1} \in$ $\mathcal{Z}_{r}(G)$ of length $l_{1}$ and consider the graph $G \backslash C_{1}$.

Obviously, $\mathcal{Z}:=\left(\mathcal{Z}_{r}(G) \backslash\left\{C_{1}\right\}\right) \in \mathcal{C}_{r-1}\left(G \backslash C_{1}\right)$. Moreover, $\bar{L}(\mathcal{Z})=\bar{m}_{r}(G)-$ $l_{1}^{2}$. But also $\left.\bar{L}(\mathcal{Z})=\min \left\{\bar{L}\left(\mathcal{Z}_{r-1}\right)\left(G \backslash C_{1}\right)\right) \mid \mathcal{Z}_{r-1} \in \mathcal{C}_{r-1}\left(G \backslash C_{1}\right)\right\}$ must hold, otherwise $\mathcal{Z}_{r}(G)$ would not be a minimizer in $\mathcal{C}_{r}(G)$, i.e, $\bar{m}_{r-1}\left(\left(G \backslash C_{1}\right)\right)=\bar{m}_{r}(G)-l_{1}^{2}$. Using the assumption, we then get $\bar{L}(\mathcal{Z})=\bar{m}_{r-1}\left(G \backslash C_{1}\right)>\bar{m}_{r}\left(G \backslash C_{1}\right)$ and, by this, $\bar{m}_{r}(G)=\bar{m}_{r-1}\left(G \backslash C_{1}\right)+l_{1}^{2}>\bar{m}_{r}\left(G \backslash C_{1}\right)+l_{1}^{2} \geq \bar{m}_{r+1}(G)$.

Remark 11. By a similar proof it can be shown that also for

$$
\bar{M}_{s}(G):=\max \left\{\bar{L}(\mathcal{Z}) \mid \mathcal{Z} \in \mathcal{C}_{s}(G)\right\}, s \geq 1 ; \quad \bar{M}_{0}(G):=|E(G)|^{2}
$$

the strict inequalities

$$
\bar{M}_{s-1}(G)>\bar{M}_{s}(G) \quad s=1,2, \ldots, \nu(G)
$$

hold. The proof is just the same but instead of taking out a cycle $C_{1} \in \overline{\mathcal{Z}}_{r}(G)$ one takes it out from $\overline{\mathcal{Z}}_{r+1}(G)$.

Theorem 10 now will be used to get a condition that a cycle packing $\mathcal{Z}$ inducing maximum traces in Eulerian $G$ is maximum.

Let $\mathcal{C}^{*}=\bigcup_{s=1}^{\nu(G)} \mathcal{C}_{s}^{*}$. By $F(\mathcal{Z})=\sum_{v \in V} \mid E(T(v) \mid$ denote the total size of the local traces.

Theorem 12. Let $G$ be Eulerian. Every cycle packing $\mathcal{Z}^{*}$ that minimizes $F$ on $\mathcal{C}^{*}$ is a maximum cycle packing of $G$, i.e, $\mathcal{Z}^{*} \in \mathcal{C}_{\nu(G)}^{*}$.

Proof. We first observe that for all $v \in V$ and for all $C_{i} \in \mathcal{Z}=\left\{C_{1}, C_{2}, \ldots, C_{s}\right\} \subset$ $\mathcal{C}_{s}^{*}$ the following is true: $v \in V\left(C_{i}\right)$ if and only if $C_{i} \in \mathcal{Z}(v)$.

Therefore, we get $F(\mathcal{Z})=\sum_{v \in V} \mid E\left(T(v)\left|=\sum_{v \in V} \sum_{C_{i} \in \mathcal{Z}(v)}\right| E\left(C_{i}\right) \mid=\right.$ $\sum_{C_{i} \in \mathcal{Z}} \sum_{v \in V\left(C_{i}\right)}\left|E\left(C_{i}\right)\right|=\sum_{C_{i} \in \mathcal{Z}}\left|V\left(C_{i}\right)\right|\left|E\left(C_{i}\right)\right|=\sum_{i=1}^{s}\left|E\left(C_{i}\right)\right|^{2}$.
Let $\mathcal{Z}^{*}$ be a minimizer of $F$ in $\mathcal{C}_{\nu(G)}^{*}$, i.e, $F\left(\mathcal{Z}^{*}\right)=\bar{m}_{\nu(G)}$. Assume that there is $\overline{\mathcal{Z}}^{*} \in \mathcal{C}_{s}^{*}$, such that $F\left(\overline{\mathcal{Z}}^{*}\right)=F\left(\mathcal{Z}^{*}\right)$, but $s<\nu(G)$. We then get $F\left(\overline{\mathcal{Z}}^{*}\right) \geq$ $\min \left\{\bar{L}\left(\mathcal{Z}^{\prime} \mid \mathcal{Z}^{\prime} \in \mathcal{C}_{s}\right\}=\bar{m}_{s}>\bar{m}_{\nu(G)}=F\left(\mathcal{Z}^{*}\right)\right.$, a contradiction.

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