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MAXIMUM CYCLE PACKING IN EULERIAN GRAPHS USING LOCAL TRACES

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Abstract

For a graph $G = (V, E)$ and a vertex $v \in V$, let $T(v)$ be a *local trace* at v, i.e. $T(v)$ is an Eulerian subgraph of G such that every walk $W(v)$, with start vertex v can be extended to an Eulerian tour in $T(v)$.

We prove that every maximum edge-disjoint cycle packing $\mathcal{Z}^*$ of G induces a maximum trace $T(v)$ at v for every $v \in V$. Moreover, if G is Eulerian then sufficient conditions are given that guarantee that the sets of cycles inducing maximum local traces of G also induce a maximum cycle packing of G.

Keywords: edge-disjoint cycle packing, local traces, extremal problems in graph theory.

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1. Introduction

We consider a finite and undirected graph G with vertex set $V(G)$ and edge set $E(G)$ that contains no loops. For a finite sequence $v_{i_1}, e_1, v_{i_2}, e_2, \ldots, e_{r-1}, v_{i_r}$ of vertices v_{i_j} and pairwise distinct edges $e_j = (v_{i_j}, v_{i_{j+1}})$ of G, the subgraph W of G with vertices $V(W) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ and edges $E(W) = \{e_1, e_2, \ldots, e_{r-1}\}$ is called a *walk* with *start vertex* v_{i_1} and *end vertex* v_{i_r}. If W is closed (i.e, $v_{i_1} = v_{i_r}$) we call it a *circuit* in G. A *path* is a walk in which all vertices v have degree $d_W(v) \leq 2$. A closed path will be called a *cycle*. A connected graph in which all vertices v have even degree is called *Eulerian*. For an Eulerian graph G, a circuit W with $E(W) = E(G)$ is called an *Eulerian tour*.

For $1 \leq i \leq k$, let $G_i \subset G$ be subgraphs of G. We say that G is *induced by* $\{G_1, G_2, \ldots, G_k\}$ if $V(G) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_k)$ and $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)$. Two subgraphs $G' = (V', E')$, $G'' = (V'', E'')$ of G are called *edge-disjoint* if $E' \cap E'' = \emptyset$. For $E' \subseteq E$ we define $G \setminus E' = (V, E \setminus E')$. For $V' \subset V$ we define $G \setminus V' = G|_{V \setminus V'}$, where $V(G|_{V \setminus V'}) = V \setminus V'$ and $E(G|_{V \setminus V'}) = \{e \in E(G)|$ both endvertices of e belong to $V\}$.

A *packing* $\mathcal{Z}(G) = \{G_1, \ldots, G_q\}$ of G is a collection of subgraphs G_i of G $(i = 1, \ldots, q)$ such that all G_i are mutually edge-disjoint and G is induced by $\{G_1, \ldots, G_q\}$. If exactly s of the G_i are cycles, $\mathcal{Z}(G)$ is called a cycle packing of cardinality s. The family of cycle-packings of G is denoted by $\mathcal{C}(G)$. If the cardinality of a cycle packing $\mathcal{Z}(G)$ is maximum, it is called a *maximum cycle packing*. Its cardinality is denoted by $\nu(G)$. If no confusion is possible we will write $\mathcal{Z}$ instead of $\mathcal{Z}(G)$ and $\mathcal{C}$ instead of $\mathcal{C}_s(G)$, respectively.

Packing edge-disjoint cycles in graphs is a classical graph-theoretical problem. There is a large amount of literature concerning conditions that are sufficient for the existence of some number of disjoint cycles which may satisfy some further restrictions. A selection of related references is given in [8]. The algorithmic problems concerning edge-disjoint cycle packings are typically hard (e.g. see [4, 5, 10]). There are papers in which practical applications of such packings are mentioned [1, 3, 6, 9].

Starting point of the paper is the attempt to obtain a maximum cycle packing of a graph G by the determination of such packings for specific subgraphs of G. In [8] such an approach was studied when the subgraphs were induced by vertex cuts.

In the present paper we study the behaviour of such packings if G is Eulerian and the subgraphs are (local) traces.

In Section 2, local traces are introduced and relations between local traces and maximum cycle packings are given. It turns out in Section 3 that under special conditions a maximum cycle packing can be constructed from maximum cycle packings of maximum local traces.

In Section 4, a mini-max theorem gives a condition whether given maximum local traces are induced by a maximum cycle packing $\mathcal{Z}^*$ of G. For this the square-length of the cycles is essential.

2. Relation Between Maximum Cycles Packings and Local Traces

In this section we will show, how to built up maximum cycle packings iteratively from maximum cycle packings of special subgraphs, if G is Eulerian. This subgraphs will be (local) traces. For special cases Theorem 10 guarantees that the so constructed cycle packing is maximum.

Let $G = (V, E)$ be an Eulerian graph. A vertex $v \in V$ is called *proper*, if every walk W, starting at v can be extended to an Euler-tour in H. An Eulerian graph that contains a proper vertex is called a *trace*. Traces were first considered by Ore in [11] and [2]. Such type of graphs can be characterized in the following way.

Proposition 1. *Let $G = (V, E)$ be an Eulerian graph. Let $v \in V$. The following statements are equivalent:*

i. *v is proper.*

ii. *If C is an arbitrary cycle in G, then $v \in V(C)$.*

iii. *The number k of components of $G \setminus \{v\}$ is determined by $k = d_G(v) - \gamma(G)$, where $\gamma(G)$ denotes the cyclomatic number of G.*

Proof. See [11]. ■

If v is a proper vertex of degree $d_G(v)$, then G is induced by $r = \frac{d(v)}{2}$ edge-disjoint cycles $\{C_1, \ldots, C_r\}$,where all C_i are passing v. Any two of these cycles $C_i, C_j, i \neq j$ have at most one other vertex in common, and there exists at most one further proper vertex $w \neq v$ in V. This is the case if and only if $d(v) = d(w)$ (see [2]).

The following simple characterization relates traces to cycle packings. In [12] it is proved

Proposition 2. *If $G = (V, E)$ is Eulerian and $d_G(v) = \Delta = \max\{d_G(u) | u \in V\}$, then $\nu(G) = \frac{1}{2}\Delta = \frac{1}{2}d_G(v)$ if and only if G is a trace with proper vertex v.*

Proof. Note that $\nu(G) \geq \frac{1}{2}\Delta$ holds since G is Eulerian.

"$\Rightarrow$": Let $\nu(G) = \frac{1}{2}\Delta = \frac{1}{2}d_G(v)$. Assume that there is a cycle $C \subseteq G$ with $v \notin V(C)$. Obviously, each of the components $G'_1, G'_2, \ldots, G'_k$ of $G \setminus E(C)$ is Eulerian. Let G'_i be that component that contains v. Then $d|_{G'_i}(v) = d(v) = \Delta$. But then, $\nu(G) \geq 1 + \sum_{j=1}^{k} \nu(G'_j) > \nu(G'_i) \geq \frac{1}{2}d_{G'_i}(v) = \frac{1}{2}d_G(v) = \frac{1}{2}\Delta$, contradicting $\nu(G) = \frac{1}{2}\Delta$. Therefore, each cycle $C \subset G$ passes v, hence by Proposition 1, v is a proper vertex.

"$\Leftarrow$": Let v be a proper vertex of G. If $\mathcal{Z}^* = \{C_1, C_2, \ldots, C_{\nu(G)}\}$ is a maximum cycle packing of G, then all cycles in $\mathcal{Z}^*$ have to pass v, i.e, $d_G(v) = 2\nu(G) \geq \Delta$. Since $d_G(v) \leq \Delta$, $\nu(G) = \frac{1}{2}\Delta = \frac{1}{2}d_G(v)$ follows. ■

Remark 3. i. For a graph G, let $\gamma(G)$ denote the cyclomatic number of G. If G is a trace with proper vertex v, then the graph $G \setminus \{v\}$ consists of $k = d_G(v) - \gamma(G) \geq 1$ components $\{B_1, B'_k, \ldots, B'_k\}$ that are all trees. Let B_i be such a component and $W_i := \{w \in B'_i | d_{B_i}(w) \text{ is odd }\}, r_i := \#W_i$. Then the graph $G_i = (V(G_i), E(G_i))$ with $V(G_i) = V(B_i) \cup \{v\}$ and $E(G_i) =$

$E(B_i) \cup \{(w,v)|w \in W_i\}$ is also a trace with proper vertex v. Obviously, $\nu(G_i) = \frac{1}{2}r_i$ and $\nu(G) = \sum_{i=1}^{k} \nu(G_i)$.

ii. If G is 2-connected and $k' := \gamma(G) - \nu(G)$, then there is a finite set $\mathcal{P}(k')$ of graphs (depending only on k' not on G) such that G arises by applying a simple extension rule to a graph in $\mathcal{P}(k')$ (see [7]). If G is a trace, then this situation is even simpler: since for each of the subgraphs G_i it holds $\gamma(G_i) - \nu(G_i) = \frac{1}{2}r_i - 1 = \gamma(K_2^{r_i}) - \nu(K_2^{r_i})$ and all edges $E(G_i)$ belong to a maximum cycle packing of G_i, G_i arises by an extension of $K_2^{r_i}$. Here $K_2^{r_i}$ is the multi-graph consisting of two vertices and r_i parallel edges.

Now, we will transfer the concept of a trace to an arbitrary graph $G = (V, E)$.

For $v \in V$, an Eulerian subgraph $T(v) = (V(T(v)), E(T(v))) \neq \emptyset$ of G is called *a local trace* (*at* v), if $v \in V(T(v))$ and v is proper with respect to $T(v)$. The number $|E(T(v))|$ is called the *size of the trace* (*at* v).

A local trace $T(v)$ is called *saturated* (at v), if there is no Eulerian subgraph $H \subset G$ such that $T(v) \subsetneq H$ and v is proper with respect to H. It is called *maximum*, if $T(v)$ is induced by $k(v)$ edge-disjoint cycles $\{C_1, C_2, \ldots, C_{k(v)}\} \subset G$ and $k(v)$ is maximum.

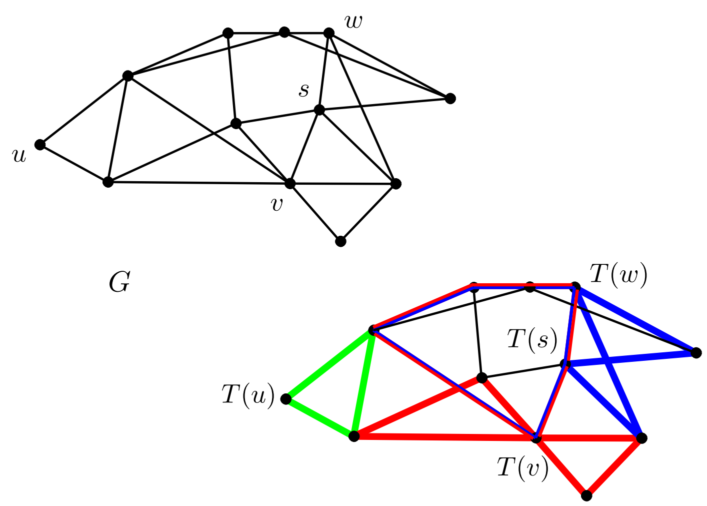

Figure 1. G together with maximum traces $T(u)$ (green colored edges), $T(v)$ (red), $T(w) = T(s)$ (blue).

Being a trace $T(v)$ at v is a local property of the graph G. Obviously, each single cycle $C \in G$ that passes v is a local trace at v. In general, local traces are not uniquely determined, even maximum local traces are not.

For Eulerian graphs we have

Lemma 4. *Let $G = (V, E)$ be Eulerian and $\mathcal{Z}^*$ a maximum cycle packing of G. For $v \in V$, let $\mathcal{Z}^*(v) := \{C_i \in \mathcal{Z}^* | v \in V(C_i)\}$. Then $\mathcal{Z}^*(v)$ induces a maximum trace $T(v)$ at v.*

Proof. Let $T(v)$ be the subgraph of G induced by the $\frac{d_G(v)}{2}$ cycles of $\mathcal{Z}^*(v)$. Obviously, $T(v)$ is Eulerian, $v \in V(T(v))$ and $d_T(v) \geq d_T(u)$ for all $u \in T(v)$. Because $\mathcal{Z}^*$ is maximum, $\mathcal{Z}^*(v)$ is also a maximum cycle packing of $T(v)$, i.e, $\nu(T(v)) = \frac{d_G(v)}{2} = \frac{d_T(v)}{2}$. Then, by Proposition 2, v is a proper vertex of $T(v)$, i.e, $T(v)$ is a maximum trace. ■

Note, that the fact that G is Eulerian is crucial, i.e, in a general situation a maximum cycle packing must not induce a maximum trace at v, even it must not induce a saturated trace

3. Getting Maximum Packings of G from Cycle Packings of Maximum Traces

An immediate question that arises is under which conditions the inverse of Lemma 4 is true. In this section such a condition is given, that allows a construction of a maximum cycle packing $\mathcal{Z}^*$ of G. The construction will use local traces of special subgraphs of G.

First, we give construction scheme to obtain a local trace at v from an arbitrary set $C(v)$ of edge-disjoint cycles that all pass v.

Lemma 5. *Let $G = (V, G)$, $v \in V$. For $r \geq 1$ let $C(v) = \{C_1, C_2, \ldots, C_r\}$, be a set of edge-disjoint cycles in G that all pass v. Then there is a trace $T(v)$, induced by r cycles $\{\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_r\}$ such that $E(T(v)) \subset E(C(v))$.*

Proof. Let G' be the graph induced by $C(v)$. If all cycles in G' pass v, then by Proposition 1 $T(v) := G'$ is a trace.

Assume that G' contains a cycle C, that does not pass v. The cycle C consists of segments $(S_1, S_2, \ldots, S_t)$, where a segment S_i is a sequence of edges such that S_i belongs to one of the cycles C_j. We can assume that the segments are organized in such a way that different subsequent segments S_i, S_{i+1} (modulo t) belong to different cycles. Note, that it may happen, that two different, non-adjacent segments share the same cycle. Let u_i and w_i be the starting vertex and end-vertex, respectively, of S_i. Now, consider any of the points $u_i = w_{i-1}$. Such a point is the endpoint of two edge-disjoint paths, namely $W_{C_k}(v, u_i)$ and $W_{C_{k'}}(w_{i-1}, v)$ for some $k \neq k'$. There are exactly two edges $e_i(1)$ and $e_i(2)$ that are incident with u_i such $e_i(1) \in W_{C_k}(v, u_i)$ and $e_i(2) \in W_{C'_k}(v, u_i)$. Now, r new edge-disjoint cycles $\{C'_1, C'_2, \ldots, C'_r\}$ are generated in G' as follows:

i. If $V(C_k) \cap V(C) = \emptyset$, set $C'_k = C_k$.

ii. If $V(C_k) \cap V(C) \neq \emptyset$, then a new circuit C'_k is constructed as follows: Start from v along the path $W_{C_k}(v_1, u_i)$ (we can assume that u_i is the first vertex on $W_{C_k}(v_1, u_i)$ in C). Then u_i is reached on the edge $e_i(1) \in W_{C_k}(v, u_i)$. Instead of following segment $S_i \in C$ we follow along $e_i(2) \in W_{C_{k'}}(v, u_i)$. If we reach v on $W_{C_{k'}}(v_1, u_i)$ without visiting another $u_j \in C$, the new cycle C'_k is defined by $C'_k = W_{C_k}(v_1, u_i) \cup W_{C_{k'}}(v_1, u_i)$.

   If we reach another vertex, say $u_j \in V(C)$, when passing along $W_{C_{k'}}(v, u_i)$ from u_i we will reach u_j on some edge $e_j(1)$ before arriving at v, we leave u_j on edge $e_j(2) \in W_{C_{k''}}(v, u_i)$, and so on. A new circuit C'_k is constructed if v is reached for the first time. As a circuit passing v, C'_k contains a cycle that passes v, here also denoted by C'_k.

It is obvious that in this way a set of r cycles $C(v)' = \{C'_1, C'_2, \ldots, C'_r\}$ is determined such that they only use edges in $E(C(v))$. They are mutually edge-disjoint, all pass v, but none of them will use any edge in C. Hence, $E(C(v)') \subset E(C(v))$. Now, we consider the graph G'' induced by $C(v)'$. If it contains a cycle C', that does not pass v, we proceed in the same manner. After a finite number of steps a set $\bar{C}(v) = \{\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_r\}$ of edge-disjoint cycles is all passing v, is constructed, such that in the induced graph $\bar{G}$ every cycle passes v. Hence $T(v) := \bar{G}$ is a trace. Obviously, $E(T(v)) \subset E(C(v))$. ■

The next lemma gives a relation between maximum and saturated traces.

Lemma 6. *Let $G = (V, E)$ and $T(v) \neq \emptyset$ be a maximum trace at v. Then $T(v)$ is saturated.*

Proof. Assume, that this is not the case. Then there is an Eulerian graph $H \subset G$ such that $T(v) \subsetneq H$ and v is proper with respect to H.

Let $T(v)$ be induced by $\{C_1, \ldots, C_{k^*}\}$ and H be induced by $\{C'_1, \ldots, C_{\bar{k}^*}\}$, respectively. Note, that $k^* = \bar{k}^* \leq \lfloor \frac{d(v)}{2} \rfloor$, otherwise $T(v)$ would not be maximal. Let $\bar{E} = \{e | e \text{ is incident with } v\} \cap E(T(v))$. Without loss of generality, we can assume that the representations of T and H, respectively, have no common cycle C. Otherwise, if there is such a cycle C, then we consider $T(v) \setminus C$ and $H \setminus C$, respectively.

We will show that H must contain a cycle $\tilde{C}$ that does not pass v, which is impossible. For this, take a cycle C_{i_1} and the two edges $e_{i_1}, e_{i_2} \in E(C_{i_1}) \cap \bar{E}$. Since $\bar{E} \subset E(H)$, there is a cycle C'_{j_1} with $e_{i_2} \in E(C'_{j_1})$. The cycle C'_{j_1} also contains an edge $e_{i_3} \in \bar{E}$. The edge e_{i_3} is then again contained in a cycle C_{i_2}, which also contains an edge $e_{i_4} \in \bar{E}$ and so on. In such a way, we get a sequence $C_{i_1}, C'_{j_1}, C_{i_2}, C'_{j_2}, \ldots$ of cycles that alternately belong to the representations of

$T(v)$ and H, respectively. Within this sequence, there must be one cycle C'_{j_k} that contains the edge e_{i_1}.

Now, let $P(v)$ be a path along C_{i_1} starting at v and using the edge e_{i_2}. Let w_{j_1} be the last vertex in $P(v)$ that belongs to $C_{i_1} \cap C'_{j_1}$. Such a vertex must exist and, obviously, $w_{j_1} \neq v$. We now construct the cycle $\tilde{C}$: starting from w_{j_1} we pass along the cycle C'_{j_1} until to the first vertex $w_{i_2} \neq v$ in C_{i_2}. From there we pass along C_{i_2} until to the first vertex $w_{j_2} \neq v$ in C'_{j_2} and so on. We proceed until we reach the vertex $w_{j_k} \neq v$ in C'_{j_k}. From there we pass along C'_{j_k} until we reach $w_{i_1} \neq v$ in C_{i_1}. From there it is possible to pass along C_{i_1} to the vertex w_{j_1}, not using v. In such a way we have constructed a cycle $\tilde{C} \subset H$ that does not pass through v, contradicting that v is proper with respect to H. ■

Note that the converse is not true in general, even if G is Eulerian. In the following figure a saturated trace $T(w)$ is drawn which is not maximum.

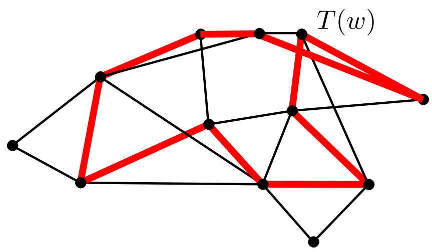

Figure 2. Saturated local trace $T(w)$ (red) in G that is not maximum.

Using a similar construction scheme as in Lemma 5 we now can give a characterization for a a maximum trace to be unique. For $v \in V$, let $\mathcal{C}(v)$ the family of sets of edge-disjoint cycles that induce a maximum trace $T(v)$ at v.

Lemma 7. *Let $G = (V, E)$ be Eulerian, $v \in V$ and $T(v) \neq \emptyset$ be a maximum trace at v. Then the following is equivalent:*

i. *$T(v)$ is unique.*

ii. *For all $C(v) \in \mathcal{C}(v)$ it holds: a cycle in $G \setminus \{v\}$ and a cycle in $C(v)$ has no common edge.*

Proof. "*i.* $\Rightarrow$ *ii.* ": Let $T(v)$ be uniquely induced by the edge-disjoint cycles $C(v) = \{C_1, C_2, \ldots, C_r\} \in \mathcal{C}(v)$. Assume there is a cycle $C \subset G \setminus \{v\}$, such that $E(C) \cap E(C(v)) \neq \emptyset$. Then C contains segments $(S_0, S_1, S_2, \ldots, S_t)$, where a segment S_i is a sequence of edges such that S_i belongs to one of the cycles C_j or S_i does not belong to $T(v)$. At least one such segment, say S_0, cannot belong to $T(v)$ since otherwise $T(v)$ would not be a trace. S_0 is now used to construct a set $C'(v) \in \mathcal{C}(v)$ that induces a maximum trace, different from $T(v)$. This will give the contradiction.

Let u and u' be the endpoints of S_0 in C. Then there are C_i and C_j such that $C_i = W_1^{(i)}(v,u) \cup W_2^{(i)}(u,v)$ and $C_j = W_1^{(j)}(v,u') \cup W_2^{(j)}(u',v)$.
If $C_i = C_j$ then $W_1^{(i)}(u,u') \subset W_1^{(i)}(v,u')$. Then set

$$\tilde{C}_i = C_i \setminus W_1^{(i)}(u,u') \cup S_0.$$

The cycles $\{C_1, C_2, \ldots, C_r\} \setminus C_i \cup \tilde{C}_i$ then induce a maximum trace at v not containing $W_1^{(i)}(u,u')$.

For the case that $C_i \neq C_j$, we distinguish two situations.

Case a. There is a vertex w different from v such that $w \in V(C_i) \cap V(C_j)$. Note that at most one such vertex can exist. If $w \in \{u,u'\}$, say $w = u'$, then set

$$\tilde{C}_i = C_i \setminus W_1^{(i)}(u,u') \cup S_0.$$

Again, the cycles $\{C_1, C_2, \ldots, C_r\} \setminus C_i \cup \tilde{C}_i$ then induce a maximum trace at v not containing $W_1^{(i)}(u,u')$.

If $w \notin \{u,u'\}$, then assume $w \in W_1^{(i)}(v,u)$ and $w \in W_1^{(j)}(v,u')$. Now, set

$$\tilde{C}_i := W_1^{(i)}(v,w) \cup W_1^{(j)}(v,w)\tilde{C}_j := W_2^{(i)}(v,u) \cup S_0 \cup W_2^{(j)}(v,u').$$

Then the cycles $\{C_1, C_2, \ldots, C_r\} \setminus \{C_i, C_j\} \cup \{\tilde{C}_i, \tilde{C}_j\}$ induce a maximum trace at v not containing $W_1^{(i)}(w,u)$ and $W_1^{(j)}(w,u')$.

Case b. The only common vertex of C_i and C_j is v. In this case we use a similar construction as in Lemma 5. We start from v along the path $W_1^{(i)}(v,u)$. Then u is reached on the edge $e_i(1) \in W_1^{(i)}(v,u)$. Instead of following $W_i^{(2)}(v,u)$ we follow along $S_0 \in C$ until reaching u' and follow the path $W_j^{(2)}(v,u')$.

If we reach v on $W_j^{(2)}(v,u')$ without visiting another $u'' \in C$, then the new cycle $\tilde{C}_i$ is defined by $\tilde{C}_j := W_2^{(i)}(v,u) \cup S_0 \cup W_2^{(j)}(v,u')$.

If we reach another vertex, say $u'' \in V(C)$, when passing along $W_j^{(2)}(v,u')$, then from u' we will reach u'' using a segment S_k before arriving at v; we then leave u'' on the segment S_{k+1} using $W_s^{(1)}(v,u'')$, and so on.

In such a way r circuits $\{\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}r\}$ are constructed (all passing v), that do not contain $W_i^{(2)}(v,u)$ and $W_j^{(2)}(v,u')$.

"*ii.* $\Rightarrow$ *i.*": First note that the components $B_1, B_2, \ldots, B_s$ of $G \setminus \{v\}$ are uniquely determined and that a subset of cycles in $C(v) \in \mathcal{C}(v)$ induce a maximum trace $T_i(v)$ for the (Eulerian) graph G_i induced by $B_i \cup \{v\}$. And vice versa.

Let $T(v)$ and $T'(v)$ be two maximum traces at v. Let $C(v), C'(v) \in \mathcal{C}$ be the sets of cycles that induce $T(v)$ and $T'(v)$, respectively.

If $G \setminus \{v\}$ contains no cycle, then non of the B_i contain a cycle, i.e, B_i is a tree for all i. The subgraphs $T_i(v), T_i'(v) \subset G_i$ are two maximum traces for G_i that, by Lemma 6, are saturated. But v is a proper vertex with respect to the graphs G_i. Hence $G_i = T_i(v) = T_i'(v)$, i.e, $T(v) = T'(v)$.

If $G \setminus \{v\}$ contains a cycle C, then by assumption, $E(C) \cap E(C(v)) = E(C) \cap E(C'(v)) = \emptyset$. We then consider the Eulerian graph $G' = G \setminus E(C)$. For G', $T(v)$ and $T'(v)$ are maximum traces at v and we can perform the same considerations as before. In the case that $G' \setminus \{v\}$ contains no cycle, we again get $T(v) = T'(v)$, otherwise we remove the cycle from G'. Proceeding in this way we will terminate with a Eulerian graph $\bar{G}$ in which $T(v)$ and $T'(v)$ are maximum traces at v and $\bar{G} \setminus \{v\}$ contains no cycle, concluding then $T(v) = T'(v)$. ■

By Lemma 7 we have proved

Proposition 8. *Let $G = (V, E)$ be Eulerian. If there is $v \in V$ such that the maximum local trace $T(v) \neq \emptyset$ is unique, then*

$$\nu(G) = \frac{d_G(v)}{2} + \nu(G \setminus \{v\})$$

and

$$\mathcal{Z}^*(G) = C(v) \cup \mathcal{Z}^*(G \setminus \{v\}).$$

In the following section, we will give a more general sufficient condition that makes the cycle packings $C(v)$ corresponding to maximum traces $T(v)$, $v \in V$, to build up a maximum cycle packing in G.

4. A Mini-max Theorem

We start with the observation that there are Eulerian graphs G with corresponding cycle packing $\mathcal{Z}_1 = \{C_1, C_2, \ldots, C_s\}$ of cardinality $s < \nu(G)$ such that G is induced by $\mathcal{Z}_1$ and for every $v \in V$ the subgraph $T(v)$ of G, induced by the cycles in $\mathcal{Z}_1(v)$, is a maximum trace.

It follows there are cases that maximum traces of G can be induced by cycle packings of G that are not maximum. In Figure 3 such an example is illustrated.

The question arises what are conditions that guarantee that a set $\{T(v) | v \in V\}$ of maximum local traces of G is induced by a maximum cycle packing $\mathcal{Z}^*$ of G.

We now investigate such a situation more generally. For $1 \leq s \leq \nu(G)$, we consider the family of cycle packings $\mathcal{C}_s^* \subset \mathcal{C}_s$ of G. A packing $\mathcal{Z}$ belongs to $\mathcal{C}_s^*$ if it is a cycle packing of cardinality s and for all $v \in V$ the subgraph $T(v)$ of G

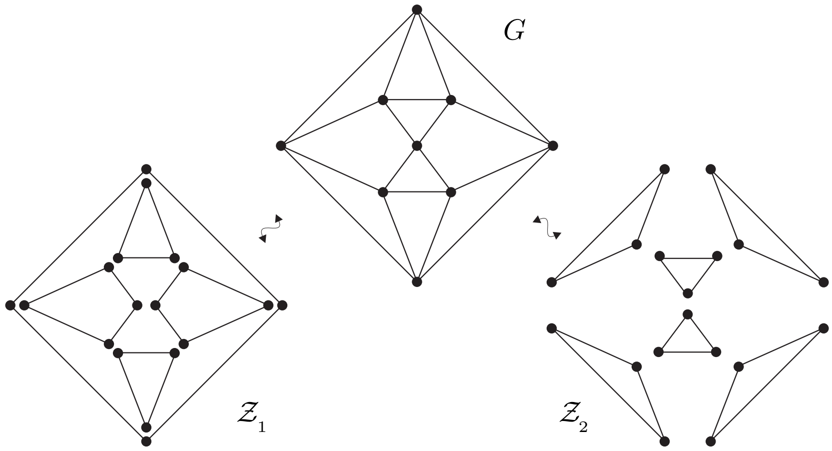

Figure 3. $|\mathcal{Z}_1| = 5, F(\mathcal{Z}_1) = 66$ whereas $|\mathcal{Z}_2| = 6 = \nu(G), F(\mathcal{Z}_2) = 54$.

induced by the cycles in $\mathcal{Z}(v)$ is a maximum trace at v. A first (simple) condition can be derived as an immediate consequence of Lemma 7.

Corollary 9. *Let $\mathcal{Z} \in \mathcal{C}_s^*$ and let $(v_0, v_1, \ldots, v_k)$ be a sequence of vertices in G. With $G_0 := G$ denote by $G_{i+1} := G_i \setminus E(\mathcal{Z}(v_i)), i = 0, 1, \ldots, k$, the sequence of subgraphs of G recursively induced by maximum traces $T_{G_i}(v_i)$ at v_i in G_i. If maximum traces $T_{G_i}(v_i) \subset G_i$ are unique (with respect to G_i) and $G_{k+1} = \emptyset$, then $s = \nu(G)$, i.e, $\mathcal{Z}$ is maximum.*

For a more general condition we first prove a theorem, which is true not only for Eulerian graphs.

For this let G be a graph with $\nu(G) \geq 1$. For $0 \leq s \leq \nu(G)$, let $\mathcal{C}_s(G)$ be the set of cycle packings of cardinality s. Then $\mathcal{C}(G) = \bigcup_s^{\nu(G)} \mathcal{C}_s(G)$ describes the set of all cycle-packings of G. Note, that $\mathcal{C}_0(G) = \emptyset$ if and only if G is a cycle. If this is not the case, then $\mathcal{C}_s(G) \neq \emptyset$ implies $\mathcal{C}_{s-1}(G) \neq \emptyset, s \geq 1$. For $s \geq 1$ a packing $\mathcal{Z} = \{C_1, C_2, \ldots, C_s, \tilde{G}_s\} \in \mathcal{C}_s(G)$ consists of s cycles C_i and a "reminder" $\tilde{G}_s$. Let $l_i = |E(C_i)|$. For $\mathcal{Z} \in \mathcal{C}_s, s \geq 1$, define

$$\bar{L}(\mathcal{Z}) = \sum_{i=1}^{s} l_i^2 + |E(\tilde{G}_s)|^2.$$

For $\mathcal{Z} \in \mathcal{C}_0$, set $\bar{L}(\mathcal{Z}) := |E(G)|^2$. We get

Theorem 10. *Let $\nu(G) \geq 1$. Every cycle packing $\mathcal{Z}^*$ that minimizes $\bar{L}$ on $\mathcal{C}(G)$ is maximum, i.e, $\mathcal{Z}^* \in \mathcal{C}_{\nu(G)}$.*

Proof. Obviously, the theorem is true if G is a cycle. Therefore, assume G is not a cycle. For $s \in \{0, 1, 2, \ldots, \nu(G)\}$ let $\bar{m}_s(G) := \min\{\bar{L}(\mathcal{Z}|\mathcal{Z} \in \mathcal{C}_s(G)\}$. We will show that

$$\bar{m}_{s-1}(G) > \bar{m}_s(G), \quad s = 1, 2, \dots, \nu(G).$$

To prove the inequality we will use the induction on $r \leq \nu(G)$. Obviously, $\bar{m}_0(G) = |E(G)|^2$.

Let $r = 1, \mathcal{C}_1(G) \neq \emptyset$. Let $\mathcal{Z}_1 \in \mathcal{C}_1(G)$, i.e, $\mathcal{Z}_1 = \{C_1, \tilde{G}_1\}$ and $l_1 = |E(C_1)|$.

Since G is not a cycle, $l_1 < |E(G)|$ and we immediately get $\bar{L}(\mathcal{Z}_1) := l_1^2 + (|E(G)| - l_1)^2 = 2l_1^2 + |E(G)|^2 - 2l_1^2|E(G)| < |E(G)|^2$, i.e, $\bar{m}_0(G) > \bar{m}_1(G)$. Now, let $r \geq 1$ such that $\mathcal{C}_r(G) \neq \emptyset$ and let us assume that for all graphs G such that $\nu(G) \leq r$ and all $r' \leq r$ the relations $\bar{m}_{r'-1}(G) > \bar{m}_{r'}(G)$ hold.

Let G be a graph such that $\mathcal{C}_{r+1}(G) \neq \emptyset$. Hence $\mathcal{C}_r(G) \neq \emptyset$. Since $\mathcal{C}_r(G) \neq \emptyset$ there exists $\mathcal{Z}_r(G) \in \mathcal{C}_r(G)$ such that $\bar{L}(\mathcal{Z}_r(G)) = \bar{m}_r(G)$. Take the cycle $C_1 \in \mathcal{Z}_r(G)$ of length l_1 and consider the graph $G \setminus C_1$.

Obviously, $\mathcal{Z} := (\mathcal{Z}_r(G) \setminus \{C_1\}) \in \mathcal{C}_{r-1}(G \setminus C_1)$. Moreover, $\bar{L}(\mathcal{Z}) = \bar{m}_r(G) - l_1^2$. But also $\bar{L}(\mathcal{Z}) = \min\{\bar{L}(\mathcal{Z}_{r-1})(G \setminus C_1))|\mathcal{Z}_{r-1} \in \mathcal{C}_{r-1}(G \setminus C_1)\}$ must hold, otherwise $\mathcal{Z}_r(G)$ would not be a minimizer in $\mathcal{C}_r(G)$, i.e, $\bar{m}_{r-1}((G \setminus C_1)) = \bar{m}_r(G) - l_1^2$. Using the assumption, we then get $\bar{L}(\mathcal{Z}) = \bar{m}_{r-1}(G \setminus C_1) > \bar{m}_r(G \setminus C_1)$ and, by this, $\bar{m}_r(G) = \bar{m}_{r-1}(G \setminus C_1) + l_1^2 > \bar{m}_r(G \setminus C_1) + l_1^2 \geq \bar{m}_{r+1}(G)$. ■

Remark 11. By a similar proof it can be shown that also for

$$\bar{M}_s(G) := \max\{\bar{L}(\mathcal{Z})|\mathcal{Z} \in \mathcal{C}_s(G)\}, s \geq 1; \quad \bar{M}_0(G) := |E(G)|^2$$

the strict inequalities

$$\bar{M}_{s-1}(G) > \bar{M}_s(G) \quad s = 1, 2, \dots, \nu(G)$$

hold. The proof is just the same but instead of taking out a cycle $C_1 \in \bar{\mathcal{Z}}_r(G)$ one takes it out from $\bar{\mathcal{Z}}_{r+1}(G)$.

Theorem 10 now will be used to get a condition that a cycle packing $\mathcal{Z}$ inducing maximum traces in Eulerian G is maximum.

Let $\mathcal{C}^* = \bigcup_{s=1}^{\nu(G)} \mathcal{C}_s^*$. By $F(\mathcal{Z}) = \sum_{v \in V} |E(T(v)|$ denote the total size of the local traces.

Theorem 12. *Let G be Eulerian. Every cycle packing $\mathcal{Z}^*$ that minimizes F on $\mathcal{C}^*$ is a maximum cycle packing of G, i.e, $\mathcal{Z}^* \in \mathcal{C}_{\nu(G)}^*$.*

Proof. We first observe that for all $v \in V$ and for all $C_i \in \mathcal{Z} = \{C_1, C_2, \dots, C_s\} \subset \mathcal{C}_s^*$ the following is true: $v \in V(C_i)$ if and only if $C_i \in \mathcal{Z}(v)$.

Therefore, we get $F(\mathcal{Z}) = \sum_{v \in V} |E(T(v)| = \sum_{v \in V} \sum_{C_i \in \mathcal{Z}(v)} |E(C_i)| = \sum_{C_i \in \mathcal{Z}} \sum_{v \in V(C_i)} |E(C_i)| = \sum_{C_i \in \mathcal{Z}} |V(C_i)||E(C_i)| = \sum_{i=1}^{s} |E(C_i)|^2$.
Let $\mathcal{Z}^*$ be a minimizer of F in $\mathcal{C}_{\nu(G)}^*$, i.e, $F(\mathcal{Z}^*) = \bar{m}_{\nu(G)}$. Assume that there is $\bar{\mathcal{Z}}^* \in \mathcal{C}_s^*$, such that $F(\bar{\mathcal{Z}}^*) = F(\mathcal{Z}^*)$, but $s < \nu(G)$. We then get $F(\bar{\mathcal{Z}}^*) \geq \min\{\bar{L}(\mathcal{Z}'|\mathcal{Z}' \in \mathcal{C}_s\} = \bar{m}_s > \bar{m}_{\nu(G)} = F(\mathcal{Z}^*)$, a contradiction. ■

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