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ON UNIQUELY HAMILTONIAN CLAW-FREE AND TRIANGLE-FREE GRAPHS

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Abstract

A graph is uniquely Hamiltonian if it contains exactly one Hamiltonian cycle. In this note, we prove that claw-free graphs with minimum degree at least 3 are not uniquely Hamiltonian. We also show that this is best possible by exhibiting uniquely Hamiltonian claw-free graphs with minimum degree 2 and arbitrary maximum degree. Finally, we show that a construction due to Entringer and Swart can be modified to construct triangle-free uniquely Hamiltonian graphs with minimum degree 3.

Keywords: Hamiltonian cycle, uniquely Hamiltonian graphs, claw-free graphs, triangle-free graphs.

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1. Introduction

All graphs in this note will be finite, undirected, and simple. A Hamiltonian cycle of a graph G is a cycle whose vertex set is precisely $V(G)$. A graph is *uniquely Hamiltonian* if it contains exactly one Hamiltonian cycle. A classic result of Smith [12] states in any 3-regular graph there are an even number of Hamiltonian cycles through any given edge; it follows that no 3-regular graph is uniquely Hamiltonian. Thomason [9] showed that Smith's Theorem can be extended to any graph whose vertices all have odd degree, by way of a clever "lollipop argument".

Smith's Theorem inspired the following conjecture:

Conjecture 1.1 (Sheehan [8]). *There is no uniquely Hamiltonian* 4*-regular graph.*

Recall that Petersen's 2-Factor Theorem states that every $(2r)$-regular graph can be decomposed into r edge-disjoint 2-factors [7]. Hence, if Conjecture 1.1 is true, then there is no uniquely Hamiltonian d-regular graph when $d \geq 3$. Thomassen [11] used the Local Lemma to show that there is no uniquely Hamiltonian d-regular graph for $d > 72$. The result was subsequently improved by Haxell, the author, and Verstraete to $d > 22$ and $d > 14$ if G has sufficiently high girth [6].

One natural strengthening of the problem is to consider uniquely Hamiltonian graphs having fixed minimum degree. Entringer and Swart [4] gave infinitely many examples of uniquely Hamiltonian graphs with minimum degree 3. Very recently, Fleischner [5] gave an infinite family of uniquely Hamiltonian graphs having minimum degree 4. The cases of higher constant minimum degree remain open. However, it was been shown by Bondy and Jackson [3] that there exists an absolute constant c for which no n-vertex graph with $\delta > c \log_2 n$ is uniquely Hamiltonian. The best result to date on the matter is a bound of $\delta > c \log_2 n + 2$ where $c \approx 1.71$ [1].

Thomassen [10] posed the following related conjecture:

Conjecture 1.2 (Thomassen [10]). *Every Hamiltonian graph G with $\delta(G) \geq 3$ has an edge $e \in E(G)$ such that both $G - e$ and G/e are Hamiltonian.*

If G is a graph containing two Hamiltonian cycles, then any edge contained in one cycle but not the other satisfies the statement of the conjecture. It was also shown by Bielak [2] that Conjecture 1.2 holds if G is also assumed to be claw-free (that is, G does not contain $K_{1,3}$ as an induced subgraph). Little else is known about Conjecture 1.2.

In this paper, we extend Bielak's result to show that claw-free graphs with $\delta \geq 3$ are not uniquely Hamiltonian. We also exhibit claw-free uniquely Hamiltonian graphs with $\delta = 2$ and Δ arbitrarily large. Finally, we show there exist triangle-free uniquely Hamiltonian graphs with $\delta = 3$.

2. Claw-Free Graphs

Bielak's proof of Conjecture 1.2 for claw-free graphs can be easily modified to prove the following:

Theorem 2.1. *If G is a claw-free graph with $\delta(G) \geq 3$, then G is not uniquely Hamiltonian.*

Proof. If G has no Hamiltonian cycle, then we are done. Let C be a Hamiltonian cycle, and consider the vertices of C in some order as they appear along C. For a vertex $x \in V(G)$, denote by x^- and x^+ the vertices preceding and following x, respectively. We also define x^{-n} and x^{+n} to be the vertices at distance n from x along C with respect to the ordering of $V(C)$.

We begin with the simple observation that if $a, b \in V(G)$ are vertices that are not joined by an edge of C and $ab^-, a^+b \in E(G)$, then G has a Hamiltonian cycle distinct from C, namely $C - \{aa^+, b^-b\} + \{ab^-, a^+b\}$. We call such a structure a "cycle exchange"; we may assume from this point forward that G has no cycle exchange. In particular, if $v_i, v_{i+1}, v_{i+2}, v_{i+3}$ are such that $v_j^+ = v_{j+1}$ for each $j = 0, 1, 2$, then either v_iv_{i+2} or $v_{i+1}v_{i+3}$ is a non-edge.

Let v be a vertex such that $v^-v^+ \notin E(G)$. Let u be a neighbour of v distinct from v^- and v^+. Since $\{u, v^-, v, v^+\}$ may not induce a claw, u must be adjacent to at least one of v^- and v^+. Without loss of generality, say $uv^+ \in E(G)$. We now consider u and its neighbours $\{v^+, u^-, u^+\}$. Since G is claw-free, one of v^+u^+, u^-u^+ or u^-v^+ is an edge of G. In the first case, G would have a cycle exchange. In the second case, $C - u^-uu^+ + u^-u^+ + vuv^+ - vv^+$ is a second Hamiltonian cycle of G. We thus proceed assuming that $u^-v^+ \in E(G)$. We now consider v^+ and its neighbours $\{u^-, v, v^{+2}\}$. As before, if vu^- or $vv^{+2} \in E(G)$, then G has a second Hamiltonian cycle, so we proceed assuming that $u^-v^{+2} \in E(G)$. By iteratively applying this argument, we see that $E(G)$ must contain the edges $vu, uv^+, v^+u^-, u^-v^{+2}, v^{+2}u^{-2}, \ldots$ with the sequence ending when the final edge added has endpoints which lie at distance 2 from one another along C.

Now, consider the vertex u and its neighbours $\{v, u^-, u^+\}$. Again, if vu^- or $u^-u^+ \in E(G)$, then G contains a second Hamiltonian cycle. We thus assume that $vu^+ \in E(G)$. A symmetric argument to the one above shows that $E(G)$ contains the edges $vu^+, u^+v^-, v^-u^{+2}, u^{+2}v^{-2}, \ldots$ with the sequence again ending when the final edge added has endpoints which lie at distance 2 from one another along C.

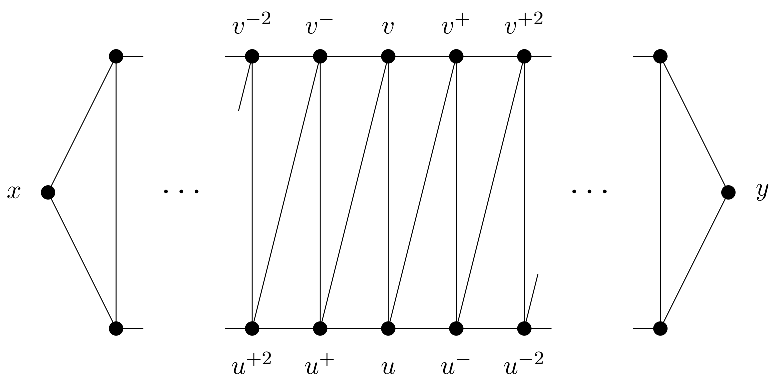

Figure 1. The structure of G.

Let $F = \{\ldots, v^{-2}u^{+2}, u^{+2}v^{-}, v^{-}u^{+}, u^{+}v, vu, uv^{+}, v^{+}u^{-}, u^{-}v^{+2}, v^{+2}u^{-2}, \ldots\}$ be the set of edges added above. Let x, y be the two vertices not incident to any edge of F. Since $\delta(G) \geq 3$, x has at least one neighbour yet to be determined. If $xy \in E(G)$, it is easy to see that G contains a Hamiltonian cycle consisting of the edges of F together with some cycle edge incident to each of x and y. To finish the proof, we relabel the vertices of $V(G) \setminus \{x, y\}$ for ease of notation. Noting that the edges of F form a Hamiltonian path of $G - \{x, y\}$, we label the vertices, in order along this path, $a_1, b_1, a_2, b_2, \ldots$ so that $a_1x \in E(C)$.

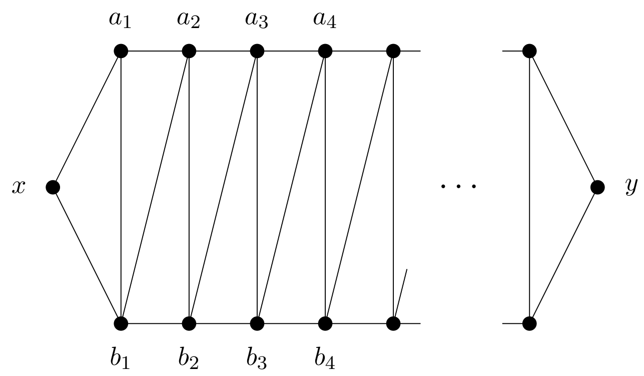

Figure 2. The structure of G, with relabelled vertices.

There are two remaining cases to consider, namely $xa_j \in E(G)$ for some j or $xb_k \in E(G)$ for some k. If $xa_j \in E(G)$, then $C - a_1a_2 \cdots a_j - xb_1b_2 \cdots b_{j-1} + xa_j + a_1b_1a_2b_2 \cdots b_{j-1}$ is a Hamiltonian cycle of G. If $xb_k \in E(G)$, then $C - a_1a_2 \cdots a_k - xb_1b_2 \cdots b_k + xb_k + a_1b_1a_2b_2 \cdots b_{k-1}a_k$ is a Hamiltonian cycle of G. Having exhausted all cases, we conclude that G is not uniquely Hamiltonian. ■

We now show that the minimum degree requirement cannot be removed from the statement of Theorem 2.1. While cycles would suffice to show this, we present a graph in Figure 3 that is claw-free, uniquely Hamiltonian, has minimum degree 2, and arbitrarily high maximum degree.

The graph in Figure 3, which we call H, is obtained from the complete graph K_k and a cycle C_{3k} for some fixed positive integer k. Let the vertices of K_k be $V(K_k) = \{v_1, \ldots, v_k\}$ and the vertices of C_{3k} be $V(C_{3k}) = \{u_1, \ldots, u_{3k}\}$. The graph H is defined as $V(H) = V(K_k) \cup V(C_{3k})$ and $E(H) = E(K_k) \cup E(C_{3k}) \cup \{v_iu_{3i-2}, v_iu_{3i-1} \ : \ 1 \leq i \leq k\}$. It is easy to check that H is claw-free, that $\delta(H) = 2$, and that $\Delta(H) = k + 1$. The edges in bold in Figure 3 show that H is Hamiltonian, and it is not hard to see that this is the only possible Hamiltonian cycle in H.

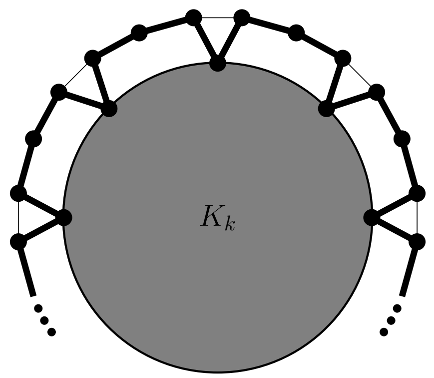

Figure 3. A uniquely Hamiltonian claw-free graph with arbitrarily large maximum degree

Theorem 2.2. *For any positive integer d, there exists a uniquely Hamiltonian claw-free graph with maximum degree at least d.*

3. Triangle-Free Graphs

In this section, we show that there exist triangle-free, uniquely Hamiltonian graphs having $\delta = 3$, and so any minimum degree result analogous to Theorem 2.1 for triangle-free graphs would be for minimum degree at least 4. Since the net (a triangle with a pendant edge on each vertex) and the bull (a triangle with a pendant edge on each of two vertices) both rely on the presence of a triangle, a similar statement holds for net-free and bull-free graphs as well.

We will make use of the following constructive result:

Theorem 3.1 (Entringer, Swart [4])**.** *For every $n = 2k$, $k \geq 11$, there exists a uniquely Hamiltonian graph on n vertices with two vertices having degree* 4 *and all others having degree* 3*.*

Let G be a graph having Hamiltonian cycle C, and let uvw be a triangle in G. We define a *C-blowup* of a triangle uvw in G to be the graph obtained by replacing the triangle uvw with the graph given in Figure 4. Note that the replacement operation does depend on which edges of the triangle are in C. We call the graph that replaces the triangle X.

The edges of C are indicated in bold in the figures on the left of Figure 4. The bold edges in the corresponding copy of X show how C extends to a Hamiltonian cycle in the graph containing the C-blowup. Note that, in a general graph containing a triangle uvw, it would be possible for a Hamiltonian cycle to use

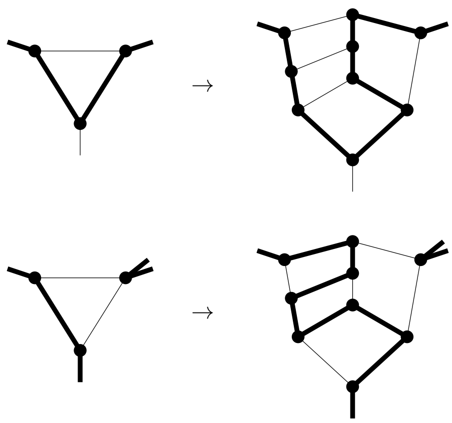

Figure 4. C-blowups of a triangle uvw.

none of the edges uv, vw, or uw. However, since we are only considering triangles which contain at least one vertex of degree 3, this situation will not arise; the only possibilities left are the two pictured in Figure 4.

Lemma 3.2. *Let G be a graph with $\delta(G) = 3$ such that every triangle contains a vertex of degree 3. Suppose that G contains a unique Hamiltonian cycle, C. If G' is the graph obtained by applying a C-blowup to each triangle of G, then G' is uniquely Hamiltonian. Furthermore, $\delta(G') = \delta(G) = 3$ and $\Delta(G') = \Delta(G)$.*

Proof. It is easy to see that, in C-blowup of a triangle uvw, the vertices u, v, w maintain the same degree and all other vertices added to the graph have degree 3. Hence, we have $\delta(G') = \delta(G) = 3$ and $\Delta(G') = \Delta(G)$.

By construction, G' is Hamiltonian. Let C' be a Hamiltonian cycle of G', and let u, v, w be vertices as in Figure 4. Clearly, C' must either:

1. contain a path that covers every vertex of X and has its ends in $\{u, v, w\}$; or
2. contain a path that covers every vertex of X except for one of $\{u, v, w\}$ and has its ends in $\{u, v, w\}$.

In either case, C' gives rise to a Hamiltonian cycle in G by reversing the C-blowup. Since G is uniquely Hamiltonian, this cycle must be precisely C. We can easily check that, for each C-blowup shown in Figure 4, the edges of the Hamiltonian cycle in the graph containing the blowup constitute the only way to extend C to a Hamiltonian cycle. Hence G' must be uniquely Hamiltonian. ■

By applying Lemma 3.2, we obtain the following corollary to Theorem 3.1.

Corollary 3.3. *There exist triangle-free uniquely Hamiltonian graphs with two vertices having degree 4 and all others having degree 3.*

Thus, if one wishes to find uniquely Hamiltonian H-free graphs where H is any graph containing a triangle (say, for instance, the bull or the net), then one must begin with graphs having minimum degree at least 4.

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