# GRAPHS WITH 3-RAINBOW INDEX $n-1$ AND $n-2$ 

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#### Abstract

Let $G=(V(G), E(G))$ be a nontrivial connected graph of order $n$ with an edge-coloring $c: E(G) \rightarrow\{1,2, \ldots, q\}, q \in \mathbb{N}$, where adjacent edges may be colored the same. A tree $T$ in $G$ is a rainbow tree if no two edges of $T$ receive the same color. For a vertex set $S \subseteq V(G)$, a tree connecting $S$ in $G$ is called an $S$-tree. The minimum number of colors that are needed in an edge-coloring of $G$ such that there is a rainbow $S$-tree for each $k$-subset $S$ of $V(G)$ is called the $k$-rainbow index of $G$, denoted by $r x_{k}(G)$, where $k$ is an integer such that $2 \leq k \leq n$. Chartrand et al. got that the $k$-rainbow index of a tree is $n-1$ and the $k$-rainbow index of a unicyclic graph is $n-1$ or $n-2$. So there is an intriguing problem: Characterize graphs with the $k$-rainbow index $n-1$ and $n-2$. In this paper, we focus on $k=3$, and characterize the graphs whose 3 -rainbow index is $n-1$ and $n-2$, respectively.


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## 1. Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notations of Bondy and Murty [1]. Let $G=(V(G), E(G))$ be a nontrivial connected graph of order $n$ with an edge-coloring $c: E(G) \rightarrow$ $\{1,2, \ldots, q\}, q \in \mathbb{N}$, where adjacent edges may be colored the same. A path of $G$ is a rainbow path if every two edges of the path have distinct colors. The graph $G$ is rainbow connected if for every two vertices $u$ and $v$ of $G$, there is a rainbow path connecting $u$ and $v$. The minimum number of colors for which there is an edge coloring of $G$ such that $G$ is rainbow connected is called the rainbow connection number, denoted by $r c(G)$. Results on the rainbow connectivity can be found in $[2,5,6,7,8,9]$. In the sequel we will simply denote the order of a graph $G$ by $|G|$, i.e., $|G|=|V(G)|$.

These concepts were introduced by Chartrand et al. in [2]. In [3], they generalized the concept of rainbow path to rainbow tree. A tree $T$ in $G$ is a rainbow tree if no two edges of $T$ receive the same color. For $S \subseteq V(G)$, a rainbow $S$-tree is a rainbow tree connecting $S$. Given a fixed integer $k$ with $2 \leq k \leq n$, the edge-coloring $c$ of $G$ is called a $k$-rainbow coloring of $G$ if for every $k$-subset $S$ of $V(G)$, there exists a rainbow $S$-tree. In this case, $G$ is called $k$-rainbow connected. The minimum number of colors that are needed in a $k$ rainbow coloring of $G$ is called the $k$-rainbow index of $G$, denoted by $r x_{k}(G)$. Clearly, when $k=2, r x_{2}(G)$ is the rainbow connection number $r c(G)$ of $G$. For every connected graph $G$ of order $n$, it is easy to see that $r x_{2}(G) \leq r x_{3}(G) \leq$ $\cdots \leq r x_{n}(G)$.

The Steiner distance $d_{G}(S)$ of a set $S$ of vertices in $G$ is the minimum size of a tree in $G$ connecting $S$. The $k$-Steiner diameter $\operatorname{sdiam}_{k}(G)$ of $G$ is the maximum Steiner distance of $S$ among all sets $S$ with $k$ vertices in $G$. Then there is a simple upper bound and lower bound for $r x_{k}(G)$.

Observation 1.1 [3]. For every connected graph $G$ of order $n \geq 3$ and each integer $k$ with $3 \leq k \leq n, k-1 \leq \operatorname{sdiam}_{k}(G) \leq r x_{k}(G) \leq n-1$.

They showed that the $k$-rainbow index of trees attains the upper bound.
Proposition 1.2 [3]. Let $T$ be a tree of order $n \geq 3$. For each integer $k$ with $3 \leq k \leq n, r x_{k}(T)=n-1$.

They also showed that the $k$-rainbow index of a unicyclic graph is $n-1$ or $n-2$.
Theorem 1.3 [3]. If $G$ is a unicyclic graph of order $n \geq 3$ and girth $g \geq 3$, then

A natural thought is that which graph of order $n$ has the $k$-rainbow index $n-1$ except for a tree and a unicyclic graph of girth 3? Furthermore, which graph of order $n$ has the $k$-rainbow index $n-2$ except for a unicyclic graph of girth at least 4? In this paper, we focus on $k=3$. In addition, some known results are mentioned.

Observation 1.4 [3]. Let $G$ be a connected graph of order $n$ containing two bridges e and $f$. For each integer $k$ with $2 \leq k \leq n$, every $k$-rainbow coloring of $G$ must assign distinct colors to $e$ and $f$.

Lemma 1.5 [4]. Let $G$ be a connected graph and $\left\{V_{1}, V_{2}, \ldots, V_{k}\right\}$ a partition of $V(G)$. If each $V_{i}$ induces a connected subgraph $H_{i}$ of $G$, then $r x_{3}(G) \leq k-1+$ $\sum_{i=1}^{k} r x_{3}\left(H_{i}\right)$.
Theorem 1.6 [4]. Let $G$ be a connected graph of order n. Then $r x_{3}(G)=2$ if and only if $G=K_{5}$ or $G$ is a 2-connected graph of order 4 or $G$ is of order 3 .

Observation 1.7 [4]. Let $G$ be a connected graph of order $n$, and $H$ be a connected spanning subgraph of $G$. Then $r x_{3}(G) \leq r x_{3}(H)$.
Let $G$ be a connected graph with $n$ vertices and $m$ edges. Define the cyclomatic number of $G$ as $c(G)=m-n+1$. A graph $G$ with $c(G)=k$ is called $k$-cyclic. According to this definition, if a graph $G$ meets $c(G)=0,1,2$ or 3 , then the graph $G$ is called acyclic (or a tree), unicyclic, bicyclic, or tricyclic, respectively.

This paper is organized as follows. In Section 2, some basic results and notations are presented. In Section 3, we characterize the graphs whose 3-rainbow index is $n-1$ and $n-2$, respectively. For the latter case, we take two steps to finish our proof: we deal with it for bicyclic graphs first, and then for tricyclic graphs.

## 2. Some Basic Results

First of all, we need more terminology and notations.
Definition 2.1. For a subgraph $H$ of $G$ and $v \in V(G)$, let $d(v, H)=\min \left\{d_{G}(v, x)\right.$ : $x \in V(H)\}$.

Next we define some new notations.
Definition 2.2. Let $G$ be a connected graph of order $n$ with $V(G)=\left\{v_{1}, v_{2}, \ldots\right.$, $\left.v_{n}\right\}$. For any edge $e$ of $G$ with end-vertices $v_{k}$ and $v_{r}$, if $k<r$ then we will write $e=v_{k} v_{r}$. It is clear that in this way any edge has a unique expression. Then, we define a lexicographic ordering between any two edges of $G$ by $v_{i} v_{j}<v_{s} v_{t}$ if and only if $i<s$ or $i=s, j<t$.

Note that, the lexicographic ordering of a connected graph is unique. Given a coloring $c$ of a connected graph $G$, denote by $c_{\ell}(G)$ a sequence of colors of the edges which are ordered by the lexicographic ordering.

For a connected graph $G$, to contract an edge $e=x y$ is to delete $e$ and replace its ends by a single vertex incident to all the edges which were incident to either $x$ or $y$. Let $G^{\prime}$ be the graph obtained by contracting some edges of $G$. Given a rainbow coloring of $G^{\prime}$, when it comes back to $G$, we keep the colors of corresponding edges of $G^{\prime}$ in $G$ and assign a new color to a new edge, which makes $G$-rainbow connected. Hence, the following lemma holds.

Lemma 2.3. Let $G$ be a connected graph, and $G^{\prime}$ be a connected graph obtained by contracting some edges of $G$. Then $r x_{3}(G) \leq r x_{3}\left(G^{\prime}\right)+|G|-\left|G^{\prime}\right|$.
Definition 2.4. Let $G_{0}$ be the graph obtained by contracting all the cut edges of $G$. Then $G_{0}$ is called the basic graph of $G$.

## 3. Main Results

### 3.1. Characterize the graphs with $r x_{3}(G)=n-1$

Theorem 3.1. Let $G$ be a connected graph of order $n$. Then $r x_{3}(G)=n-1$ if and only if $G$ is a tree or $G$ is a unicyclic graph with girth 3 .
Proof. If $G$ is a tree or a unicyclic graph with girth 3, then by Proposition 1.2 and Theorem 1.3, $r x_{3}(G)=n-1$. Conversely, suppose $G$ is a graph with $r x_{3}(G)=n-1$ but not a tree. Then $G$ must contain cycles. Let $\left\{V_{1}, V_{2}, \ldots, V_{k}\right\}$ be a partition of $V(G)$ and each $V_{i}$ induces a connected subgraph $H_{i}$ of $G$. If $G$ contains a cycle of length $r$ at least 4, then let $H_{1}$ be the $r$-cycle, and each other subgraph a single vertex. We color $H_{1}$ with $r-2$ dedicated colors, then by Lemma 1.5, $r x_{3}(G) \leq n-r+r x_{3}\left(H_{1}\right)=n-2$. Suppose then $G$ contains at least two triangles $C_{1}$ and $C_{2}$. If $C_{1}$ and $C_{2}$ have a vertex in common, then let $H_{1}$ be the union of $C_{1}$ and $C_{2}$, and each other subgraph a single vertex. We color both $C_{1}$ and $C_{2}$ with the same three dedicated colors, thus $r x_{3}(G) \leq n-5+r x_{3}\left(H_{1}\right)=$ $n-2$. If $C_{1}$ and $C_{2}$ are vertex disjoint, then let $H_{1}=C_{1}, H_{2}=C_{2}$, and each other subgraph a single vertex. We color $H_{1}$ with three new colors and $H_{2}$ with the same three colors as $H_{1}$, thus $r x_{3}(G) \leq n-5+r x_{3}\left(H_{1}\right)+r x_{3}\left(H_{2}\right)=n-2$. Combining the above two cases, $G$ is a unicyclic graph with girth 3 . Therefore, the result holds.

### 3.2. Characterize the graphs with $r x_{3}(G)=n-2$

Next, we characterize the graphs whose 3 -rainbow index is $n-2$. We begin with a useful theorem from [4].

A 3-sun is a graph constructed from a cycle $C_{6}=v_{1} v_{2} \cdots v_{6} v_{1}$ by adding three edges $v_{2} v_{4}, v_{2} v_{6}$ and $v_{4} v_{6}$.


Figure 1. The basic graphs for Lemma 3.3.

Theorem 3.2 [4]. Let $G$ be a 2-edge-connected graph of order $n(n \geq 4)$. Then $r x_{3}(G) \leq n-2$, with equality if and only if $G=C_{n}$ or $G$ is a spanning subgraph of one of the following graphs: a 3-sun, $K_{5}-e, K_{4}, G_{1}, G_{2}, H_{1}, H_{2}, H_{3}$, where $G_{1}, G_{2}$ are defined in Figure 1 and $H_{1}, H_{2}, H_{3}$ are defined in Figure 2.

Since all the 2-edge-connected graphs with the 3-rainbow index $n-2$ have been characterized in Theorem 3.2, it remains to characterize the graphs with 3rainbow index $n-2$ which have cut edges. Notice that the cut edges of a graph must be assigned distinct colors. Thus our main purpose is to check out how the addition of cut edges to $G$ affects the 3-rainbow index of a 2-connected graph $G$ when $r x_{3}(G)=n-2$. In other words, share the colors of cut edges with the colors of the non-cut edges as many as possible.

Given a connected graph $G$ of order $n$, and a coloring $c$ of $G$, we always let $A_{1}$ be the set of colors assigned to the non-cut edges of $G$ and $A_{2}$ the set of colors assigned to the cut edges of $G$. For each positive integer $k$, let $N_{k}=\{1,2, \ldots, k\}$. We always set that $A_{2}=N_{s}$, where $s$ is the number of cut edges of $G$. Note that $A_{1}$ and $A_{2}$ may intersect and suppose $\left|A_{1} \cap A_{2}\right|=p$. We can interchange the colors of cut edges suitably such that $A_{1} \cap A_{2}=\{1,2, \ldots, p\}$. Set $A_{1} \backslash A_{2}=\left\{a_{1}, \ldots, a_{t}\right\}$, $t \leq m-s$ and $a_{j} \in N_{|c|}$.

For a connected graph $G$, a block is a maximal 2-connected subgraph. In this paper, we regard $K_{2}$ other than a block. An internal cut edge is a cut edge which is on the unique path joining some two blocks. Denote the cut edges of $G$ by $e_{1}=x_{1} y_{1}, \ldots, e_{p}=x_{p} y_{p}$ and the colors of these cut edges by $1, \ldots, p$, respectively. Moreover, if $x_{i} y_{i}$ is not an internal cut edge, we always set $d\left(x_{i}, B\right) \leq d\left(y_{i}, B\right)$ where $B$ is an arbitrary block.

Let $H$ be a connected subgraph of $G$. Denote by $i \in H$ if the color $i$ appears in $H$. Given a graph $G$, let $G_{0}$ be its basic graph. Deleting the corresponding edges of $G_{0}$ in $G$, we obtain a forest. Each component corresponds to a vertex $v$ in $G_{0}$, denoted by $T(v)$. Denote by $U(v)$ the number of leaves of $T(v)$ in $G$ and $U(G)=\sum_{v \in V(G)} U(v)$. Let $W(v)$ be the number of edges of $T(v)$ whose colors are appeared in $A_{1}$, that is, $W(v)=\left|c(T(v)) \cap A_{1}\right|$.


Figure 2. The basic graphs for Lemma 3.4.

### 3.2.1. Bicyclic graphs with $r x_{3}(G)=n-2$

First, we introduce some graph classes. Let $G_{i}$ be the graphs shown in Figure 1. Define by $\mathcal{G}_{i}^{*}$ the set of graphs whose basic graph is $G_{i}$, where $1 \leq i \leq 6$. Set

$$
\begin{aligned}
\mathcal{G}_{1} & =\left\{G \in \mathcal{G}_{1}^{*}: U\left(v_{3}\right) \leq 1\right\}, \\
\mathcal{G}_{2} & =\left\{G \in \mathcal{G}_{2}^{*}: U\left(v_{3}\right)+U\left(v_{i}\right) \leq 1, i=4,6\right\}, \\
\mathcal{G}_{3} & =\left\{G \in \mathcal{G}_{3}^{*}: U\left(v_{i}\right)+U\left(v_{j}\right) \leq 2, v_{i} v_{j} \in E\left(G_{3}\right)\right\}, \\
\mathcal{G}_{4} & =\left\{G \in \mathcal{G}_{4}^{*}: U\left(v_{i}\right) \leq 2, i=1,3\right\}, \\
\mathcal{G}_{5} & =\left\{G \in \mathcal{G}_{5}^{*}: U\left(v_{2}\right)+U\left(v_{3}\right) \leq 2, U\left(v_{4}\right)+U\left(v_{5}\right) \leq 2\right\}, \\
\mathcal{G}_{6} & =\left\{G \in \mathcal{G}_{6}^{*}: U\left(v_{2}\right)=U\left(v_{6}\right)=0, U\left(v_{4}\right) \leq 1, U\left(v_{4}\right)+U\left(v_{i}\right) \leq 2, i=3,5\right\}, \\
\text { and set } \mathcal{G} & =\mathcal{G}_{1} \cup \mathcal{G}_{2} \cup \cdots \cup \mathcal{G}_{6} .
\end{aligned}
$$

Lemma 3.3. Let $G$ be a connected bicyclic graph of order $n$. Then $r x_{3}(G)=n-2$ if and only if $G \in \mathcal{G}$.

Proof. Suppose that $G$ is a graph with $r x_{3}(G)=n-2$ but $G \notin \mathcal{G}$. Let $G_{0}$ be the basic graph of $G$. Then $G_{0}$ is a 2-edge-connected bicyclic graph. If $G_{0} \neq G_{i}$, then by Theorem 3.2, $r x_{3}\left(G_{0}\right) \leq\left|G_{0}\right|-3$. Moreover, by Lemma 2.3, we have $r x_{3}(G) \leq r x_{3}\left(G_{0}\right)+|G|-\left|G_{0}\right| \leq n-3$. Hence $G_{0}=G_{i}$. Next we show that if $G \in \mathcal{G}_{i}^{*} \backslash \mathcal{G}_{i}$, then $r x_{3}(G) \leq n-3$, where $1 \leq i \leq 6$. As pointed out before, all the cut edges of $G$ are colored with $1,2, \ldots$. We only provide a coloring $c_{\ell}$ of $G_{0}$, namely, color the corresponding edges of $G$, with parts of
colors used in cut edges, and the position of cut edges will be determined as follows: $\{1,2, \ldots, q\} \subseteq T(v)$ means to assign colors $\{1,2, \ldots, q\}$ to $q$ leaves of $T(v)$ arbitrarily. If $G \in \mathcal{G}_{1}^{*} \backslash \mathcal{G}_{1}$, then $U\left(v_{3}\right) \geq 2$, set $c_{\ell}\left(G_{1}\right)=1 a_{1} a_{2} a_{2} a_{1} 2$ and $\{1,2\} \subseteq T\left(v_{3}\right)$. If $G \in \mathcal{G}_{2}^{*} \backslash \mathcal{G}_{2}$, then $U\left(v_{3}\right)+U\left(v_{4}\right) \geq 2$ or $U\left(v_{3}\right)+U\left(v_{6}\right) \geq 2$. By contracting $v_{3} v_{4}$ or $v_{3} v_{6}$, we obtain a graph $G^{\prime}$ belonging to $\mathcal{G}_{1}^{*} \backslash \mathcal{G}_{1}$. Then the coloring of $G$ can be obtained easily from $G^{\prime}$ by Lemma 2.3. If $G \in \mathcal{G}_{3}^{*} \backslash \mathcal{G}_{3}$, then there is an edge $v_{i} v_{j} \in E\left(G_{3}\right)$ such that $U\left(v_{i}\right)+U\left(v_{j}\right) \geq 3$. By symmetry, there exist four cases for $G$ : (1) $U\left(v_{1}\right) \geq 3$; (2) $U\left(v_{1}\right) \geq 2, U\left(v_{2}\right) \geq 1$; (3) $U\left(v_{1}\right) \geq 1$, $U\left(v_{2}\right) \geq 2$; (4) $U\left(v_{2}\right) \geq 3$. Set $c_{\ell}\left(G_{3}\right)=a_{1} a_{2} a_{2} 123$ and set $\{1,2,3\} \subseteq T\left(v_{1}\right)$ for (1); $\{1\} \subseteq T\left(v_{2}\right),\{2,3\} \subseteq T\left(v_{1}\right)$ for (2); $\{1\} \subseteq T\left(v_{1}\right),\{2,3\} \subseteq T\left(v_{2}\right)$ for (3); $\{1,2,3\} \subseteq T\left(v_{2}\right)$ for (4). If $G \in \mathcal{G}_{4}^{*} \backslash \mathcal{G}_{4}$, then $U\left(v_{1}\right) \geq 3$ or $U\left(v_{3}\right) \geq 3$. By symmetry, suppose $U\left(v_{1}\right) \geq 3$ and set $c_{\ell}\left(G_{4}\right)=123 a_{1} a_{1}$ and $\{1,2,3\} \subseteq T\left(v_{1}\right)$. If $G \in \mathcal{G}_{5}^{*} \backslash \mathcal{G}_{5}$, then by contracting $v_{2} v_{3}$ or $v_{4} v_{5}$, we obtain a graph $G^{\prime}$ belonging to $\mathcal{G}_{4}^{*} \backslash \mathcal{G}_{4}$. Now consider $G \in \mathcal{G}_{6}^{*} \backslash \mathcal{G}_{6}$. Then $U\left(v_{2}\right) \geq 1$, or $U\left(v_{6}\right) \geq 1$, or $U\left(v_{4}\right) \geq 2$, or $U\left(v_{4}\right)+U\left(v_{3}\right) \geq 3$, or $U\left(v_{4}\right)+U\left(v_{5}\right) \geq 3$. For the last two cases, it belongs to $\mathcal{G}_{5}^{*} \backslash \mathcal{G}_{5}$ by contracting $v_{3} v_{4}$ or $v_{4} v_{5}$. If $U\left(v_{2}\right) \geq 1$, set $c_{\ell}\left(G_{6}\right)=a_{3} a_{2} a_{4} a_{4} a_{2} a_{3} 1$ and $\{1\} \subseteq T\left(v_{2}\right)$. If $U\left(v_{4}\right) \geq 2$, set $c_{\ell}\left(G_{6}\right)=a_{3} 1 a_{2} 2 a_{1} a_{2} a_{1}$ and $\{1,2\} \subseteq T\left(v_{4}\right)$. It is not hard to check that the colorings above make $G$ rainbow connected with $n-3$ colors, thus $r x_{3}(G) \leq n-3$.

Conversely, let $G$ be a bicyclic graph such that $G \in \mathcal{G}$. Assume, to the contrary, that $r x_{3}(G) \leq n-3$. Then there exists a rainbow coloring $c$ such that $A_{1} \cup A_{2}=N_{n-3}$. By Theorem 3.2, we focus on the graphs with cut edges and $\left|A_{1} \cap A_{2}\right| \geq 1$. We write $d_{G_{i}}(u, v, w)$ to mean that the number of edges of a $\{u, v, w\}$-tree in $G$ which correspond to the edges of $G_{i}$, the basic graph of $G$. We divide into three cases.

Case 1. $G \in \mathcal{G}_{1} \cup \mathcal{G}_{2}$. First, we assume that $G \in \mathcal{G}_{2}$ and we give the following claims. If there is a nontrivial path $P_{\ell}$ connecting $B_{1}$ and $B_{2}$ in $G$, then denote its ends by $v_{3}^{\prime}\left(\in B_{1}\right)$ and $v_{3}^{\prime \prime}\left(\in B_{2}\right)$.

Claim 1. Each block $B_{i}$ has at most one edge which use the color from $A_{2}$, where $i \in\{1,2\}$. Moreover, if a color of $A_{2}$ appears in $B_{i}$, then the other edges of $B_{i}$ must be assigned with different colors in $A_{1} \backslash A_{2}$.

Proof. Suppose two edges of $B_{1}$ are colored with 1,2 , respectively. We also set $d\left(x_{i}, B_{1}\right) \leq d\left(y_{i}, B_{1}\right)$, where $x_{i} y_{i}$ belongs to $P_{\ell}$. Since the cut edges colored with 1 and 2 should be contained in the rainbow tree whose vertices contain $y_{1}$ and $y_{2}$, by deleting the edges assigned with 1 and 2 in $B_{1}, G$ is disconnected. Let $w$ be a vertex in the component that does not contain $y_{1}$. Then there is no rainbow tree connecting $\left\{y_{1}, y_{2}, w\right\}$, a contradiction. We can take the similar argument for the other cases when two edges of $B_{1}\left(B_{2}\right)$ are colored with 1 or two edges of $B_{2}$ are colored with 1,2 , respectively.

Now suppose $1 \in B_{i} \cap A_{2}$ and two edges of $B_{i}$ have the same color $a_{1}$. Let
$w_{1}, w_{2}$ be the end-vertices of the edge assigned with 1 , then $\left\{y_{1}, w_{1}, w_{2}\right\}$ has no rainbow tree.

Claim 2. The colors of the path $P_{\ell}$ cannot appear in $A_{1}$.
Proof. Assume $e$ is the edge of $P_{\ell}$ colored with 1 . The color 1 cannot appear in $B_{1}$. Otherwise, suppose the three edges of $B_{1}$ are assigned with $1, a_{1}$ and $a_{2}$, respectively. Consider $\left\{v_{1}, v_{2}, v_{5}\right\}$, then $c\left(v_{3}^{\prime \prime} v_{4}\right), c\left(v_{4} v_{5}\right) \in\left\{2, a_{3}\right\}$ or $c\left(v_{3}^{\prime \prime} v_{6}\right), c\left(v_{5} v_{6}\right) \in\left\{2, a_{3}\right\}$. Without loss of generality, suppose $c\left(v_{3}^{\prime \prime} v_{4}\right), c\left(v_{4} v_{5}\right) \in$ $\left\{2, a_{3}\right\}$, then by Claim $1, c\left(v_{3}^{\prime \prime} v_{6}\right), c\left(v_{5} v_{6}\right) \in\left\{a_{1}, a_{2}\right\}$, thus $\left\{v_{1}, v_{2}, v_{6}\right\}$ has no rainbow tree. On the other hand, 1 cannot be in $B_{2}$. It is easy to see that neither $c\left(v_{3}^{\prime \prime} v_{4}\right)$ nor $c\left(v_{3}^{\prime \prime} v_{6}\right)$ can be 1 by considering $\left\{v_{1}, v_{2}, v_{6}\right\}$. If $c\left(v_{5} v_{6}\right)=1$, consider $\left\{v_{1}, v_{5}, v_{6}\right\},\left\{v_{2}, v_{5}, v_{6}\right\}$, then $c\left(v_{1} v_{3}^{\prime}\right), c\left(v_{2} v_{3}^{\prime}\right) \in A_{2}$, a contradiction to Claim 1 .

By Claim 1, we have $1 \leq\left|A_{1} \cap A_{2}\right| \leq 2$ and only colors 1 and 2 can exist in $A_{1}$. We should discuss all the situations according to which cut edges are colored with 1, 2. By the definition of $G, U\left(v_{3}\right)=1$ or $U\left(v_{3}\right)=0$. By similarity, we only deal with the former case. First, assume $\left|A_{1} \cap A_{2}\right|=1$, then $A_{1}=$ $\left\{1, a_{1}, a_{2}, a_{3}\right\}$. We consider the subcase when $1 \in T\left(v_{3}\right)$. In this subcase we claim that the color 1 appears in neither $B_{1}$ nor $B_{2}$. Indeed, if $c\left(v_{3}^{\prime \prime} v_{6}\right)=1$, since then every tree whose vertices contain $y_{1}$ must contain the cut edge colored with 1 , $d_{G_{2}}\left(y_{1}, v_{1}, v_{6}\right)=4$. Thus $\left\{y_{1}, v_{1}, v_{6}\right\}$ has no rainbow tree. If now $c\left(v_{5} v_{6}\right)=1$, then consider $\left\{y_{1}, v_{5}, v_{6}\right\},\left\{y_{1}, v_{5}, v_{1}\right\},\left\{y_{1}, v_{5}, v_{2}\right\}$ successively, we have $c\left(v_{1} v_{3}^{\prime}\right)=$ $c\left(v_{2} v_{3}^{\prime}\right)=c\left(v_{3}^{\prime \prime} v_{6}\right)$, leading to a contradiction when considering $\left\{v_{1}, v_{2}, v_{6}\right\}$. Else if $c\left(v_{1} v_{3}^{\prime}\right)=1$, then $\left\{y_{1}, v_{1}, v_{5}\right\}$ has no rainbow tree. The last possibility is that $c\left(v_{1} v_{2}\right)=1$, we may set $c\left(v_{1} v_{3}\right)=a_{1}, c\left(v_{2} v_{3}\right)=a_{2}$. Consider $\left\{y_{1}, v_{1}, v_{4}\right\}$, $\left\{y_{1}, v_{2}, v_{4}\right\},\left\{y_{1}, v_{1}, v_{6}\right\},\left\{y_{1}, v_{2}, v_{6}\right\}$ successively, we have $c\left(v_{3}^{\prime \prime} v_{4}\right)=c\left(v_{3}^{\prime \prime} v_{6}\right)=a_{3}$ and 1 cannot appear in $B_{2}$, hence $\left\{v_{1}, v_{4}, v_{6}\right\}$ has no rainbow tree. The other subcases are similar.

Thus $\left|A_{1} \cap A_{2}\right|=2, A_{1}=\left\{1,2, a_{1}, a_{2}, a_{3}\right\}$. By Claim 1 , set $1 \in B_{1}, 2 \in B_{2}$, and the other edges in each block have distinct colors. If $1,2 \in T\left(v_{3}\right)$, assume that $d\left(y_{1}, T\left(v_{3}\right)\right)>d\left(y_{2}, T\left(v_{3}\right)\right)$, there always exist two vertices which come from different blocks such that there is no rainbow tree connecting them and $y_{1}$. If $1 \in T\left(v_{3}\right), 2 \in T\left(v_{1}\right)$, the most difficult case is that $c\left(v_{1} v_{2}\right)=1, c\left(v_{5} v_{6}\right)=2$. In this case, consider $\left\{y_{2}, v_{5}, v_{6}\right\}$, forcing that one of $v_{1} v_{3}^{\prime}, v_{3}^{\prime \prime} v_{4}, v_{3}^{\prime \prime} v_{6}, v_{4} v_{5}$ is colored with 1 , contradicting Claim 1 . With an analogous argument, we would get a contradiction if 1,2 are in other cut edges of $G$.

For $G \in \mathcal{G}_{1}$, it can be obtained by contracting an edge of a graph in $\mathcal{G}_{2}$. Then by Lemma $2.3, r x_{3}(G) \geq n-2$.

Case 2. $G \in \mathcal{G}_{3}$. First note that each path from $v_{1}$ to $v_{5}$ in $G_{3}$ can have at most one color in $A_{2}$. Thus $\left|A_{1} \cap A_{2}\right| \leq 3$. On the other hand, noticing that $d_{G_{3}}\left(v_{2}, v_{3}, v_{4}\right)=3>2$, all the cases satisfying $W\left(v_{1}\right)=W\left(v_{5}\right)=0$ and $W\left(v_{2}\right)$, $W\left(v_{3}\right), W\left(v_{4}\right) \leq 1$ are easy to get a contradiction, so we omit them here.

First, assume $\left|A_{1} \cap A_{2}\right|=1$, then $A_{1}=\left\{1, a_{1}, a_{2}\right\}$. If $1 \in T\left(v_{1}\right)$, then consider $\left\{y_{1}, v_{2}, v_{3}\right\}$, $\left\{y_{1}, v_{2}, v_{4}\right\}$ and $\left\{y_{1}, v_{3}, v_{4}\right\}$ successively, $v_{1} v_{2}, v_{1} v_{3}, v_{1} v_{4}$ must be colored with distinct colors from $A_{1} \backslash\{1\}$, which is impossible.

Assume now $\left|A_{1} \cap A_{2}\right|=2$, then $A_{1}=\left\{1,2, a_{1}, a_{2}\right\}$. If $1,2 \in T\left(v_{1}\right)$, then consider $\left\{y_{1}, y_{2}, v_{5}\right\}$, without loss of generality, set $c\left(v_{1} v_{2}\right)=a_{1}, c\left(v_{2} v_{5}\right)=a_{2}$. Thus $c\left(v_{1} v_{3}\right)$ can be neither 1 nor 2 , otherwise there is no rainbow $\left\{y_{1}, y_{2}, v_{3}\right\}$ tree. On the other hand, $c\left(v_{1} v_{3}\right)$ cannot be $a_{1}$, otherwise $\left.c\left(v_{3} v_{5}\right)=i, i=1,2\right)$, then $\left\{y_{i}, v_{2}, v_{3}\right\}$ has no rainbow tree. Meanwhile, $v_{1} v_{3}$ cannot be colored with $a_{2}$, otherwise $c\left(v_{3} v_{5}\right)=i(i=1,2)$, then $\left\{y_{i}, v_{3}, v_{5}\right\}$ has no rainbow tree. If $1 \in T\left(v_{1}\right)$, $2 \in T\left(v_{5}\right)$, then every path from $v_{1}$ to $v_{5}$ must color $\left\{a_{1}, a_{2}\right\}$, a contradiction to $\left|A_{1} \cap A_{2}\right|=3$. If $1,2 \in T\left(v_{2}\right)$, then by the same reason, we conclude that $c\left(v_{1} v_{2}\right)$, $c\left(v_{2} v_{5}\right) \notin\{1,2\}$ and we may set $c\left(v_{1} v_{3}\right)=1$. But now $\left\{y_{1}, v_{1}, v_{3}\right\}$ has no rainbow tree. If $1 \in T\left(v_{1}\right), 2 \in T\left(v_{2}\right)$. By considering $\left\{y_{1}, y_{2}, v_{3}\right\},\left\{y_{1}, y_{2}, v_{4}\right\},\left\{y_{1}, y_{2}, v_{5}\right\}$, we may set $c\left(v_{1} v_{3}\right)=c\left(v_{1} v_{4}\right)=c\left(v_{2} v_{5}\right)=a_{1}$, this force $c\left(v_{3} v_{5}\right)=i(i=1,2)$. However, there is no rainbow tree connecting $\left\{y_{i}, v_{3}, v_{5}\right\}$.

Thus $\left|A_{1} \cap A_{2}\right|=3, A_{1}=\left\{1,2,3, a_{1}, a_{2}\right\}$. If $1,2,3 \in T\left(v_{1}\right)$, since $U\left(v_{1}\right) \leq 2$, we may assume that $y_{2}$ is on the unique path from $y_{1}$ to $v_{1}$. Thus one path from $v_{1}$ to $v_{5}$ must be colored with $\left\{a_{1}, a_{2}\right\}$, a contradiction to $\left|A_{1} \cap A_{2}\right|=3$. If $1,2 \in T\left(v_{2}\right), 3 \in T\left(v_{5}\right)$, and without loss of generality, $y_{2}$ is on the unique path from $y_{1}$ to $v_{2}$. Considering $\left\{y_{1}, v_{3}, y_{3}\right\}$ and $\left\{y_{1}, v_{4}, y_{3}\right\}$, we may set $c\left(v_{1} v_{2}\right)=a_{1}$, $c\left(v_{1} v_{3}\right)=c\left(v_{1} v_{4}\right)=a_{2}$. But there is no rainbow $\left\{y_{1}, v_{3}, v_{4}\right\}$-tree. Each other case is similar or easier.

Case 3. $G \in \mathcal{G}_{4} \cup \mathcal{G}_{5} \cup \mathcal{G}_{6}$. First let $G \in \mathcal{G}_{6}$. Similarly, each path from $v_{2}$ to $v_{6}$ in $G_{6}$ can have at most one color in $A_{2}$. Thus we have $1 \leq\left|A_{1} \cap A_{2}\right| \leq 3$. Assume first $\left|A_{1} \cap A_{2}\right|=1, A_{1}=\left\{1, a_{1}, a_{2}, a_{3}\right\}$.

We only focus on the case that $1 \in T\left(v_{4}\right)$. To make sure there are rainbow trees connecting $\left\{y_{1}, v_{1}, v_{3}\right\}$ and $\left\{y_{1}, v_{1}, v_{5}\right\}$, only $c\left(v_{2} v_{6}\right)$ can be 1 , but now $\left\{y_{1}, v_{2}, v_{6}\right\}$ has no rainbow tree.

Assume then $\left|A_{1} \cap A_{2}\right|=2, A_{1}=\left\{1,2, a_{1}, a_{2}, a_{3}\right\}$. If $1,2 \in T\left(v_{1}\right)$, we may set $c\left(v_{1} v_{2}\right)=a_{1}, c\left(v_{2} v_{3}\right)=a_{2}, c\left(v_{3} v_{4}\right)=a_{3}$ by considering $\left\{y_{1}, y_{2}, v_{4}\right\} . c\left(v_{5} v_{6}\right)$ can be neither 1 nor 2 , otherwise $\left\{y_{1}, y_{2}, v_{5}\right\}$ has no rainbow tree. Moreover, $c\left(v_{4} v_{5}\right)$ can be neither 1 nor 2 , otherwise when $c\left(v_{4} v_{5}\right)=i(i=1,2)$, there is no rainbow $\left\{y_{i}, v_{4}, v_{5}\right\}$-tree. Thus $v_{1} v_{6}, v_{2} v_{6}$ must use colors $\{1,2\}$, but now $\left\{y_{1}, y_{2}, v_{6}\right\}$ has no rainbow tree. If $1,2 \in T\left(v_{3}\right)$, first we claim that at most one edge of the triangle $v_{1} v_{2} v_{6}$ uses a color from $\{1,2\}$. Otherwise, if $c\left(v_{1} v_{2}\right), c\left(v_{2} v_{6}\right) \in\{1,2\}$, then $\left\{y_{1}, y_{2}, v_{1}\right\}$ has no rainbow tree. If $c\left(v_{1} v_{6}\right), c\left(v_{2} v_{6}\right) \in\{1,2\}$, then the rest non-cut edges must color $\left\{a_{1}, a_{2}, a_{3}\right\}$. It is easy to verify that either $\left\{y_{1}, v_{1}, v_{5}\right\}$ or $\left\{y_{1}, v_{1}, v_{4}\right\}$ has no rainbow tree. So the longest path from $v_{2}$ to $v_{6}$ has an edge colored with 1 or 2 . However, we will show that it is impossible. It is easy to check that $c\left(v_{2} v_{3}\right), c\left(v_{3} v_{4}\right) \notin\{1,2\}$. If $c\left(v_{5} v_{6}\right) \in\{1,2\}$, then we may set $c\left(v_{5} v_{6}\right)=1$. Consider $\left\{y_{1}, v_{1}, v_{5}\right\}$ and $\left\{y_{1}, v_{5}, v_{6}\right\}$, then $c\left(v_{1} v_{2}\right)=c\left(v_{2} v_{6}\right)=2$, a contradiction.

It is similar to check that $c\left(v_{4} v_{5}\right)$ cannot be 1 or 2 , a contradiction.
Now assume that $\left|A_{1} \cap A_{2}\right|=3, A_{1}=\left\{1,2,3, a_{1}, a_{2}, a_{3}\right\}$. If $1 \in T\left(v_{1}\right)$, then $2,3 \in T\left(v_{3}\right)$. Again, we may set $c\left(v_{1} v_{2}\right)=a_{1}, c\left(v_{2} v_{3}\right)=a_{2}$. Thus $c\left(v_{1} v_{6}\right), c\left(v_{2} v_{6}\right) \in\{1,2,3\}$. If $c\left(v_{1} v_{6}\right), c\left(v_{2} v_{6}\right) \in\{1, i\}$, then there is a contradiction by considering $\left\{y_{1}, y_{i}, v_{6}\right\}(i=2,3)$. Thus we may set that $c\left(v_{1} v_{6}\right)=2$, $c\left(v_{2} v_{6}\right)=3$. By considering $\left\{y_{1}, y_{3}, v_{4}\right\}$ and $\left\{y_{1}, y_{3}, v_{5}\right\}$, we get that $c\left(v_{3} v_{4}\right)=$ $c\left(v_{5} v_{6}\right)=a_{3}, c\left(v_{4} v_{5}\right)=1$, but now $\left\{y_{1}, v_{4}, v_{5}\right\}$ has no rainbow tree. If $1 \in T\left(v_{3}\right)$, $2,3 \in T\left(v_{5}\right)$, then we set $v_{3} v_{4}=a_{1}, v_{4} v_{5}=a_{2}$. If $c\left(v_{2} v_{6}\right)=i, c\left(v_{5} v_{6}\right)=j$, $i, j \in\{1,2,3\}$, then $\left\{y_{i}, y_{j}, v_{6}\right\}$ has no rainbow tree. The only possibility is $c\left(v_{2} v_{3}\right)=2, c\left(v_{2} v_{6}\right)=3, c\left(v_{5} v_{6}\right)=a_{3}$. However, $\left\{y_{1}, y_{2}, v_{1}\right\}$ has no rainbow tree.

For $G \in \mathcal{G}_{5}$, notice that $\left|A_{1} \cap A_{2}\right| \leq 3$. If $U\left(v_{2}\right)=0$ or $U\left(v_{5}\right)=0$, then $G$ can be obtained by contracting an edge of a graph in $\mathcal{G}_{6}$. Then by Lemma 2.3, $r x_{3}(G) \geq n-2$. Thus we need to consider the case when $W\left(v_{2}\right) \geq 1$ and $W\left(v_{5}\right) \geq$ 1. If $\left|A_{1} \cap A_{2}\right|=2$, then suppose $1 \in T\left(v_{2}\right), 2 \in T\left(v_{5}\right)$. Consider $\left\{y_{1}, y_{2}, v_{3}\right\}$, $\left\{y_{1}, y_{2}, v_{4}\right\}$, we have $c\left(v_{2} v_{3}\right), c\left(v_{2} v_{5}\right), c\left(v_{4} v_{5}\right) \in\left\{a_{1}, a_{2}\right\}$ and $c\left(v_{2} v_{3}\right)=c\left(v_{4} v_{5}\right)$. But now $c\left(v_{3} v_{4}\right)=i \quad(i=1,2)$, then there is no rainbow $\left\{y_{i}, v_{3}, v_{4}\right\}$-tree. If $\left|A_{1} \cap A_{2}\right|=3$, then $A_{1}=\left\{1,2,3, a_{1}, a_{2}\right\}$. If $1 \in T\left(v_{1}\right), 2 \in T\left(v_{2}\right), 3 \in T\left(v_{5}\right)$, then consider $\left\{y_{1}, y_{2}, y_{3}\right\}$, we have that two of $v_{1} v_{2}, v_{1} v_{5}, v_{2} v_{5}$ have colors outside $A_{2}$, contradicting to $\left|A_{1} \cap A_{2}\right|=3$. If $1 \in T\left(v_{2}\right), 2 \in T\left(v_{5}\right), 3 \in T\left(v_{2}\right)$ and we may assume that $y_{3}$ is on the unique path from $y_{1}$ to $v_{2}$. Then consider $\left\{y_{1}, v_{3}, y_{3}\right\}$ and $\left\{y_{1}, v_{4}, y_{3}\right\}$, we have $c\left(v_{2} v_{3}\right)=c\left(v_{4} v_{5}\right)$, thus $c\left(v_{3} v_{4}\right)$ cannot be in $A_{2}$, contradicting to $\left|A_{1} \cap A_{2}\right|=3$. If $1 \in T\left(v_{2}\right), 2 \in T\left(v_{5}\right), 3 \in T\left(v_{i}\right)(i=3,4)$, then consider $\left\{y_{1}, y_{2}, y_{3}\right\}$, we have that $c\left(v_{2} v_{5}\right)$ is in $A_{1} \backslash A_{2}$, contradicting $\left|A_{1} \cap A_{2}\right|=3$.

Finally, for a graph $G$ belonging to $\mathcal{G}_{4}$, it can be obtained by contracting an edge of a graph in $\mathcal{G}_{3} \cup \mathcal{G}_{6}$. Then by Lemma 2.3, $r x_{3}(G) \geq n-2$.

Combining all the cases above, we have $r x_{3}(G) \geq n-2$ for $G \in \mathcal{G}$. By Theorem 3.1, it follows that $r x_{3}(G)=n-2$.

### 3.2.2. $\quad$ Tricyclic graphs with $r x_{3}(G)=n-2$

Define by $\mathcal{H}_{i}^{*}$ the set of graphs whose basic graph is $H_{i}$, where $H_{i}$ is shown in Figure 2 and $1 \leq i \leq 8$.

Now, we introduce another graph class $\mathcal{H}$. Set $\mathcal{H}=\mathcal{H}_{1} \cup \mathcal{H}_{2} \cup \cdots \cup \mathcal{H}_{8}$, where
$\mathcal{H}_{1}=\left\{G \in \mathcal{H}_{1}^{*}: U(G)=0\right\}$,
$\mathcal{H}_{2}=\left\{G \in \mathcal{H}_{2}^{*}: U\left(v_{i}\right) \leq 1, U\left(v_{j}\right)=0, i=5,6, j=1,3,4\right\}$,
$\mathcal{H}_{3}=\left\{G \in \mathcal{H}_{3}^{*}: U\left(v_{2}\right) \leq 1, U\left(v_{5}\right)+U\left(v_{6}\right) \leq 1, U\left(v_{i}\right)=0, i=1,3,4\right\}$,
$\mathcal{H}_{4}=\left\{G \in \mathcal{H}_{4}^{*}: U\left(v_{i}\right) \leq 1, U\left(v_{j}\right) \leq 2, U\left(v_{i}\right)+U\left(v_{j}\right) \leq 1, U\left(v_{j}\right)+U\left(v_{k}\right) \leq 3\right.$, $i=1,5, j, k=2,3,4\}$,
$\mathcal{H}_{5}=\left\{G \in \mathcal{H}_{5}^{*}: U\left(v_{i}\right) \leq 1, U\left(v_{j}\right)=0, i=1,3,5, j=2,4,6\right\}$,
$\mathcal{H}_{6}=\left\{G \in \mathcal{H}_{6}^{*}: U\left(v_{3}\right)=0, U\left(v_{i}\right) \leq 1, U\left(v_{1}\right)+U\left(v_{5}\right) \leq 1, i=1,2,4,5\right\}$,
$\mathcal{H}_{7}=\left\{G \in \mathcal{H}_{2}^{*}: U\left(v_{2}\right)+U\left(v_{4}\right) \leq 1, U\left(v_{3}\right)+U\left(v_{5}\right) \leq 1, U\left(v_{5}\right)+U\left(v_{1}\right) \leq 1\right.$,

$$
\begin{aligned}
& \left.U\left(v_{j}\right)+U\left(v_{j+1}\right) \leq 1, j=1,2,4\right\} \\
\mathcal{H}_{8}= & \left\{G \in \mathcal{H}_{8}^{*}: U\left(v_{i}\right) \leq 2, U\left(v_{i}\right)+U\left(v_{j}\right)+U\left(v_{k}\right) \leq 3, i, j, k=1,2,3,4\right\}
\end{aligned}
$$

Lemma 3.4. Let $G$ be a connected tricyclic graph of order $n$. Then $r x_{3}(G)=$ $n-2$ if and only if $G \in \mathcal{H}$.

Proof. Suppose that $r x_{3}(G)=n-2$ but $G \notin \mathcal{H}$. Let $G_{0}$ be the basic graph of $G$. Similar to Lemma 3.3, we have $G_{0}=H_{i}$ and we can rainbow color $G$ with $n-3$ colors for $G \in \mathcal{H}_{i}^{*} \backslash \mathcal{H}_{i}, i=1, \ldots, 8$.

If $G \in \mathcal{H}_{1}^{*} \backslash \mathcal{H}_{1}$, then if $U\left(v_{2}\right) \geq 1$, set $c_{\ell}\left(H_{1}\right)=a_{4} 1 a_{1} a_{2} a_{2} a_{4} a_{3} a_{3} a_{4}$ and $\{1\} \subseteq T\left(v_{2}\right)$; if $U\left(v_{3}\right) \geq 1$, set $c_{\ell}\left(H_{1}\right)=a_{4} a_{1} a_{1} 1 a_{3} a_{4} a_{2} a_{2} a_{4}$ and $\{1\} \subseteq T\left(v_{3}\right)$; if $U\left(v_{4}\right) \geq 1$, set $c_{\ell}\left(H_{1}\right)=a_{3} a_{2} a_{2} a_{1} 1 a_{3} a_{4} a_{4} a_{1}$ and $\{1\} \subseteq T\left(v_{4}\right)$.

If $G \in \mathcal{H}_{2}^{*} \backslash \mathcal{H}_{2}$, then if $U\left(v_{3}\right) \geq 1\left(U\left(v_{1}\right) \geq 1\right.$ is similar), set $c_{\ell}\left(H_{2}\right)=$ $a_{3} a_{2} 1 a_{2} a_{3} a_{1} a_{1} a_{2}$ and $\{1\} \subseteq T\left(v_{3}\right)$; if $U\left(v_{4}\right) \geq 1$, set $c_{\ell}\left(H_{2}\right)=a_{1} a_{3} a_{2} 1 a_{2} a_{3} a_{3} a_{1}$ and $\{1\} \subseteq T\left(v_{4}\right)$; if $U\left(v_{6}\right) \geq 2\left(U\left(v_{5}\right) \geq 2\right.$ is similar), set $c_{\ell}\left(H_{2}\right)=a_{3} 1 a_{2} 2 a_{2} a_{3} a_{1} 1$ and $\{1,2\} \subseteq T\left(v_{3}\right)$.

If $G \in \mathcal{H}_{3}^{*} \backslash \mathcal{H}_{3}$, then if $U\left(v_{2}\right) \geq 2$, set $c_{\ell}\left(H_{3}\right)=a_{1} 2 a_{2} a_{3} 2 a_{2} a_{1} 1$ and $\{1,2\} \subseteq$ $T\left(v_{2}\right)$; if $U\left(v_{5}\right) \geq 2$, set $c_{\ell}\left(H_{3}\right)=2 a_{2} a_{3} 1 a_{2} a_{1} 1 a_{3}$ and $\{1,2\} \subseteq T\left(v_{5}\right)$; if $U\left(v_{5}\right)+$ $U\left(v_{6}\right) \geq 2$, set $c_{\ell}\left(H_{3}\right)=1 a_{1} 2 a_{2} a_{3} a_{1} a_{2} a_{3}$ and $\{1\} \subseteq T\left(v_{6}\right),\{2\} \subseteq T\left(v_{5}\right)$; if $U\left(v_{3}\right) \geq 1$, set $c_{\ell}\left(H_{3}\right)=a_{1} a_{2} a_{2} 1 a_{1} a_{3} a_{3} a_{2}$ and $\{1\} \subseteq T\left(v_{3}\right)$; if $U\left(v_{4}\right) \geq 1$, set $c_{\ell}\left(H_{3}\right)=a_{2} a_{1} 1 a_{1} a_{2} a_{3} a_{3} a_{2}$ and $\{1\} \subseteq T\left(v_{4}\right)$.

If $G \in \mathcal{H}_{4}^{*} \backslash \mathcal{H}_{4}$, then there are four cases for the graph $G$ :
(1) $U\left(v_{i}\right) \geq 3(i=2,3,4)$;
(2) $U\left(v_{i}\right) \geq 2(i=1,5)$;
(3) $U\left(v_{i}\right)+U\left(v_{j}\right) \geq 2(i \in\{1,5\}, j \in\{2,3,4\})$;
(4) $U\left(v_{i}\right) \geq 2$ and $U\left(v_{j}\right) \geq 2, i, j \in\{2,3,4\}$.

If $G$ is a graph in Case (1), then there exists a graph in $\mathcal{G}_{3}^{*} \backslash \mathcal{G}_{3}$ which is a subgraph of $G$. Thus the result is obvious. If $U\left(v_{1}\right) \geq 2$, set $c_{\ell}\left(H_{4}\right)=a_{2} a_{2} a_{1} a_{1} 21 a_{2}$ and $\{1,2\} \subseteq T\left(v_{1}\right)$; if $U\left(v_{1}\right)+U\left(v_{2}\right) \geq 2$, set $c_{\ell}\left(H_{4}\right)=a_{1} a_{2} a_{2} 2 a_{2} 1 a_{1}$ and $\{1\} \subseteq T\left(v_{1}\right)$, $\{2\} \subseteq T\left(v_{2}\right)$; if $U\left(v_{2}\right) \geq 2$ and $U\left(v_{4}\right) \geq 2$, set $c_{\ell}\left(H_{4}\right)=3421 a_{2} a_{2} a_{1}$ and $\{1,2\} \subseteq$ $T\left(v_{2}\right),\{3,4\} \subseteq T\left(v_{4}\right)$.

If $G \in \mathcal{H}_{5}^{*} \backslash \mathcal{H}_{5}$, then $U\left(v_{i}\right) \geq 1(i=2,4,6)$ or $U\left(v_{i}\right) \geq 2(i=1,3,5)$. If $G$ is a graph in the former case, there exists a graph in $\mathcal{G}_{6}^{*} \backslash \mathcal{G}_{6}$ which is a subgraph of $G$. If $U\left(v_{3}\right) \geq 2$, set $c_{\ell}\left(H_{5}\right)=a_{2} 2 a_{1} a_{2} a_{3} 1 a_{3} a_{2}$ and $\{1,2\} \subseteq T\left(v_{3}\right)$; if $U\left(v_{5}\right) \geq 2$, set $c_{\ell}\left(H_{5}\right)=1 a_{2} 2 a_{1} a_{3} a_{2} a_{3} a_{1}$ and $\{1,2\} \subseteq T\left(v_{5}\right)$.

If $G \in \mathcal{H}_{6}^{*} \backslash \mathcal{H}_{6}$, then $U\left(v_{3}\right) \geq 1$ or $U\left(v_{i}\right) \geq 2(i=1,2,4,5)$ or $U\left(v_{1}\right)+U\left(v_{5}\right) \geq$ 2. If $U\left(v_{3}\right) \geq 1$, set $c_{\ell}\left(H_{6}\right)=a_{2} a_{1} 1 a_{1} a_{2} a_{2} a_{1}$ and $\{1\} \subseteq T\left(v_{3}\right)$; if $U\left(v_{1}\right) \geq 2$, set $c_{\ell}\left(H_{6}\right)=a_{2} a_{1} a_{1} 1 a_{2} 12$ and $\{1,2\} \subseteq T\left(v_{1}\right)$; if $U\left(v_{2}\right) \geq 2$, set $c_{\ell}\left(H_{6}\right)=a_{2} 1 a_{1} a_{1} a_{2} 12$ and $\{1,2\} \subseteq T\left(v_{2}\right)$; if $U\left(v_{1}\right)+U\left(v_{5}\right) \geq 2$, set $c_{\ell}\left(H_{6}\right)=a_{1} a_{1} a_{2} a_{2} 21 a_{1}$ and $\{1\} \subseteq$ $T\left(v_{1}\right),\{2\} \subseteq T\left(v_{5}\right)$.

If $G \in \mathcal{H}_{7}^{*} \backslash \mathcal{H}_{7}$, then $U\left(v_{i}\right) \geq 2 \quad(i=1,2,3,4,5)$ or $U\left(v_{1}\right)+U\left(v_{2}\right) \geq 2$ or $U\left(v_{2}\right)+U\left(v_{3}\right) \geq 2$ or $U\left(v_{4}\right)+U\left(v_{5}\right) \geq 2$ or $U\left(v_{2}\right)+U\left(v_{4}\right) \geq 2$ or $U\left(v_{3}\right)+$
$U\left(v_{5}\right) \geq 2$ or $U\left(v_{1}\right)+U\left(v_{5}\right) \geq 2$. If $U\left(v_{1}\right) \geq 2$, set $c_{\ell}\left(H_{7}\right)=a_{1} a_{2} a_{2} 12 a_{1} a_{1}$ and $\{1,2\} \subseteq T\left(v_{3}\right)$; if $U\left(v_{2}\right) \geq 2$, set $c_{\ell}\left(H_{7}\right)=a_{2} a_{1} a_{1} a_{1} a_{2} 12$ and $\{1,2\} \subseteq T\left(v_{2}\right)$; if $U\left(v_{3}\right) \geq 2$, set $c_{\ell}\left(H_{7}\right)=a_{2} a_{1} a_{1} 2 a_{1} a_{2} 1$ and $\{1,2\} \subseteq T\left(v_{3}\right)$; if $U\left(v_{1}\right)+U\left(v_{2}\right) \geq 2$, set $c_{\ell}\left(H_{7}\right)=a_{2} a_{1} a_{1} a_{1} a_{2} 21$ and $\{1\} \subseteq T\left(v_{1}\right),\{2\} \subseteq T\left(v_{2}\right)$; if $U\left(v_{2}\right)+U\left(v_{3}\right) \geq 2$, set $c_{\ell}\left(H_{7}\right)=a_{2} a_{1} a_{1} 2 a_{2} a_{2} 1$ and $\{1\} \subseteq T\left(v_{2}\right),\{2\} \subseteq T\left(v_{3}\right)$; if $U\left(v_{2}\right)+U\left(v_{4}\right) \geq 2$, set $c_{\ell}\left(H_{7}\right)=a_{2} 12 a_{1} a_{2} a_{1} a_{2}$ and $\{1\} \subseteq T\left(v_{2}\right),\{2\} \subseteq T\left(v_{4}\right)$.

If $G \in \mathcal{H}_{8}^{*} \backslash \mathcal{H}_{8}$, then $U\left(v_{i}\right) \geq 3 \quad(i=1,2,3,4)$ or $U\left(v_{i}\right)+U\left(v_{j}\right)+U\left(v_{k}\right) \geq$ $4, i, j, k=1,2,3,4$. If $G$ is a graph in the former case, then a graph belonging to $\mathcal{G}_{4}^{*} \backslash \mathcal{G}_{4}$ is a subgraph of $G$. If $U\left(v_{1}\right)+U\left(v_{2}\right)+U\left(v_{4}\right) \geq 4$, set $c_{\ell}\left(H_{8}\right)=1 a_{1} a_{1} 423$ and $\{1,2\} \subseteq T\left(v_{1}\right),\{3\} \subseteq T\left(v_{2}\right),\{4\} \subseteq T\left(v_{4}\right)$; if $U\left(v_{2}\right)+U\left(v_{3}\right) \geq 4$, set $c_{\ell}\left(H_{8}\right)=$ $12 a_{1} a_{1} 34$ and $\{1,2\} \subseteq T\left(v_{2}\right),\{3,4\} \subseteq T\left(v_{3}\right)$.

It is not hard to check that the colorings above make $G$ rainbow connected with $n-3$ colors, thus $r x_{3}(G) \leq n-3$.

Conversely, let $G$ be a tricyclic graph such that $G \in \mathcal{H}$. Similar to Lemma 3.3 , we only need to consider the case that $G$ has cut edges and $\left|A_{1} \cap A_{2}\right| \geq 1$. Assume, to the contrary, that $r x_{3}(G) \leq n-3$. Then there exists a rainbow coloring $c$ of $G$ using colors in $N_{n-3}$.

For $G \in \mathcal{H}_{1}$, if there is a nontrivial path $P^{\prime}$ connecting $B_{1}$ and $B_{2}$ in $G$, then denote its ends by $v_{3}^{\prime}\left(\in B_{1}\right)$ and $v_{3}^{\prime \prime}\left(\in B_{2}\right)$ and if there is a nontrivial path $P^{\prime \prime}$ connecting $B_{2}$ and $B_{3}$ in $G$, then denote its ends by $v_{5}^{\prime}\left(\in B_{2}\right)$ and $v_{5}^{\prime \prime}\left(\in B_{3}\right)$. Similar to Claim 2 in Lemma 3.3, the colors in the path $P^{\prime}$ and $P^{\prime \prime}$ cannot appear in $A_{1}$, which implies $\left|A_{1} \cap A_{2}\right|=0$, contradicting $\left|A_{1} \cap A_{2}\right| \geq 1$. For $G \in \mathcal{H}_{5}$, notice that $d_{H_{5}}\left(v_{1}, v_{3}, v_{5}\right)=4$ and $\left|A_{1} \backslash A_{2}\right|=3$, the result holds. The same argument applies to the case when $G \in \mathcal{H}_{6}$. Thus, we mainly discuss the rest cases for $G$ as follows.

Case 1. $G \in \mathcal{H}_{2}$. We have $1 \leq\left|A_{1} \cap A_{2}\right| \leq 4$ and $\left|A_{1} \backslash A_{2}\right|=3$. If there is a nontrivial path $P^{\prime}$ connecting $B_{1}$ and $B_{2}$ in $G$, then denote its ends by $v_{4}^{\prime}\left(\in B_{1}\right)$ and $v_{4}^{\prime \prime}\left(\in B_{2}\right)$. We can also claim that $c\left(P^{\prime}\right) \cap A_{1}=\emptyset$. Noticing that $d_{H_{2}}\left(v_{2}, v_{5}, v_{6}\right)=4>3$, we only check the case when $W\left(v_{2}\right) \geq 2$. Since the case of $\left|A_{1} \cap A_{2}\right|=1$ or $\left|A_{1} \cap A_{2}\right|=4$ is easy to check, we consider the remaining two cases. Assume $\left|A_{1} \cap A_{2}\right|=2, A_{1}=\left\{1,2, a_{1}, a_{2}, a_{3}\right\}$ and $1,2 \in T\left(v_{2}\right)$, consider $\left\{y_{1}, y_{2}, v_{5}\right\}$ and we may set $c\left(v_{2} v_{3}\right)=a_{1}, c\left(v_{3} v_{4}^{\prime}\right)=a_{2}, c\left(v_{4}^{\prime \prime} v_{5}\right)=a_{3}$. If 1 and 2 are in $B_{1}$, and 1 appears in $v_{1} v_{2}$ or $v_{1} v_{4}^{\prime}$, then we have $c\left(v_{4}^{\prime \prime} v_{6}\right)=a_{3}$ by considering $\left\{y_{1}, y_{2}, v_{6}\right\}$, and thus $c\left(v_{5} v_{6}\right) \notin A_{2}$, but now $\left\{y_{1}, v_{5}, v_{6}\right\}$ has no rainbow tree. So one of 1,2 , say 1 , is in $B_{2}$ and $c\left(v_{5} v_{6}\right)=1$. Now we have $c\left(v_{4}^{\prime \prime} v_{6}\right) \neq a_{3}$ and $c\left(v_{1} v_{2}\right), c\left(v_{1} v_{4}^{\prime}\right), c\left(v_{4}^{\prime \prime} v_{6}\right) \in\left\{a_{1}, a_{2}, a_{3}\right\}$ by considering $\left\{y_{1}, y_{2}, v_{6}\right\}$. Then every $\left\{y_{1}, v_{5}, v_{6}\right\}$-tree of size 5 cannot have the color 2 . Thus there is no rainbow $\left\{y_{1}, v_{5}, v_{6}\right\}$-tree.

Assume then $\left|A_{1} \cap A_{2}\right|=3$ and $A_{1}=\left\{1,2, a_{1}, a_{2}, a_{3}\right\}$. If $1,2,3 \in T\left(v_{2}\right)$, first we claim that $v_{2} v_{3}, v_{3} v_{4}^{\prime}$ cannot use colors from $A_{2}$ both. Otherwise assume $c\left(v_{2} v_{3}\right)=1, c\left(v_{3} v_{4}\right)=2, c\left(v_{1} v_{2}\right)=a_{1}, c\left(v_{1} v_{4}^{\prime}\right)=a_{2}$, and by considering
$\left\{y_{1}, y_{2}, v_{5}\right\}$ and $\left\{y_{1}, y_{2}, v_{6}\right\}$, we have $c\left(v_{4}^{\prime \prime} v_{5}\right)=c\left(v_{4}^{\prime \prime} v_{6}\right)=a_{3}$, and $c\left(v_{5} v_{6}\right)$ can be $a_{1}$ or $a_{2}$. However, there is no rainbow $\left\{y_{1}, v_{5}, v_{6}\right\}$-tree or $\left\{y_{2}, v_{5}, v_{6}\right\}$-tree. With the same reason, we conclude that exactly one edge of the unique 4 -cycle of $H_{2}$ can be colored with a color from $A_{2}$. Thus there are two cases by symmetry. If $c\left(v_{1} v_{2}\right)=1, c\left(v_{1} v_{3}\right)=2, c\left(v_{5} v_{6}\right)=3$, then consider $y_{1}$ and $y_{2}$, together with $v_{5}, v_{6}$ respectively, we have $c\left(v_{4}^{\prime \prime} v_{5}\right)=c\left(v_{4}^{\prime \prime} v_{6}\right)$, which is impossible. If $c\left(v_{1} v_{4}^{\prime}\right)=1, c\left(v_{1} v_{3}\right)=2, c\left(v_{5} v_{6}\right)=3$, then consider $y_{1}$ and $y_{3}$, together with $v_{5}, v_{6}$ respectively. suppose $c\left(v_{4}^{\prime \prime} v_{5}\right)=a_{1}, c\left(v_{4}^{\prime \prime} v_{6}\right)=a_{2}$ and $c\left(v_{3} v_{4}^{\prime}\right)=a_{3}$, $c\left(v_{1} v_{2}\right), c\left(v_{2} v_{3}\right) \in\left\{a_{1}, a_{2}\right\}$, but now there is no rainbow $\left\{y_{3}, v_{5}, v_{6}\right\}$-tree. If $1,2 \in T\left(v_{2}\right), 3 \in T\left(v_{6}\right)$, similarly set $c\left(v_{1} v_{2}\right)=a_{1}, c\left(v_{1} v_{4}^{\prime}\right)=a_{2}, c\left(v_{4}^{\prime \prime} v_{6}\right)=a_{3}$. First we can easily claim that the color 3 cannot appear in $B_{2}$. Thus there are three possibilities for the color 3. If $c\left(v_{3} v_{4}^{\prime}\right)=3$, then consider $\left\{y_{1}, y_{3}, v_{3}\right\}$ and $\left\{y_{2}, y_{3}, v_{3}\right\}$, we have $c\left(v_{1} v_{3}\right), c\left(v_{2} v_{3}\right) \in\{1,2\}$. Consider $\left\{y_{1}, y_{2}, v_{5}\right\}$, one of $c\left(v_{4}^{\prime \prime} v_{5}\right)$ and $c\left(v_{5} v_{6}\right)$ is $a_{3}$, but now there is no rainbow $\left\{y_{2}, y_{3}, v_{5}\right\}$-tree. The case when $c\left(v_{1} v_{3}\right)=3$ is similar to the case of $c\left(v_{3} v_{4}^{\prime}\right)=3$. If $c\left(v_{2} v_{3}\right)=3$, then similarly we get $c\left(v_{1} v_{3}\right), c\left(v_{2} v_{3}\right) \in\{1,2\}$ and one of $c\left(v_{4}^{\prime \prime} v_{5}\right)$ and $c\left(v_{5} v_{6}\right)$ is $a_{3}$. Consider $\left\{y_{1}, y_{3}, v_{5}\right\}$, this forces one of $c\left(v_{4}^{\prime \prime} v_{5}\right)$ and $c\left(v_{5} v_{6}\right)$ is 2, it is impossible.

Case 2. $G \in \mathcal{H}_{3}$. Then $1 \leq\left|A_{1} \cap A_{2}\right| \leq 4$ and $\left|A_{1} \backslash A_{2}\right|=3$. If there is a nontrivial path $P^{\prime}$ connecting $B_{1}$ and $B_{2}$ in $G$, then denote its ends by $v_{4}^{\prime}\left(\in B_{1}\right)$ and $v_{4}^{\prime \prime}\left(\in B_{2}\right)$. Similarly, it is easy to check that $c\left(P^{\prime}\right) \cap A_{1}=\emptyset$. We only focus on the case that $\left|A_{1} \cap A_{2}\right|=1$, where $A_{1}=\left\{1, a_{1}, a_{2}, a_{3}\right\}$. If $1 \in T\left(v_{6}\right)$, consider $\left\{y_{1}, v_{1}, v_{3}\right\}$, set $c\left(v_{4}^{\prime \prime} v_{6}\right)=a_{1}, c\left(v_{1} v_{4}^{\prime}\right)=a_{2}, c\left(v_{3} v_{4}^{\prime}\right)=a_{3}$. Then $c\left(v_{5} v_{6}\right) \neq 1$, otherwise there is no rainbow tree connecting $\left\{y_{1}, v_{1}, v_{5}\right\}$ or $\left\{y_{1}, v_{3}, v_{5}\right\}$ depending on $c\left(v_{4}^{\prime \prime} v_{5}\right)$. Similarly, $c\left(v_{4}^{\prime \prime} v_{5}\right) \neq 1$. Next $c\left(v_{2} v_{4}^{\prime}\right) \neq 1$ by considering $\left\{y_{1}, v_{2}, v_{5}\right\}$. Suppose $c\left(v_{1} v_{2}\right)=1$, to make sure there is a rainbow $\left\{y_{1}, v_{1}, v_{2}\right\}$-tree and $\left\{y_{1}, v_{2}, v_{3}\right\}$-tree, we have $c\left(v_{2} v_{4}^{\prime}\right)=a_{3}$ and $c\left(v_{2} v_{3}\right)=a_{2}$. But now $\left\{v_{2}, v_{3}, v_{5}\right\}$ has no rainbow tree. If $1 \in T\left(v_{2}\right)$, then consider $\left\{y_{1}, v_{5}, v_{6}\right\}$, $\left\{y_{1}, v_{1}, v_{5}\right\},\left\{y_{1}, v_{1}, v_{6}\right\},\left\{y_{1}, v_{3}, v_{5}\right\},\left\{y_{1}, v_{3}, v_{6}\right\}$ successively. Set $c\left(v_{2} v_{4}^{\prime}\right)=a_{1}$, then $c\left(v_{1} v_{2}\right), c\left(v_{1} v_{4}^{\prime}\right), c\left(v_{2} v_{3}\right), c\left(v_{3} v_{4}^{\prime}\right), c\left(v_{4}^{\prime \prime} v_{5}\right)$ and $c\left(v_{4}^{\prime \prime} v_{6}\right)$ can only be $a_{2}$ or $a_{3}$. It is easy to check that $\left\{v_{2}, v_{3}, v_{5}\right\}$ has no rainbow tree.

Case 3. $G \in \mathcal{H}_{4} . \quad 1 \leq\left|A_{1} \cap A_{2}\right| \leq 4$ and $\left|A_{1} \backslash A_{2}\right|=2$. First notice that $d_{H_{4}}\left(v_{2}, v_{3}, v_{4}\right)=3$, the case that $W\left(v_{1}\right)=W\left(v_{5}\right)=0, W\left(v_{2}\right), W\left(v_{3}\right)$, $W\left(v_{4}\right) \leq 1$ is evident. Assume $\left|A_{1} \cap A_{2}\right|=1$, then $1 \in T\left(v_{1}\right)$, the case is similar with the case that $G \in \mathcal{G}_{3}$ in Lemma 3.3. Assume now $\left|A_{1} \cap A_{2}\right|=2$, if $1 \in T\left(v_{1}\right), 2 \in T\left(v_{5}\right)$, by considering all the trees containing $y_{1}$ and $y_{2}$, without loss of generality, set $c\left(v_{1} v_{5}\right)=a_{1}, c\left(v_{1} v_{2}\right)=1(2), c\left(v_{2} v_{5}\right)=a_{2}$. Moreover, by considering $\left\{y_{1}\left(y_{2}\right), v_{2}, v_{3}\right\}$ and $\left\{y_{1}\left(y_{2}\right), v_{2}, v_{4}\right\}$, the remaining two paths of length 2 from $v_{1}$ to $v_{5}$ must be colored with 2(1), $a_{2}$, respectively. However, there is no rainbow $\left\{y_{2}\left(y_{1}\right), v_{3}, v_{4}\right\}$-tree. If $1,2 \in T\left(v_{2}\right)$, by considering $\left\{y_{1}, y_{2}, v_{4}\right\}$, set $c\left(v_{2} v_{5}\right)=a_{1}, c\left(v_{4} v_{5}\right)=a_{2}$. Since the two possible rainbow trees connecting $\left\{y_{1}, v_{3}, v_{4}\right\}$ and $\left\{y_{2}, v_{3}, v_{4}\right\}$ are the same, we may set $c\left(v_{3} v_{5}\right)=1$. It is easy to see
that $c\left(v_{1} v_{2}\right), c\left(v_{1} v_{3}\right)$ cannot use colors from $A_{2}$ by considering $\left\{y_{1}, y_{2}, v_{3}\right\}$, and $c\left(v_{1} v_{4}\right)=2$ by considering $\left\{y_{1}, v_{3}, v_{4}\right\}$. But now if $c\left(v_{1} v_{5}\right)=1$ or $c\left(v_{1} v_{5}\right)=2$, there is no rainbow $\left\{y_{1}, v_{2}, v_{5}\right\}$-tree or $\left\{y_{2}, v_{1}, v_{4}\right\}$-tree, respectively.

Assume $\left|A_{1} \cap A_{2}\right|=3$, then $1,2 \in T\left(v_{2}\right), 3 \in T\left(v_{4}\right)$. Similarly as above, we may set $c\left(v_{2} v_{5}\right)=a_{1}, c\left(v_{4} v_{5}\right)=a_{2}, c\left(v_{3} v_{5}\right)=1, c\left(v_{1} v_{5}\right)=3$, one of $c\left(v_{1} v_{2}\right)$, $c\left(v_{1} v_{4}\right)$ is 2 . However, there is no rainbow $\left\{y_{2}, y_{3}, v_{1}\right\}$-tree.

Finally assume $\left|A_{1} \cap A_{2}\right|=4$ and $1,2 \in T\left(v_{2}\right), 3 \in T\left(v_{3}\right), 4 \in T\left(v_{4}\right)$, consider $\left\{y_{1}, y_{3}, y_{4}\right\}$ and $\left\{y_{2}, y_{3}, y_{4}\right\}$, at least four of the non-cut edges must be colored with $\left\{a_{1}, a_{2}\right\}$. This contradicts $\left|A_{1} \cap A_{2}\right|=4$.

Case 4. $G \in \mathcal{H}_{7}$. Since $d_{H_{7}}\left(v_{1}, v_{3}, v_{4}\right)=3$, we only focus on the case $1 \in T\left(v_{2}\right)$. Consider all the three vertices containing $y_{1}$, it is not hard to obtain a contradiction.

Case 5. $G \in \mathcal{H}_{8}$. First notice $d_{H_{8}}\left(v_{1}, v_{2}, v_{3}\right)=2$, the case that $W\left(v_{1}\right), W\left(v_{2}\right)$, $W\left(v_{3}\right), W\left(v_{4}\right) \leq 1$ is evident. Assume $\left|A_{1} \cap A_{2}\right|=2$, then $1,2 \in T\left(v_{1}\right)$. Consider $\left\{y_{1}, y_{2}, v_{2}\right\},\left\{y_{1}, y_{2}, v_{3}\right\},\left\{y_{1}, y_{2}, v_{4}\right\}$ successively, we have $c\left(v_{1} v_{2}\right)=c\left(v_{1} v_{3}\right)=$ $c\left(v_{1} v_{4}\right)=a_{1}$. However, there is no rainbow tree connecting $\left\{y_{1}, v_{2}, v_{3}\right\}$ or $\left\{y_{2}, v_{2}, v_{3}\right\}$, a contradiction. Now focus on $\left|A_{1} \cap A_{2}\right|=3$, then $1,2 \in T\left(v_{1}\right)$, $3 \in T\left(v_{2}\right)$. Consider $\left\{y_{1}, y_{3}, v_{3}\right\},\left\{y_{1}, y_{3}, v_{4}\right\},\left\{y_{2}, y_{3}, v_{3}\right\}$ and $\left\{y_{2}, y_{3}, v_{4}\right\}$ successively, $c\left(v_{1} v_{3}\right), c\left(v_{1} v_{4}\right), c\left(v_{2} v_{3}\right), c\left(v_{2} v_{4}\right)$ must be 1 or 2 . Again, there is no rainbow $\left\{y_{1}, y_{2}, v_{3}\right\}$-tree.

By the detailed analysis above, we have $r x_{3}(G) \geq n-2$ for $G \in \mathcal{H}$. By Theorem 3.1, it follows that $r x_{3}(G)=n-2$.

### 3.2.3. Characterize the graphs with $r x_{3}(G)=n-2$

We begin with a lemma about a connected 5-cyclic graph.
Lemma 3.5. Let $G$ be a connected 5 -cyclic graph of order $n$. Then $r x_{3}(G)=n-2$ if and only if $G=K_{5}-e$.

Proof. Let $G \neq K_{5}-e$ and $r x_{3}(G)=n-2$, by Lemma 2.3 and Theorem 3.2, $r x_{3}(G) \leq n-3$, a contradiction. Conversely, suppose $G=K_{5}-e$, by Theorem 1.6, $r x_{3}(G) \geq 3$, on the other hand, $r x_{3}(G) \leq r x_{3}\left(C_{5}\right)=3$. Thus $r x_{3}(G)=n-2$.

For $n \geq 3$, the wheel $W_{n}$ is a graph constructed by joining a vertex $v_{0}$ to every vertex of a cycle $C_{n}: v_{1}, v_{2}, \ldots, v_{n}, v_{n+1}=v_{1}$.

A third graph class is defined as follows. Let $\mathcal{J}_{1}$ be a class of graphs such that every graph is obtained from a graph in $\mathcal{H}_{5}$ by adding an edge $v_{4} v_{6}$. Let $\mathcal{J}_{2}$ be a class of graphs such that every graph is obtained from a graph in $\mathcal{H}_{7}$ where $U\left(v_{2}\right)=0$ and $U\left(v_{5}\right)=0$ by adding an edge $v_{2} v_{5}$. Set $\mathcal{J}=\left\{\mathcal{J}_{1}, \mathcal{J}_{2}, W_{4}\right\}$.

Now we are ready to show our second main theorem of this paper.

Theorem 3.6. Let $G$ be a connected graph of order $n(n \geq 6)$. Then $r x_{3}(G)=$ $n-2$ if and only if $G$ is unicyclic with the girth at least 4 or $G \in \mathcal{G} \cup \mathcal{H} \cup \mathcal{J}$ or $G=K_{5}-e$.

Proof. Let $G$ be a $t$-cyclic graph with $r x_{3}(G)=n-2$, but not a graph listed in the theorem. By Proposition 1.2, Theorem 1.3, Lemma 3.3 and Lemma 3.4, we need to consider the cases $t \geq 4$. If $t=4$, by Theorem 3.2, the basic graph of $G$ should be a 3 -sun or the basic graph of $\mathcal{J}_{2}$ or $W_{4}$. If $G \notin \mathcal{J}_{1}$ or $G \notin \mathcal{J}_{2}$, then by the similar arguments with Lemma 3.3, we have $r x_{3}(G) \leq n-3$, a contradiction. Suppose the basic graph of $G$ is $W_{4}$ and there are some cut edges in $G$. If $U\left(v_{0}\right) \geq$ 1 , then a graph belonging to $\mathcal{G}_{6}^{*} \backslash \mathcal{G}_{6}$ and satisfying $U\left(v_{3}\right) \geq 1$ is a subgraph of $G$. If $U\left(v_{1}\right) \geq 1$ (other cases are similar), then set $c_{\ell}\left(W_{4}\right)=a_{2} 1 a_{1} a_{1} a_{1} a_{2} a_{2} a_{1}$ and $\{1\} \subseteq T\left(v_{1}\right)$. If $t \geq 5$, by Theorem 3.2, the basic graph of $G$ should be $K_{5}-e$, since $n \geq 6$, by the similar argument with $t=4$, we have $r x_{3}(G) \leq n-3$, a contradiction.

Conversely, by Theorem 1.3, Theorem 1.6, Lemma 3.3, Lemma 3.4 and Lemma 3.5, suppose $G$ is a graph such that $G \in \mathcal{J}_{1}$ or $G \in \mathcal{J}_{2}$. Assume, to the contrary, that $r x_{3}(G) \leq n-3$. Then there exists a rainbow coloring $c$ of $G$ using $n-3$ colors. Both cases can be considered similar to the case that $G \in \mathcal{H}_{5}$ or $G \in \mathcal{H}_{7}$ in Lemma 3.4, a contradiction.

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