*Discussiones Mathematicae*
*Graph Theory 35 (2015) 105–120*
*doi:10.7151/dmgt.1783*

# GRAPHS WITH 3-RAINBOW INDEX $n-1$ AND $n-2$

Xueliang Li[1,2,(a)], Ingo Schiermeyer[(b)],

Kang Yang[1,(a)] and Yan Zhao[1,(a)]

[(a)] *Center for Combinatorics and LPMC-TJKLC*
*Nankai University*
*Tianjin 300071, China*

[(b)] *Institut für Diskrete Mathematik und Algebra*
*Technische Universität Bergakademie Freiberg*
*09596 Freiberg, Germany*

**e-mail:** lxl@nankai.edu.cn
Ingo.Schiermeyer@tu-freiberg.de
yangkang@mail.nankai.edu.cn
zhaoyan2010@mail.nankai.edu.cn

**Abstract**

Let $G = (V(G), E(G))$ be a nontrivial connected graph of order $n$ with an edge-coloring $c : E(G) \to \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A tree $T$ in $G$ is a *rainbow tree* if no two edges of $T$ receive the same color. For a vertex set $S \subseteq V(G)$, a tree connecting $S$ in $G$ is called an $S$-tree. The minimum number of colors that are needed in an edge-coloring of $G$ such that there is a rainbow $S$-tree for each $k$-subset $S$ of $V(G)$ is called the $k$-rainbow index of $G$, denoted by $rx_k(G)$, where $k$ is an integer such that $2 \leq k \leq n$. Chartrand *et al.* got that the $k$-rainbow index of a tree is $n-1$ and the $k$-rainbow index of a unicyclic graph is $n-1$ or $n-2$. So there is an intriguing problem: Characterize graphs with the $k$-rainbow index $n-1$ and $n-2$. In this paper, we focus on $k = 3$, and characterize the graphs whose 3-rainbow index is $n-1$ and $n-2$, respectively.

**Keywords:** rainbow $S$-tree, $k$-rainbow index.

**2010 Mathematics Subject Classification:** 05C05, 05C15, 05C75.

---

[1]Supported by NSFC Nos. 11371205 and 11071130.
[2]Corresponding author.

## 1. Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notations of Bondy and Murty [1]. Let $G = (V(G), E(G))$ be a nontrivial connected graph of order $n$ with an edge-coloring $c : E(G) \to \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A path of $G$ is a *rainbow path* if every two edges of the path have distinct colors. The graph $G$ is *rainbow connected* if for every two vertices $u$ and $v$ of $G$, there is a rainbow path connecting $u$ and $v$. The minimum number of colors for which there is an edge coloring of $G$ such that $G$ is rainbow connected is called the *rainbow connection number*, denoted by $rc(G)$. Results on the rainbow connectivity can be found in [2, 5, 6, 7, 8, 9]. In the sequel we will simply denote the order of a graph $G$ by $|G|$, i.e., $|G| = |V(G)|$.

These concepts were introduced by Chartrand *et al.* in [2]. In [3], they generalized the concept of rainbow path to rainbow tree. A tree $T$ in $G$ is a *rainbow tree* if no two edges of $T$ receive the same color. For $S \subseteq V(G)$, a *rainbow $S$-tree* is a rainbow tree connecting $S$. Given a fixed integer $k$ with $2 \leq k \leq n$, the edge-coloring $c$ of $G$ is called a *$k$-rainbow coloring* of $G$ if for every $k$-subset $S$ of $V(G)$, there exists a rainbow $S$-tree. In this case, $G$ is called *$k$-rainbow connected*. The minimum number of colors that are needed in a *$k$-rainbow coloring* of $G$ is called the *$k$-rainbow index* of $G$, denoted by $rx_k(G)$. Clearly, when $k = 2$, $rx_2(G)$ is the rainbow connection number $rc(G)$ of $G$. For every connected graph $G$ of order $n$, it is easy to see that $rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$.

The *Steiner distance* $d_G(S)$ of a set $S$ of vertices in $G$ is the minimum size of a tree in $G$ connecting $S$. The *$k$-Steiner diameter* $sdiam_k(G)$ of $G$ is the maximum Steiner distance of $S$ among all sets $S$ with $k$ vertices in $G$. Then there is a simple upper bound and lower bound for $rx_k(G)$.

**Observation 1.1** [3]**.** *For every connected graph $G$ of order $n \geq 3$ and each integer $k$ with $3 \leq k \leq n$, $k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1$.*

They showed that the $k$-rainbow index of trees attains the upper bound.

**Proposition 1.2** [3]**.** *Let $T$ be a tree of order $n \geq 3$. For each integer $k$ with $3 \leq k \leq n$, $rx_k(T) = n - 1$.*

They also showed that the $k$-rainbow index of a unicyclic graph is $n - 1$ or $n - 2$.

**Theorem 1.3** [3]**.** *If $G$ is a unicyclic graph of order $n \geq 3$ and girth $g \geq 3$, then*

$$rx_k(G) = \begin{cases} n-2, & k = 3 \quad \textit{and} \quad g \geq 4, \\ n-1, & g = 3 \quad \textit{or} \quad 4 \leq k \leq n. \end{cases} \tag{1}$$

A natural thought is that which graph of order $n$ has the $k$-rainbow index $n-1$ except for a tree and a unicyclic graph of girth 3? Furthermore, which graph of order $n$ has the $k$-rainbow index $n-2$ except for a unicyclic graph of girth at least 4? In this paper, we focus on $k=3$. In addition, some known results are mentioned.

**Observation 1.4** [3]**.** *Let $G$ be a connected graph of order $n$ containing two bridges $e$ and $f$. For each integer $k$ with $2 \leq k \leq n$, every $k$-rainbow coloring of $G$ must assign distinct colors to $e$ and $f$.*

**Lemma 1.5** [4]**.** *Let $G$ be a connected graph and $\{V_1, V_2, \ldots, V_k\}$ a partition of $V(G)$. If each $V_i$ induces a connected subgraph $H_i$ of $G$, then $rx_3(G) \leq k-1+\sum_{i=1}^{k} rx_3(H_i)$.*

**Theorem 1.6** [4]**.** *Let $G$ be a connected graph of order $n$. Then $rx_3(G)=2$ if and only if $G=K_5$ or $G$ is a 2-connected graph of order 4 or $G$ is of order 3.*

**Observation 1.7** [4]**.** *Let $G$ be a connected graph of order $n$, and $H$ be a connected spanning subgraph of $G$. Then $rx_3(G) \leq rx_3(H)$.*

Let $G$ be a connected graph with $n$ vertices and $m$ edges. Define the *cyclomatic number* of $G$ as $c(G)=m-n+1$. A graph $G$ with $c(G)=k$ is called *$k$-cyclic*. According to this definition, if a graph $G$ meets $c(G)=0$, 1, 2 or 3, then the graph $G$ is called acyclic (or a tree), unicyclic, bicyclic, or tricyclic, respectively.

This paper is organized as follows. In Section 2, some basic results and notations are presented. In Section 3, we characterize the graphs whose 3-rainbow index is $n-1$ and $n-2$, respectively. For the latter case, we take two steps to finish our proof: we deal with it for bicyclic graphs first, and then for tricyclic graphs.

## 2. Some Basic Results

First of all, we need more terminology and notations.

**Definition 2.1.** For a subgraph $H$ of $G$ and $v \in V(G)$, let $d(v,H)=\min\{d_G(v,x): x \in V(H)\}$.

Next we define some new notations.

**Definition 2.2.** Let $G$ be a connected graph of order $n$ with $V(G)=\{v_1, v_2, \ldots, v_n\}$. For any edge $e$ of $G$ with end-vertices $v_k$ and $v_r$, if $k<r$ then we will write $e=v_kv_r$. It is clear that in this way any edge has a unique expression. Then, we define a *lexicographic ordering* between any two edges of $G$ by $v_iv_j < v_sv_t$ if and only if $i<s$ or $i=s$, $j<t$.

Note that, the lexicographic ordering of a connected graph is unique. Given a coloring $c$ of a connected graph $G$, denote by $c_\ell(G)$ a sequence of colors of the edges which are ordered by the lexicographic ordering.

For a connected graph $G$, to *contract* an edge $e = xy$ is to delete $e$ and replace its ends by a single vertex incident to all the edges which were incident to either $x$ or $y$. Let $G^{'}$ be the graph obtained by contracting some edges of $G$. Given a rainbow coloring of $G^{'}$, when it comes back to $G$, we keep the colors of corresponding edges of $G^{'}$ in $G$ and assign a new color to a new edge, which makes $G$ 3-rainbow connected. Hence, the following lemma holds.

**Lemma 2.3.** *Let $G$ be a connected graph, and $G^{'}$ be a connected graph obtained by contracting some edges of $G$. Then $rx_3(G) \leq rx_3(G^{'}) + |G| - |G^{'}|$.*

**Definition 2.4.** Let $G_0$ be the graph obtained by contracting all the cut edges of $G$. Then $G_0$ is called the *basic graph* of $G$.

## 3. Main Results

### 3.1. Characterize the graphs with $rx_3(G) = n - 1$

**Theorem 3.1.** *Let $G$ be a connected graph of order $n$. Then $rx_3(G) = n - 1$ if and only if $G$ is a tree or $G$ is a unicyclic graph with girth 3.*

***Proof.*** If $G$ is a tree or a unicyclic graph with girth 3, then by Proposition 1.2 and Theorem 1.3, $rx_3(G) = n - 1$. Conversely, suppose $G$ is a graph with $rx_3(G) = n-1$ but not a tree. Then $G$ must contain cycles. Let $\{V_1, V_2, \ldots, V_k\}$ be a partition of $V(G)$ and each $V_i$ induces a connected subgraph $H_i$ of $G$. If $G$ contains a cycle of length $r$ at least 4, then let $H_1$ be the $r$-cycle, and each other subgraph a single vertex. We color $H_1$ with $r-2$ dedicated colors, then by Lemma 1.5, $rx_3(G) \leq n-r+rx_3(H_1) = n-2$. Suppose then $G$ contains at least two triangles $C_1$ and $C_2$. If $C_1$ and $C_2$ have a vertex in common, then let $H_1$ be the union of $C_1$ and $C_2$, and each other subgraph a single vertex. We color both $C_1$ and $C_2$ with the same three dedicated colors, thus $rx_3(G) \leq n-5+rx_3(H_1) = n-2$. If $C_1$ and $C_2$ are vertex disjoint, then let $H_1 = C_1$, $H_2 = C_2$, and each other subgraph a single vertex. We color $H_1$ with three new colors and $H_2$ with the same three colors as $H_1$, thus $rx_3(G) \leq n - 5 + rx_3(H_1) + rx_3(H_2) = n - 2$. Combining the above two cases, $G$ is a unicyclic graph with girth 3. Therefore, the result holds. ■

### 3.2. Characterize the graphs with $rx_3(G) = n - 2$

Next, we characterize the graphs whose 3-rainbow index is $n-2$. We begin with a useful theorem from [4].

A 3-*sun* is a graph constructed from a cycle $C_6 = v_1v_2\cdots v_6v_1$ by adding three edges $v_2v_4$, $v_2v_6$ and $v_4v_6$.

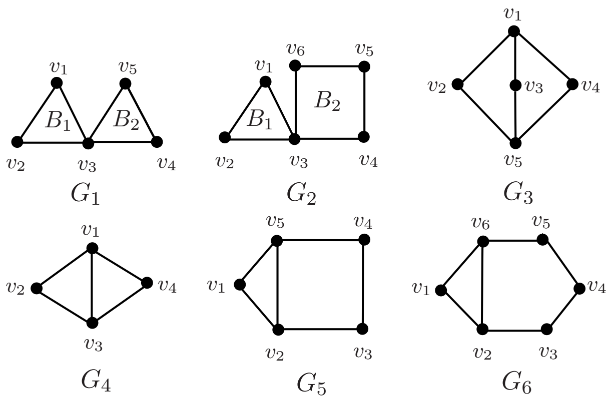


Figure 1. The basic graphs for Lemma 3.3.

**Theorem 3.2** [4]**.** *Let $G$ be a 2-edge-connected graph of order $n$ $(n \geq 4)$. Then $rx_3(G) \leq n-2$, with equality if and only if $G = C_n$ or $G$ is a spanning subgraph of one of the following graphs: a 3-sun, $K_5 - e$, $K_4$, $G_1$, $G_2$, $H_1$, $H_2$, $H_3$, where $G_1$, $G_2$ are defined in Figure 1 and $H_1$, $H_2$, $H_3$ are defined in Figure 2.*

Since all the 2-edge-connected graphs with the 3-rainbow index $n-2$ have been characterized in Theorem 3.2, it remains to characterize the graphs with 3-rainbow index $n-2$ which have cut edges. Notice that the cut edges of a graph must be assigned distinct colors. Thus our main purpose is to check out how the addition of cut edges to $G$ affects the 3-rainbow index of a 2-connected graph $G$ when $rx_3(G) = n-2$. In other words, share the colors of cut edges with the colors of the non-cut edges as many as possible.

Given a connected graph $G$ of order $n$, and a coloring $c$ of $G$, we always let $A_1$ be the set of colors assigned to the non-cut edges of $G$ and $A_2$ the set of colors assigned to the cut edges of $G$. For each positive integer $k$, let $N_k = \{1, 2, \ldots, k\}$. We always set that $A_2 = N_s$, where $s$ is the number of cut edges of $G$. Note that $A_1$ and $A_2$ may intersect and suppose $|A_1 \cap A_2| = p$. We can interchange the colors of cut edges suitably such that $A_1 \cap A_2 = \{1, 2, \ldots, p\}$. Set $A_1 \setminus A_2 = \{a_1, \ldots, a_t\}$, $t \leq m-s$ and $a_j \in N_{|c|}$.

For a connected graph $G$, a *block* is a maximal 2-connected subgraph. In this paper, we regard $K_2$ other than a block. An *internal cut edge* is a cut edge which is on the unique path joining some two blocks. Denote the cut edges of $G$ by $e_1 = x_1y_1, \ldots, e_p = x_py_p$ and the colors of these cut edges by $1, \ldots, p$, respectively. Moreover, if $x_iy_i$ is not an internal cut edge, we always set $d(x_i, B) \leq d(y_i, B)$ where $B$ is an arbitrary block.

Let $H$ be a connected subgraph of $G$. Denote by $i \in H$ if the color $i$ appears in $H$. Given a graph $G$, let $G_0$ be its basic graph. Deleting the corresponding edges of $G_0$ in $G$, we obtain a forest. Each component corresponds to a vertex $v$ in $G_0$, denoted by $T(v)$. Denote by $U(v)$ the number of leaves of $T(v)$ in $G$ and $U(G) = \sum_{v \in V(G)} U(v)$. Let $W(v)$ be the number of edges of $T(v)$ whose colors are appeared in $A_1$, that is, $W(v) = |c(T(v)) \cap A_1|$.

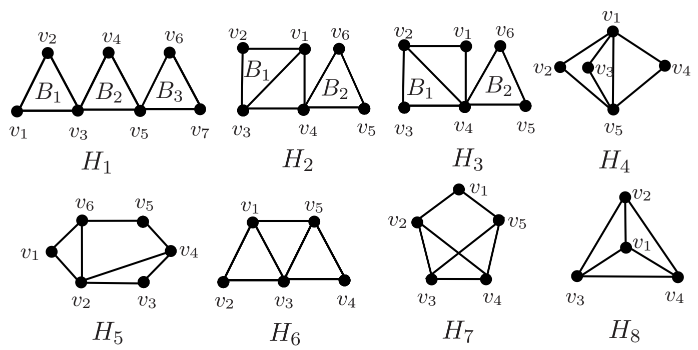


Figure 2. The basic graphs for Lemma 3.4.

#### 3.2.1. Bicyclic graphs with $rx_3(G) = n - 2$

First, we introduce some graph classes. Let $G_i$ be the graphs shown in Figure 1. Define by $\mathcal{G}_i^*$ the set of graphs whose basic graph is $G_i$, where $1 \le i \le 6$. Set

$\mathcal{G}_1 = \{G \in \mathcal{G}_1^* : U(v_3) \le 1\}$,

$\mathcal{G}_2 = \{G \in \mathcal{G}_2^* : U(v_3) + U(v_i) \le 1,\ i = 4, 6\}$,

$\mathcal{G}_3 = \{G \in \mathcal{G}_3^* : U(v_i) + U(v_j) \le 2,\ v_iv_j \in E(G_3)\}$,

$\mathcal{G}_4 = \{G \in \mathcal{G}_4^* : U(v_i) \le 2,\ i = 1, 3\}$,

$\mathcal{G}_5 = \{G \in \mathcal{G}_5^* : U(v_2) + U(v_3) \le 2,\ U(v_4) + U(v_5) \le 2\}$,

$\mathcal{G}_6 = \{G \in \mathcal{G}_6^* : U(v_2) = U(v_6) = 0,\ U(v_4) \le 1,\ U(v_4) + U(v_i) \le 2,\ i = 3, 5\}$,

and set $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_6$.

**Lemma 3.3.** *Let $G$ be a connected bicyclic graph of order $n$. Then $rx_3(G) = n-2$ if and only if $G \in \mathcal{G}$.*

***Proof.*** Suppose that $G$ is a graph with $rx_3(G) = n - 2$ but $G \notin \mathcal{G}$. Let $G_0$ be the basic graph of $G$. Then $G_0$ is a 2-edge-connected bicyclic graph. If $G_0 \ne G_i$, then by Theorem 3.2, $rx_3(G_0) \le |G_0| - 3$. Moreover, by Lemma 2.3, we have $rx_3(G) \le rx_3(G_0) + |G| - |G_0| \le n - 3$. Hence $G_0 = G_i$. Next we show that if $G \in \mathcal{G}_i^* \setminus \mathcal{G}_i$, then $rx_3(G) \le n - 3$, where $1 \le i \le 6$. As pointed out before, all the cut edges of $G$ are colored with $1, 2, \ldots$. We only provide a coloring $c_\ell$ of $G_0$, namely, color the corresponding edges of $G$, with parts of colors used in cut edges, and the position of cut edges will be determined as follows: $\{1,2,\ldots,q\} \subseteq T(v)$ means to assign colors $\{1,2,\ldots,q\}$ to $q$ leaves of $T(v)$ arbitrarily. If $G \in \mathcal{G}_1^* \setminus \mathcal{G}_1$, then $U(v_3) \geq 2$, set $c_\ell(G_1) = 1a_1a_2a_2a_12$ and $\{1,2\} \subseteq T(v_3)$. If $G \in \mathcal{G}_2^* \setminus \mathcal{G}_2$, then $U(v_3)+U(v_4) \geq 2$ or $U(v_3)+U(v_6) \geq 2$. By contracting $v_3v_4$ or $v_3v_6$, we obtain a graph $G'$ belonging to $\mathcal{G}_1^* \setminus \mathcal{G}_1$. Then the coloring of $G$ can be obtained easily from $G'$ by Lemma 2.3. If $G \in \mathcal{G}_3^* \setminus \mathcal{G}_3$, then there is an edge $v_iv_j \in E(G_3)$ such that $U(v_i)+U(v_j) \geq 3$. By symmetry, there exist four cases for $G$: (1) $U(v_1) \geq 3$; (2) $U(v_1) \geq 2$, $U(v_2) \geq 1$; (3) $U(v_1) \geq 1$, $U(v_2) \geq 2$; (4) $U(v_2) \geq 3$. Set $c_\ell(G_3) = a_1a_2a_2123$ and set $\{1,2,3\} \subseteq T(v_1)$ for (1); $\{1\} \subseteq T(v_2)$, $\{2,3\} \subseteq T(v_1)$ for (2); $\{1\} \subseteq T(v_1)$, $\{2,3\} \subseteq T(v_2)$ for (3); $\{1,2,3\} \subseteq T(v_2)$ for (4). If $G \in \mathcal{G}_4^* \setminus \mathcal{G}_4$, then $U(v_1) \geq 3$ or $U(v_3) \geq 3$. By symmetry, suppose $U(v_1) \geq 3$ and set $c_\ell(G_4) = 123a_1a_1$ and $\{1,2,3\} \subseteq T(v_1)$. If $G \in \mathcal{G}_5^* \setminus \mathcal{G}_5$, then by contracting $v_2v_3$ or $v_4v_5$, we obtain a graph $G'$ belonging to $\mathcal{G}_4^* \setminus \mathcal{G}_4$. Now consider $G \in \mathcal{G}_6^* \setminus \mathcal{G}_6$. Then $U(v_2) \geq 1$, or $U(v_6) \geq 1$, or $U(v_4) \geq 2$, or $U(v_4)+U(v_3) \geq 3$, or $U(v_4)+U(v_5) \geq 3$. For the last two cases, it belongs to $\mathcal{G}_5^* \setminus \mathcal{G}_5$ by contracting $v_3v_4$ or $v_4v_5$. If $U(v_2) \geq 1$, set $c_\ell(G_6) = a_3a_2a_4a_4a_2a_31$ and $\{1\} \subseteq T(v_2)$. If $U(v_4) \geq 2$, set $c_\ell(G_6) = a_31a_22a_1a_2a_1$ and $\{1,2\} \subseteq T(v_4)$. It is not hard to check that the colorings above make $G$ rainbow connected with $n-3$ colors, thus $rx_3(G) \leq n-3$.

Conversely, let $G$ be a bicyclic graph such that $G \in \mathcal{G}$. Assume, to the contrary, that $rx_3(G) \leq n-3$. Then there exists a rainbow coloring $c$ such that $A_1 \cup A_2 = N_{n-3}$. By Theorem 3.2, we focus on the graphs with cut edges and $|A_1 \cap A_2| \geq 1$. We write $d_{G_i}(u,v,w)$ to mean that the number of edges of a $\{u,v,w\}$-tree in $G$ which correspond to the edges of $G_i$, the basic graph of $G$. We divide into three cases.

*Case* 1. $G \in \mathcal{G}_1 \cup \mathcal{G}_2$. First, we assume that $G \in \mathcal{G}_2$ and we give the following claims. If there is a nontrivial path $P_\ell$ connecting $B_1$ and $B_2$ in $G$, then denote its ends by $v_3'(\in B_1)$ and $v_3''(\in B_2)$.

**Claim 1.** *Each block $B_i$ has at most one edge which use the color from $A_2$, where $i \in \{1,2\}$. Moreover, if a color of $A_2$ appears in $B_i$, then the other edges of $B_i$ must be assigned with different colors in $A_1 \setminus A_2$.*

***Proof.*** Suppose two edges of $B_1$ are colored with 1, 2, respectively. We also set $d(x_i,B_1) \leq d(y_i,B_1)$, where $x_iy_i$ belongs to $P_\ell$. Since the cut edges colored with 1 and 2 should be contained in the rainbow tree whose vertices contain $y_1$ and $y_2$, by deleting the edges assigned with 1 and 2 in $B_1$, $G$ is disconnected. Let $w$ be a vertex in the component that does not contain $y_1$. Then there is no rainbow tree connecting $\{y_1,y_2,w\}$, a contradiction. We can take the similar argument for the other cases when two edges of $B_1$ ($B_2$) are colored with 1 or two edges of $B_2$ are colored with 1, 2, respectively.

Now suppose $1 \in B_i \cap A_2$ and two edges of $B_i$ have the same color $a_1$. Let

$w_1$, $w_2$ be the end-vertices of the edge assigned with 1, then $\{y_1, w_1, w_2\}$ has no rainbow tree. □

**Claim 2.** *The colors of the path $P_\ell$ cannot appear in $A_1$.*

***Proof.*** Assume $e$ is the edge of $P_\ell$ colored with 1. The color 1 cannot appear in $B_1$. Otherwise, suppose the three edges of $B_1$ are assigned with 1, $a_1$ and $a_2$, respectively. Consider $\{v_1, v_2, v_5\}$, then $c(v_3^{''}v_4), c(v_4v_5) \in \{2, a_3\}$ or $c(v_3^{''}v_6), c(v_5v_6) \in \{2, a_3\}$. Without loss of generality, suppose $c(v_3^{''}v_4), c(v_4v_5) \in \{2, a_3\}$, then by Claim 1, $c(v_3^{''}v_6), c(v_5v_6) \in \{a_1, a_2\}$, thus $\{v_1, v_2, v_6\}$ has no rainbow tree. On the other hand, 1 cannot be in $B_2$. It is easy to see that neither $c(v_3^{''}v_4)$ nor $c(v_3^{''}v_6)$ can be 1 by considering $\{v_1, v_2, v_6\}$. If $c(v_5v_6) = 1$, consider $\{v_1, v_5, v_6\}$, $\{v_2, v_5, v_6\}$, then $c(v_1v_3^{'}), c(v_2v_3^{'}) \in A_2$, a contradiction to Claim 1.□

By Claim 1, we have $1 \leq |A_1 \cap A_2| \leq 2$ and only colors 1 and 2 can exist in $A_1$. We should discuss all the situations according to which cut edges are colored with 1, 2. By the definition of $G$, $U(v_3) = 1$ or $U(v_3) = 0$. By similarity, we only deal with the former case. First, assume $|A_1 \cap A_2| = 1$, then $A_1 = \{1, a_1, a_2, a_3\}$. We consider the subcase when $1 \in T(v_3)$. In this subcase we claim that the color 1 appears in neither $B_1$ nor $B_2$. Indeed, if $c(v_3''v_6) = 1$, since then every tree whose vertices contain $y_1$ must contain the cut edge colored with 1, $d_{G_2}(y_1, v_1, v_6) = 4$. Thus $\{y_1, v_1, v_6\}$ has no rainbow tree. If now $c(v_5v_6) = 1$, then consider $\{y_1, v_5, v_6\}$, $\{y_1, v_5, v_1\}$, $\{y_1, v_5, v_2\}$ successively, we have $c(v_1v_3') = c(v_2v_3') = c(v_3''v_6)$, leading to a contradiction when considering $\{v_1, v_2, v_6\}$. Else if $c(v_1v_3') = 1$, then $\{y_1, v_1, v_5\}$ has no rainbow tree. The last possibility is that $c(v_1v_2) = 1$, we may set $c(v_1v_3) = a_1$, $c(v_2v_3) = a_2$. Consider $\{y_1, v_1, v_4\}$, $\{y_1, v_2, v_4\}$, $\{y_1, v_1, v_6\}$, $\{y_1, v_2, v_6\}$ successively, we have $c(v_3''v_4) = c(v_3''v_6) = a_3$ and 1 cannot appear in $B_2$, hence $\{v_1, v_4, v_6\}$ has no rainbow tree. The other subcases are similar.

Thus $|A_1 \cap A_2| = 2$, $A_1 = \{1, 2, a_1, a_2, a_3\}$. By Claim 1, set $1 \in B_1$, $2 \in B_2$, and the other edges in each block have distinct colors. If $1, 2 \in T(v_3)$, assume that $d(y_1, T(v_3)) > d(y_2, T(v_3))$, there always exist two vertices which come from different blocks such that there is no rainbow tree connecting them and $y_1$. If $1 \in T(v_3)$, $2 \in T(v_1)$, the most difficult case is that $c(v_1v_2) = 1$, $c(v_5v_6) = 2$. In this case, consider $\{y_2, v_5, v_6\}$, forcing that one of $v_1v_3'$, $v_3''v_4$, $v_3''v_6$, $v_4v_5$ is colored with 1, contradicting Claim 1. With an analogous argument, we would get a contradiction if 1, 2 are in other cut edges of $G$.

For $G \in \mathcal{G}_1$, it can be obtained by contracting an edge of a graph in $\mathcal{G}_2$. Then by Lemma 2.3, $rx_3(G) \geq n - 2$.

*Case* 2. $G \in \mathcal{G}_3$. First note that each path from $v_1$ to $v_5$ in $G_3$ can have at most one color in $A_2$. Thus $|A_1 \cap A_2| \leq 3$. On the other hand, noticing that $d_{G_3}(v_2, v_3, v_4) = 3 > 2$, all the cases satisfying $W(v_1) = W(v_5) = 0$ and $W(v_2)$, $W(v_3)$, $W(v_4) \leq 1$ are easy to get a contradiction, so we omit them here.

First, assume $|A_1 \cap A_2| = 1$, then $A_1 = \{1, a_1, a_2\}$. If $1 \in T(v_1)$, then consider $\{y_1, v_2, v_3\}$, $\{y_1, v_2, v_4\}$ and $\{y_1, v_3, v_4\}$ successively, $v_1v_2$, $v_1v_3$, $v_1v_4$ must be colored with distinct colors from $A_1 \setminus \{1\}$, which is impossible.

Assume now $|A_1 \cap A_2| = 2$, then $A_1 = \{1, 2, a_1, a_2\}$. If $1, 2 \in T(v_1)$, then consider $\{y_1, y_2, v_5\}$, without loss of generality, set $c(v_1v_2) = a_1$, $c(v_2v_5) = a_2$. Thus $c(v_1v_3)$ can be neither 1 nor 2, otherwise there is no rainbow $\{y_1, y_2, v_3\}$-tree. On the other hand, $c(v_1v_3)$ cannot be $a_1$, otherwise $c(v_3v_5) = i$, $i = 1, 2)$, then $\{y_i, v_2, v_3\}$ has no rainbow tree. Meanwhile, $v_1v_3$ cannot be colored with $a_2$, otherwise $c(v_3v_5) = i$ $(i = 1, 2)$, then $\{y_i, v_3, v_5\}$ has no rainbow tree. If $1 \in T(v_1)$, $2 \in T(v_5)$, then every path from $v_1$ to $v_5$ must color $\{a_1, a_2\}$, a contradiction to $|A_1 \cap A_2| = 3$. If $1, 2 \in T(v_2)$, then by the same reason, we conclude that $c(v_1v_2)$, $c(v_2v_5) \notin \{1, 2\}$ and we may set $c(v_1v_3) = 1$. But now $\{y_1, v_1, v_3\}$ has no rainbow tree. If $1 \in T(v_1)$, $2 \in T(v_2)$. By considering $\{y_1, y_2, v_3\}$, $\{y_1, y_2, v_4\}$, $\{y_1, y_2, v_5\}$, we may set $c(v_1v_3) = c(v_1v_4) = c(v_2v_5) = a_1$, this force $c(v_3v_5) = i$ $(i = 1, 2)$. However, there is no rainbow tree connecting $\{y_i, v_3, v_5\}$.

Thus $|A_1 \cap A_2| = 3$, $A_1 = \{1, 2, 3, a_1, a_2\}$. If $1, 2, 3 \in T(v_1)$, since $U(v_1) \leq 2$, we may assume that $y_2$ is on the unique path from $y_1$ to $v_1$. Thus one path from $v_1$ to $v_5$ must be colored with $\{a_1, a_2\}$, a contradiction to $|A_1 \cap A_2| = 3$. If $1, 2 \in T(v_2)$, $3 \in T(v_5)$, and without loss of generality, $y_2$ is on the unique path from $y_1$ to $v_2$. Considering $\{y_1, v_3, y_3\}$ and $\{y_1, v_4, y_3\}$, we may set $c(v_1v_2) = a_1$, $c(v_1v_3) = c(v_1v_4) = a_2$. But there is no rainbow $\{y_1, v_3, v_4\}$-tree. Each other case is similar or easier.

*Case* 3. $G \in \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_6$. First let $G \in \mathcal{G}_6$. Similarly, each path from $v_2$ to $v_6$ in $G_6$ can have at most one color in $A_2$. Thus we have $1 \leq |A_1 \cap A_2| \leq 3$. Assume first $|A_1 \cap A_2| = 1$, $A_1 = \{1, a_1, a_2, a_3\}$.

We only focus on the case that $1 \in T(v_4)$. To make sure there are rainbow trees connecting $\{y_1, v_1, v_3\}$ and $\{y_1, v_1, v_5\}$, only $c(v_2v_6)$ can be 1, but now $\{y_1, v_2, v_6\}$ has no rainbow tree.

Assume then $|A_1 \cap A_2| = 2$, $A_1 = \{1, 2, a_1, a_2, a_3\}$. If $1, 2 \in T(v_1)$, we may set $c(v_1v_2) = a_1$, $c(v_2v_3) = a_2$, $c(v_3v_4) = a_3$ by considering $\{y_1, y_2, v_4\}$. $c(v_5v_6)$ can be neither 1 nor 2, otherwise $\{y_1, y_2, v_5\}$ has no rainbow tree. Moreover, $c(v_4v_5)$ can be neither 1 nor 2, otherwise when $c(v_4v_5) = i$ $(i = 1, 2)$, there is no rainbow $\{y_i, v_4, v_5\}$-tree. Thus $v_1v_6$, $v_2v_6$ must use colors $\{1, 2\}$, but now $\{y_1, y_2, v_6\}$ has no rainbow tree. If $1, 2 \in T(v_3)$, first we claim that at most one edge of the triangle $v_1v_2v_6$ uses a color from $\{1, 2\}$. Otherwise, if $c(v_1v_2)$, $c(v_2v_6) \in \{1, 2\}$, then $\{y_1, y_2, v_1\}$ has no rainbow tree. If $c(v_1v_6)$, $c(v_2v_6) \in \{1, 2\}$, then the rest non-cut edges must color $\{a_1, a_2, a_3\}$. It is easy to verify that either $\{y_1, v_1, v_5\}$ or $\{y_1, v_1, v_4\}$ has no rainbow tree. So the longest path from $v_2$ to $v_6$ has an edge colored with 1 or 2. However, we will show that it is impossible. It is easy to check that $c(v_2v_3), c(v_3v_4) \notin \{1, 2\}$. If $c(v_5v_6) \in \{1, 2\}$, then we may set $c(v_5v_6) = 1$. Consider $\{y_1, v_1, v_5\}$ and $\{y_1, v_5, v_6\}$, then $c(v_1v_2) = c(v_2v_6) = 2$, a contradiction.

It is similar to check that $c(v_4v_5)$ cannot be 1 or 2, a contradiction.

Now assume that $|A_1 \cap A_2| = 3$, $A_1 = \{1, 2, 3, a_1, a_2, a_3\}$. If $1 \in T(v_1)$, then $2, 3 \in T(v_3)$. Again, we may set $c(v_1v_2) = a_1$, $c(v_2v_3) = a_2$. Thus $c(v_1v_6), c(v_2v_6) \in \{1, 2, 3\}$. If $c(v_1v_6), c(v_2v_6) \in \{1, i\}$, then there is a contradiction by considering $\{y_1, y_i, v_6\}$ $(i = 2, 3)$. Thus we may set that $c(v_1v_6) = 2$, $c(v_2v_6) = 3$. By considering $\{y_1, y_3, v_4\}$ and $\{y_1, y_3, v_5\}$, we get that $c(v_3v_4) = c(v_5v_6) = a_3$, $c(v_4v_5) = 1$, but now $\{y_1, v_4, v_5\}$ has no rainbow tree. If $1 \in T(v_3)$, $2, 3 \in T(v_5)$, then we set $v_3v_4 = a_1$, $v_4v_5 = a_2$. If $c(v_2v_6) = i$, $c(v_5v_6) = j$, $i, j \in \{1, 2, 3\}$, then $\{y_i, y_j, v_6\}$ has no rainbow tree. The only possibility is $c(v_2v_3) = 2$, $c(v_2v_6) = 3$, $c(v_5v_6) = a_3$. However, $\{y_1, y_2, v_1\}$ has no rainbow tree.

For $G \in \mathcal{G}_5$, notice that $|A_1 \cap A_2| \leq 3$. If $U(v_2) = 0$ or $U(v_5) = 0$, then $G$ can be obtained by contracting an edge of a graph in $\mathcal{G}_6$. Then by Lemma 2.3, $rx_3(G) \geq n-2$. Thus we need to consider the case when $W(v_2) \geq 1$ and $W(v_5) \geq 1$. If $|A_1 \cap A_2| = 2$, then suppose $1 \in T(v_2)$, $2 \in T(v_5)$. Consider $\{y_1, y_2, v_3\}$, $\{y_1, y_2, v_4\}$, we have $c(v_2v_3)$, $c(v_2v_5)$, $c(v_4v_5) \in \{a_1, a_2\}$ and $c(v_2v_3) = c(v_4v_5)$. But now $c(v_3v_4) = i$ $(i = 1, 2)$, then there is no rainbow $\{y_i, v_3, v_4\}$-tree. If $|A_1 \cap A_2| = 3$, then $A_1 = \{1, 2, 3, a_1, a_2\}$. If $1 \in T(v_1)$, $2 \in T(v_2)$, $3 \in T(v_5)$, then consider $\{y_1, y_2, y_3\}$, we have that two of $v_1v_2$, $v_1v_5$, $v_2v_5$ have colors outside $A_2$, contradicting to $|A_1 \cap A_2| = 3$. If $1 \in T(v_2)$, $2 \in T(v_5)$, $3 \in T(v_2)$ and we may assume that $y_3$ is on the unique path from $y_1$ to $v_2$. Then consider $\{y_1, v_3, y_3\}$ and $\{y_1, v_4, y_3\}$, we have $c(v_2v_3) = c(v_4v_5)$, thus $c(v_3v_4)$ cannot be in $A_2$, contradicting to $|A_1 \cap A_2| = 3$. If $1 \in T(v_2)$, $2 \in T(v_5)$, $3 \in T(v_i)$ $(i = 3, 4)$, then consider $\{y_1, y_2, y_3\}$, we have that $c(v_2v_5)$ is in $A_1 \setminus A_2$, contradicting $|A_1 \cap A_2| = 3$.

Finally, for a graph $G$ belonging to $\mathcal{G}_4$, it can be obtained by contracting an edge of a graph in $\mathcal{G}_3 \cup \mathcal{G}_6$. Then by Lemma 2.3, $rx_3(G) \geq n - 2$.

Combining all the cases above, we have $rx_3(G) \geq n - 2$ for $G \in \mathcal{G}$. By Theorem 3.1, it follows that $rx_3(G) = n - 2$. ■

### 3.2.2. Tricyclic graphs with $rx_3(G) = n - 2$

Define by $\mathcal{H}_i^*$ the set of graphs whose basic graph is $H_i$, where $H_i$ is shown in Figure 2 and $1 \leq i \leq 8$.

Now, we introduce another graph class $\mathcal{H}$. Set $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots \cup \mathcal{H}_8$, where

$\mathcal{H}_1 = \{G \in \mathcal{H}_1^* : U(G) = 0\}$,

$\mathcal{H}_2 = \{G \in \mathcal{H}_2^* : U(v_i) \leq 1,\ U(v_j) = 0,\ i = 5, 6,\ j = 1, 3, 4\}$,

$\mathcal{H}_3 = \{G \in \mathcal{H}_3^* : U(v_2) \leq 1,\ U(v_5) + U(v_6) \leq 1,\ U(v_i) = 0,\ i = 1, 3, 4\}$,

$\mathcal{H}_4 = \{G \in \mathcal{H}_4^* : U(v_i) \leq 1,\ U(v_j) \leq 2,\ U(v_i)+U(v_j) \leq 1,\ U(v_j)+U(v_k) \leq 3,$
$i = 1, 5,\ j, k = 2, 3, 4\}$,

$\mathcal{H}_5 = \{G \in \mathcal{H}_5^* : U(v_i) \leq 1,\ U(v_j) = 0,\ i = 1, 3, 5,\ j = 2, 4, 6\}$,

$\mathcal{H}_6 = \{G \in \mathcal{H}_6^* : U(v_3) = 0,\ U(v_i) \leq 1,\ U(v_1) + U(v_5) \leq 1,\ i = 1, 2, 4, 5\}$,

$\mathcal{H}_7 = \{G \in \mathcal{H}_2^* : U(v_2) + U(v_4) \leq 1,\ U(v_3) + U(v_5) \leq 1,\ U(v_5) + U(v_1) \leq 1,$

$$U(v_j)+U(v_{j+1})\le 1,\ j=1,2,4\},$$
$$\mathcal{H}_8=\{G\in\mathcal{H}_8^*: U(v_i)\le 2,\ U(v_i)+U(v_j)+U(v_k)\le 3,\ i,j,k=1,2,3,4\}.$$

**Lemma 3.4.** *Let $G$ be a connected tricyclic graph of order $n$. Then $rx_3(G)=n-2$ if and only if $G\in\mathcal{H}$.*

***Proof.*** Suppose that $rx_3(G)=n-2$ but $G\notin\mathcal{H}$. Let $G_0$ be the basic graph of $G$. Similar to Lemma 3.3, we have $G_0=H_i$ and we can rainbow color $G$ with $n-3$ colors for $G\in\mathcal{H}_i^*\setminus\mathcal{H}_i$, $i=1,\ldots,8$.

If $G\in\mathcal{H}_1^*\setminus\mathcal{H}_1$, then if $U(v_2)\ge 1$, set $c_\ell(H_1)=a_41a_1a_2a_2a_4a_3a_3a_4$ and $\{1\}\subseteq T(v_2)$; if $U(v_3)\ge 1$, set $c_\ell(H_1)=a_4a_1a_11a_3a_4a_2a_2a_4$ and $\{1\}\subseteq T(v_3)$; if $U(v_4)\ge 1$, set $c_\ell(H_1)=a_3a_2a_2a_11a_3a_4a_4a_1$ and $\{1\}\subseteq T(v_4)$.

If $G\in\mathcal{H}_2^*\setminus\mathcal{H}_2$, then if $U(v_3)\ge 1$ ($U(v_1)\ge 1$ is similar), set $c_\ell(H_2)=a_3a_21a_2a_3a_1a_1a_2$ and $\{1\}\subseteq T(v_3)$; if $U(v_4)\ge 1$, set $c_\ell(H_2)=a_1a_3a_21a_2a_3a_3a_1$ and $\{1\}\subseteq T(v_4)$; if $U(v_6)\ge 2$ ($U(v_5)\ge 2$ is similar), set $c_\ell(H_2)=a_31a_22a_2a_3a_11$ and $\{1,2\}\subseteq T(v_3)$.

If $G\in\mathcal{H}_3^*\setminus\mathcal{H}_3$, then if $U(v_2)\ge 2$, set $c_\ell(H_3)=a_12a_2a_32a_2a_11$ and $\{1,2\}\subseteq T(v_2)$; if $U(v_5)\ge 2$, set $c_\ell(H_3)=2a_2a_31a_2a_11a_3$ and $\{1,2\}\subseteq T(v_5)$; if $U(v_5)+U(v_6)\ge 2$, set $c_\ell(H_3)=1a_12a_2a_3a_1a_2a_3$ and $\{1\}\subseteq T(v_6)$, $\{2\}\subseteq T(v_5)$; if $U(v_3)\ge 1$, set $c_\ell(H_3)=a_1a_2a_21a_1a_3a_3a_2$ and $\{1\}\subseteq T(v_3)$; if $U(v_4)\ge 1$, set $c_\ell(H_3)=a_2a_11a_1a_2a_3a_3a_2$ and $\{1\}\subseteq T(v_4)$.

If $G\in\mathcal{H}_4^*\setminus\mathcal{H}_4$, then there are four cases for the graph $G$:

(1) $U(v_i)\ge 3$ $(i=2,3,4)$;

(2) $U(v_i)\ge 2$ $(i=1,5)$;

(3) $U(v_i)+U(v_j)\ge 2$ $(i\in\{1,5\},\ j\in\{2,3,4\})$;

(4) $U(v_i)\ge 2$ and $U(v_j)\ge 2$, $i,j\in\{2,3,4\}$.

If $G$ is a graph in Case (1), then there exists a graph in $\mathcal{G}_3^*\setminus\mathcal{G}_3$ which is a subgraph of $G$. Thus the result is obvious. If $U(v_1)\ge 2$, set $c_\ell(H_4)=a_2a_2a_1a_121a_2$ and $\{1,2\}\subseteq T(v_1)$; if $U(v_1)+U(v_2)\ge 2$, set $c_\ell(H_4)=a_1a_2a_22a_21a_1$ and $\{1\}\subseteq T(v_1)$, $\{2\}\subseteq T(v_2)$; if $U(v_2)\ge 2$ and $U(v_4)\ge 2$, set $c_\ell(H_4)=3421a_2a_2a_1$ and $\{1,2\}\subseteq T(v_2)$, $\{3,4\}\subseteq T(v_4)$.

If $G\in\mathcal{H}_5^*\setminus\mathcal{H}_5$, then $U(v_i)\ge 1$ $(i=2,4,6)$ or $U(v_i)\ge 2$ $(i=1,3,5)$. If $G$ is a graph in the former case, there exists a graph in $\mathcal{G}_6^*\setminus\mathcal{G}_6$ which is a subgraph of $G$. If $U(v_3)\ge 2$, set $c_\ell(H_5)=a_22a_1a_2a_31a_3a_2$ and $\{1,2\}\subseteq T(v_3)$; if $U(v_5)\ge 2$, set $c_\ell(H_5)=1a_22a_1a_3a_2a_3a_1$ and $\{1,2\}\subseteq T(v_5)$.

If $G\in\mathcal{H}_6^*\setminus\mathcal{H}_6$, then $U(v_3)\ge 1$ or $U(v_i)\ge 2$ $(i=1,2,4,5)$ or $U(v_1)+U(v_5)\ge 2$. If $U(v_3)\ge 1$, set $c_\ell(H_6)=a_2a_11a_1a_2a_2a_1$ and $\{1\}\subseteq T(v_3)$; if $U(v_1)\ge 2$, set $c_\ell(H_6)=a_2a_1a_11a_212$ and $\{1,2\}\subseteq T(v_1)$; if $U(v_2)\ge 2$, set $c_\ell(H_6)=a_21a_1a_1a_212$ and $\{1,2\}\subseteq T(v_2)$; if $U(v_1)+U(v_5)\ge 2$, set $c_\ell(H_6)=a_1a_1a_2a_221a_1$ and $\{1\}\subseteq T(v_1)$, $\{2\}\subseteq T(v_5)$.

If $G\in\mathcal{H}_7^*\setminus\mathcal{H}_7$, then $U(v_i)\ge 2$ $(i=1,2,3,4,5)$ or $U(v_1)+U(v_2)\ge 2$ or $U(v_2)+U(v_3)\ge 2$ or $U(v_4)+U(v_5)\ge 2$ or $U(v_2)+U(v_4)\ge 2$ or $U(v_3)+$

$U(v_5) \geq 2$ or $U(v_1) + U(v_5) \geq 2$. If $U(v_1) \geq 2$, set $c_\ell(H_7) = a_1a_2a_212a_1a_1$ and $\{1,2\} \subseteq T(v_3)$; if $U(v_2) \geq 2$, set $c_\ell(H_7) = a_2a_1a_1a_1a_212$ and $\{1,2\} \subseteq T(v_2)$; if $U(v_3) \geq 2$, set $c_\ell(H_7) = a_2a_1a_12a_1a_21$ and $\{1,2\} \subseteq T(v_3)$; if $U(v_1) + U(v_2) \geq 2$, set $c_\ell(H_7) = a_2a_1a_1a_1a_221$ and $\{1\} \subseteq T(v_1)$, $\{2\} \subseteq T(v_2)$; if $U(v_2) + U(v_3) \geq 2$, set $c_\ell(H_7) = a_2a_1a_12a_2a_21$ and $\{1\} \subseteq T(v_2)$, $\{2\} \subseteq T(v_3)$; if $U(v_2) + U(v_4) \geq 2$, set $c_\ell(H_7) = a_212a_1a_2a_1a_2$ and $\{1\} \subseteq T(v_2)$, $\{2\} \subseteq T(v_4)$.

If $G \in \mathcal{H}_8^* \setminus \mathcal{H}_8$, then $U(v_i) \geq 3$ $(i = 1,2,3,4)$ or $U(v_i) + U(v_j) + U(v_k) \geq 4$, $i,j,k = 1,2,3,4$. If $G$ is a graph in the former case, then a graph belonging to $\mathcal{G}_4^* \setminus \mathcal{G}_4$ is a subgraph of $G$. If $U(v_1) + U(v_2) + U(v_4) \geq 4$, set $c_\ell(H_8) = 1a_1a_1423$ and $\{1,2\} \subseteq T(v_1)$, $\{3\} \subseteq T(v_2)$, $\{4\} \subseteq T(v_4)$; if $U(v_2)+U(v_3) \geq 4$, set $c_\ell(H_8) = 12a_1a_134$ and $\{1,2\} \subseteq T(v_2)$, $\{3,4\} \subseteq T(v_3)$.

It is not hard to check that the colorings above make $G$ rainbow connected with $n-3$ colors, thus $rx_3(G) \leq n-3$.

Conversely, let $G$ be a tricyclic graph such that $G \in \mathcal{H}$. Similar to Lemma 3.3, we only need to consider the case that $G$ has cut edges and $|A_1 \cap A_2| \geq 1$. Assume, to the contrary, that $rx_3(G) \leq n-3$. Then there exists a rainbow coloring $c$ of $G$ using colors in $N_{n-3}$.

For $G \in \mathcal{H}_1$, if there is a nontrivial path $P'$ connecting $B_1$ and $B_2$ in $G$, then denote its ends by $v_3'(\in B_1)$ and $v_3''(\in B_2)$ and if there is a nontrivial path $P''$ connecting $B_2$ and $B_3$ in $G$, then denote its ends by $v_5'(\in B_2)$ and $v_5''(\in B_3)$. Similar to Claim 2 in Lemma 3.3, the colors in the path $P^{'}$ and $P^{''}$ cannot appear in $A_1$, which implies $|A_1 \cap A_2| = 0$, contradicting $|A_1 \cap A_2| \geq 1$. For $G \in \mathcal{H}_5$, notice that $d_{H_5}(v_1,v_3,v_5) = 4$ and $|A_1 \setminus A_2| = 3$, the result holds. The same argument applies to the case when $G \in \mathcal{H}_6$. Thus, we mainly discuss the rest cases for $G$ as follows.

*Case* 1. $G \in \mathcal{H}_2$. We have $1 \leq |A_1 \cap A_2| \leq 4$ and $|A_1 \setminus A_2| = 3$. If there is a nontrivial path $P'$ connecting $B_1$ and $B_2$ in $G$, then denote its ends by $v_4'(\in B_1)$ and $v_4''(\in B_2)$. We can also claim that $c(P') \cap A_1 = \emptyset$. Noticing that $d_{H_2}(v_2,v_5,v_6) = 4 > 3$, we only check the case when $W(v_2) \geq 2$. Since the case of $|A_1 \cap A_2| = 1$ or $|A_1 \cap A_2| = 4$ is easy to check, we consider the remaining two cases. Assume $|A_1 \cap A_2| = 2$, $A_1 = \{1,2,a_1,a_2,a_3\}$ and $1,2 \in T(v_2)$, consider $\{y_1,y_2,v_5\}$ and we may set $c(v_2v_3) = a_1$, $c(v_3v_4') = a_2$, $c(v_4''v_5) = a_3$. If 1 and 2 are in $B_1$, and 1 appears in $v_1v_2$ or $v_1v_4'$, then we have $c(v_4''v_6) = a_3$ by considering $\{y_1,y_2,v_6\}$, and thus $c(v_5v_6) \notin A_2$, but now $\{y_1,v_5,v_6\}$ has no rainbow tree. So one of 1, 2, say 1, is in $B_2$ and $c(v_5v_6) = 1$. Now we have $c(v_4''v_6) \neq a_3$ and $c(v_1v_2), c(v_1v_4'), c(v_4''v_6) \in \{a_1,a_2,a_3\}$ by considering $\{y_1,y_2,v_6\}$. Then every $\{y_1,v_5,v_6\}$-tree of size 5 cannot have the color 2. Thus there is no rainbow $\{y_1,v_5,v_6\}$-tree.

Assume then $|A_1 \cap A_2| = 3$ and $A_1 = \{1,2,a_1,a_2,a_3\}$. If $1,2,3 \in T(v_2)$, first we claim that $v_2v_3, v_3v_4'$ cannot use colors from $A_2$ both. Otherwise assume $c(v_2v_3) = 1$, $c(v_3v_4) = 2$, $c(v_1v_2) = a_1$, $c(v_1v_4') = a_2$, and by considering

$\{y_1, y_2, v_5\}$ and $\{y_1, y_2, v_6\}$, we have $c(v_4''v_5) = c(v_4''v_6) = a_3$, and $c(v_5v_6)$ can be $a_1$ or $a_2$. However, there is no rainbow $\{y_1, v_5, v_6\}$-tree or $\{y_2, v_5, v_6\}$-tree. With the same reason, we conclude that exactly one edge of the unique 4-cycle of $H_2$ can be colored with a color from $A_2$. Thus there are two cases by symmetry. If $c(v_1v_2) = 1$, $c(v_1v_3) = 2$, $c(v_5v_6) = 3$, then consider $y_1$ and $y_2$, together with $v_5$, $v_6$ respectively, we have $c(v_4''v_5) = c(v_4''v_6)$, which is impossible. If $c(v_1v_4') = 1$, $c(v_1v_3) = 2$, $c(v_5v_6) = 3$, then consider $y_1$ and $y_3$, together with $v_5$, $v_6$ respectively. suppose $c(v_4''v_5) = a_1$, $c(v_4''v_6) = a_2$ and $c(v_3v_4') = a_3$, $c(v_1v_2), c(v_2v_3) \in \{a_1, a_2\}$, but now there is no rainbow $\{y_3, v_5, v_6\}$-tree. If $1, 2 \in T(v_2)$, $3 \in T(v_6)$, similarly set $c(v_1v_2) = a_1$, $c(v_1v_4') = a_2$, $c(v_4''v_6) = a_3$. First we can easily claim that the color 3 cannot appear in $B_2$. Thus there are three possibilities for the color 3. If $c(v_3v_4') = 3$, then consider $\{y_1, y_3, v_3\}$ and $\{y_2, y_3, v_3\}$, we have $c(v_1v_3), c(v_2v_3) \in \{1, 2\}$. Consider $\{y_1, y_2, v_5\}$, one of $c(v_4''v_5)$ and $c(v_5v_6)$ is $a_3$, but now there is no rainbow $\{y_2, y_3, v_5\}$-tree. The case when $c(v_1v_3) = 3$ is similar to the case of $c(v_3v_4') = 3$. If $c(v_2v_3) = 3$, then similarly we get $c(v_1v_3), c(v_2v_3) \in \{1, 2\}$ and one of $c(v_4''v_5)$ and $c(v_5v_6)$ is $a_3$. Consider $\{y_1, y_3, v_5\}$, this forces one of $c(v_4''v_5)$ and $c(v_5v_6)$ is 2, it is impossible.

*Case* 2. $G \in \mathcal{H}_3$. Then $1 \leq |A_1 \cap A_2| \leq 4$ and $|A_1 \setminus A_2| = 3$. If there is a nontrivial path $P'$ connecting $B_1$ and $B_2$ in $G$, then denote its ends by $v_4'(\in B_1)$ and $v_4''(\in B_2)$. Similarly, it is easy to check that $c(P') \cap A_1 = \emptyset$. We only focus on the case that $|A_1 \cap A_2| = 1$, where $A_1 = \{1, a_1, a_2, a_3\}$. If $1 \in T(v_6)$, consider $\{y_1, v_1, v_3\}$, set $c(v_4''v_6) = a_1$, $c(v_1v_4') = a_2$, $c(v_3v_4') = a_3$. Then $c(v_5v_6) \neq 1$, otherwise there is no rainbow tree connecting $\{y_1, v_1, v_5\}$ or $\{y_1, v_3, v_5\}$ depending on $c(v_4''v_5)$. Similarly, $c(v_4''v_5) \neq 1$. Next $c(v_2v_4') \neq 1$ by considering $\{y_1, v_2, v_5\}$. Suppose $c(v_1v_2) = 1$, to make sure there is a rainbow $\{y_1, v_1, v_2\}$-tree and $\{y_1, v_2, v_3\}$-tree, we have $c(v_2v_4') = a_3$ and $c(v_2v_3) = a_2$. But now $\{v_2, v_3, v_5\}$ has no rainbow tree. If $1 \in T(v_2)$, then consider $\{y_1, v_5, v_6\}$, $\{y_1, v_1, v_5\}$, $\{y_1, v_1, v_6\}$, $\{y_1, v_3, v_5\}$, $\{y_1, v_3, v_6\}$ successively. Set $c(v_2v_4') = a_1$, then $c(v_1v_2)$, $c(v_1v_4')$, $c(v_2v_3)$, $c(v_3v_4')$, $c(v_4''v_5)$ and $c(v_4''v_6)$ can only be $a_2$ or $a_3$. It is easy to check that $\{v_2, v_3, v_5\}$ has no rainbow tree.

*Case* 3. $G \in \mathcal{H}_4$. $1 \leq |A_1 \cap A_2| \leq 4$ and $|A_1 \setminus A_2| = 2$. First notice that $d_{H_4}(v_2, v_3, v_4) = 3$, the case that $W(v_1) = W(v_5) = 0$, $W(v_2)$, $W(v_3)$, $W(v_4) \leq 1$ is evident. Assume $|A_1 \cap A_2| = 1$, then $1 \in T(v_1)$, the case is similar with the case that $G \in \mathcal{G}_3$ in Lemma 3.3. Assume now $|A_1 \cap A_2| = 2$, if $1 \in T(v_1)$, $2 \in T(v_5)$, by considering all the trees containing $y_1$ and $y_2$, without loss of generality, set $c(v_1v_5) = a_1$, $c(v_1v_2) = 1(2)$, $c(v_2v_5) = a_2$. Moreover, by considering $\{y_1(y_2), v_2, v_3\}$ and $\{y_1(y_2), v_2, v_4\}$, the remaining two paths of length 2 from $v_1$ to $v_5$ must be colored with $2(1)$, $a_2$, respectively. However, there is no rainbow $\{y_2(y_1), v_3, v_4\}$-tree. If $1, 2 \in T(v_2)$, by considering $\{y_1, y_2, v_4\}$, set $c(v_2v_5) = a_1$, $c(v_4v_5) = a_2$. Since the two possible rainbow trees connecting $\{y_1, v_3, v_4\}$ and $\{y_2, v_3, v_4\}$ are the same, we may set $c(v_3v_5) = 1$. It is easy to see that $c(v_1v_2)$, $c(v_1v_3)$ cannot use colors from $A_2$ by considering $\{y_1, y_2, v_3\}$, and $c(v_1v_4) = 2$ by considering $\{y_1, v_3, v_4\}$. But now if $c(v_1v_5) = 1$ or $c(v_1v_5) = 2$, there is no rainbow $\{y_1, v_2, v_5\}$-tree or $\{y_2, v_1, v_4\}$-tree, respectively.

Assume $|A_1 \cap A_2| = 3$, then $1, 2 \in T(v_2)$, $3 \in T(v_4)$. Similarly as above, we may set $c(v_2v_5) = a_1$, $c(v_4v_5) = a_2$, $c(v_3v_5) = 1$, $c(v_1v_5) = 3$, one of $c(v_1v_2)$, $c(v_1v_4)$ is 2. However, there is no rainbow $\{y_2, y_3, v_1\}$-tree.

Finally assume $|A_1 \cap A_2| = 4$ and $1, 2 \in T(v_2)$, $3 \in T(v_3)$, $4 \in T(v_4)$, consider $\{y_1, y_3, y_4\}$ and $\{y_2, y_3, y_4\}$, at least four of the non-cut edges must be colored with $\{a_1, a_2\}$. This contradicts $|A_1 \cap A_2| = 4$.

*Case* 4. $G \in \mathcal{H}_7$. Since $d_{H_7}(v_1, v_3, v_4) = 3$, we only focus on the case $1 \in T(v_2)$. Consider all the three vertices containing $y_1$, it is not hard to obtain a contradiction.

*Case* 5. $G \in \mathcal{H}_8$. First notice $d_{H_8}(v_1, v_2, v_3) = 2$, the case that $W(v_1)$, $W(v_2)$, $W(v_3)$, $W(v_4) \leq 1$ is evident. Assume $|A_1 \cap A_2| = 2$, then $1, 2 \in T(v_1)$. Consider $\{y_1, y_2, v_2\}$, $\{y_1, y_2, v_3\}$, $\{y_1, y_2, v_4\}$ successively, we have $c(v_1v_2) = c(v_1v_3) = c(v_1v_4) = a_1$. However, there is no rainbow tree connecting $\{y_1, v_2, v_3\}$ or $\{y_2, v_2, v_3\}$, a contradiction. Now focus on $|A_1 \cap A_2| = 3$, then $1, 2 \in T(v_1)$, $3 \in T(v_2)$. Consider $\{y_1, y_3, v_3\}$, $\{y_1, y_3, v_4\}$, $\{y_2, y_3, v_3\}$ and $\{y_2, y_3, v_4\}$ successively, $c(v_1v_3)$, $c(v_1v_4)$, $c(v_2v_3)$, $c(v_2v_4)$ must be 1 or 2. Again, there is no rainbow $\{y_1, y_2, v_3\}$-tree.

By the detailed analysis above, we have $rx_3(G) \geq n - 2$ for $G \in \mathcal{H}$. By Theorem 3.1, it follows that $rx_3(G) = n - 2$. ■

### 3.2.3. Characterize the graphs with $rx_3(G) = n - 2$

We begin with a lemma about a connected 5-cyclic graph.

**Lemma 3.5.** *Let $G$ be a connected 5-cyclic graph of order $n$. Then $rx_3(G) = n-2$ if and only if $G = K_5 - e$.*

***Proof.*** Let $G \neq K_5 - e$ and $rx_3(G) = n - 2$, by Lemma 2.3 and Theorem 3.2, $rx_3(G) \leq n - 3$, a contradiction. Conversely, suppose $G = K_5 - e$, by Theorem 1.6, $rx_3(G) \geq 3$, on the other hand, $rx_3(G) \leq rx_3(C_5) = 3$. Thus $rx_3(G) = n-2$. ■

For $n \geq 3$, the *wheel* $W_n$ is a graph constructed by joining a vertex $v_0$ to every vertex of a cycle $C_n : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$.

A third graph class is defined as follows. Let $\mathcal{J}_1$ be a class of graphs such that every graph is obtained from a graph in $\mathcal{H}_5$ by adding an edge $v_4v_6$. Let $\mathcal{J}_2$ be a class of graphs such that every graph is obtained from a graph in $\mathcal{H}_7$ where $U(v_2) = 0$ and $U(v_5) = 0$ by adding an edge $v_2v_5$. Set $\mathcal{J} = \{\mathcal{J}_1, \mathcal{J}_2, W_4\}$.

Now we are ready to show our second main theorem of this paper.

**Theorem 3.6.** *Let $G$ be a connected graph of order $n$ ($n \geq 6$). Then $rx_3(G) = n-2$ if and only if $G$ is unicyclic with the girth at least 4 or $G \in \mathcal{G} \cup \mathcal{H} \cup \mathcal{J}$ or $G = K_5 - e$.*

***Proof.*** Let $G$ be a $t$-cyclic graph with $rx_3(G) = n-2$, but not a graph listed in the theorem. By Proposition 1.2, Theorem 1.3, Lemma 3.3 and Lemma 3.4, we need to consider the cases $t \geq 4$. If $t = 4$, by Theorem 3.2, the basic graph of $G$ should be a 3-sun or the basic graph of $\mathcal{J}_2$ or $W_4$. If $G \notin \mathcal{J}_1$ or $G \notin \mathcal{J}_2$, then by the similar arguments with Lemma 3.3, we have $rx_3(G) \leq n-3$, a contradiction. Suppose the basic graph of $G$ is $W_4$ and there are some cut edges in $G$. If $U(v_0) \geq 1$, then a graph belonging to $\mathcal{G}_6^* \setminus \mathcal{G}_6$ and satisfying $U(v_3) \geq 1$ is a subgraph of $G$. If $U(v_1) \geq 1$ (other cases are similar), then set $c_\ell(W_4) = a_2 1 a_1 a_1 a_1 a_2 a_2 a_1$ and $\{1\} \subseteq T(v_1)$. If $t \geq 5$, by Theorem 3.2, the basic graph of $G$ should be $K_5 - e$, since $n \geq 6$, by the similar argument with $t = 4$, we have $rx_3(G) \leq n-3$, a contradiction.

Conversely, by Theorem 1.3, Theorem 1.6, Lemma 3.3, Lemma 3.4 and Lemma 3.5, suppose $G$ is a graph such that $G \in \mathcal{J}_1$ or $G \in \mathcal{J}_2$. Assume, to the contrary, that $rx_3(G) \leq n-3$. Then there exists a rainbow coloring $c$ of $G$ using $n-3$ colors. Both cases can be considered similar to the case that $G \in \mathcal{H}_5$ or $G \in \mathcal{H}_7$ in Lemma 3.4, a contradiction. ■

**Acknowledgement**

The authors are very grateful to the referees for their valuable comments and suggestions which helped to improve the presentation of the paper.

## References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory (GTM 244, Springer, 2008).

[2] G. Chartrand, G. Johns, K. McKeon and P. Zhang, *Rainbow connection in graphs*, Math. Bohem. **133** (2008) 85–98.

[3] G. Chartrand, F. Okamoto and P. Zhang, *Rainbow trees in graphs and generalized connectivity*, Networks **55** (2010) 360–367.
doi:10.1002/net.20339

[4] L. Chen, X. Li, K. Yang and Y. Zhao, *The 3-rainbow index of a graph*, Discuss. Math. Graph Theory **35** (2015) 81–94.
doi:10.7151/dmgt.1780

[5] Y. Caro, A. Lev, Y. Roditty, Zs. Tuza and R. Yuster, *On rainbow connection*, Electron. J. Combin. **15** (2008) #R57.

[6] G. Chartrand, S. Kappor, L. Lesniak and D. Lick, *Generalized connectivity in graphs*, Bull. Bombay Math. Colloq. **2** (1984) 1–6.

[7] G. Chartrand, G. Johns, K. McKeon and P. Zhang, *The rainbow connectivity of a graph*, Networks **54** (2009) 75–81.
doi:10.1002/net.20296

[8] X. Li, Y. Shi and Y. Sun, *Rainbow connections of graphs: A survey*, Graphs Combin. **29** (2013) 1–38.
doi:10.1007/s00373-012-1243-2

[9] X. Li and Y. Sun, Rainbow Connections of Graphs (Springer Briefs in Math., Springer, 2012).

Received 23 September 2013
Revised 21 February 2014
Accepted 24 February 2014