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Note

# FRACTIONAL ASPECTS OF THE ERDŐS-FABER-LOVÁSZ CONJECTURE

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**Abstract**

The Erdős-Faber-Lovász conjecture is the statement that every graph that is the union of $n$ cliques of size $n$ intersecting pairwise in at most one vertex has chromatic number $n$. Kahn and Seymour proved a fractional version of this conjecture, where the chromatic number is replaced by the fractional chromatic number. In this note we investigate similar fractional relaxations of the Erdős-Faber-Lovász conjecture, involving variations of the fractional chromatic number. We exhibit some relaxations that can be proved in the spirit of the Kahn-Seymour result, and others that are equivalent to the original conjecture.

**Keywords:** Erdős-Faber-Lovász Conjecture, fractional chromatic number.

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## 1. Introduction

The Erdős-Faber-Lovász conjecture is the following statement.

**Conjecture 1.** *If a graph $G$ is the union of $n$ cliques of size $n$ such that any two of these $n$ cliques intersect in at most one vertex, then $\chi(G) = n$.*

For an integer $n$, let $\mathbb{EFL}_n$ denote the class of graphs that are constructed as the union of $n$ cliques of size $n$ such that any two of these $n$ cliques intersect in at most one vertex. Then $\mathbb{EFL}_n$ contains finitely many isomorphism classes of graphs, and Conjecture 1 is equivalent to the statement that

$$\max\{\chi(G) : G \in \mathbb{EFL}_n\} = n$$

for all $n$. Linear bounds of the type

$$\max\{\chi(G) : G \in \mathbb{EFL}_n\} \leq cn$$

are known; in particular,

$$\lim_{n\to\infty} \max\{\chi(G) : G \in \mathbb{EFL}_n\}/n = 1$$

(see [3]). In another direction, relaxations of the chromatic number have been considered. The fact that the clique number $\omega(G)$ of a graph $G$ in $\mathbb{EFL}_n$ is $n$ is not obvious, it is a consequence of the theorem of de Bruijn and Erdős [1] on clique decompositions of complete graphs. A strengthening of this result was proved by Kahn and Seymour using the fractional chromatic number.

**Theorem 2** [4]**.** *For every graph $G$ in $\mathbb{EFL}_n$, $\chi_{\mathrm{f}}(G) = n$.*

The definition of the fractional chromatic number $\chi_{\mathrm{f}}$ will be given in the next section, along with that of other relevant fractional parameters. We will then examine fractional relaxations of the Erdős-Faber-Lovász conjecture. We exhibit some that are provable in the spirit of the proof of Theorem 2, and others that are equivalent to the original conjecture. We will conclude with open problems.

## 2. Fractional Cover Parameters

Let $\mathbb{F}$ be a class of graphs, closed under isomorphism. For a graph $G$, let $\mathbb{F}(G)$ be the set of induced subgraphs of $G$ belonging to $\mathbb{F}$. A *fractional* $\mathbb{F}$*-cover* of $G$ is a function $f : \mathbb{F}(G) \to [0,1]$ such that $\sum_{u\in V(H)} f(H) \geq 1$ for all $u \in V(G)$, and its *weight* is $w(f) = \sum_{H\in\mathbb{F}(G)} f(H)$. The *fractional* $\mathbb{F}$*-cover number* of $G$, denoted $\mathbb{F}\text{-cover}_{\mathrm{f}}(G)$ is the minimum possible weight of a fractional $\mathbb{F}$-cover of $G$. Thus $\mathbb{F}\text{-cover}_{\mathrm{f}}(G)$ is finite if and only if every vertex of $G$ is in some member of $\mathbb{F}(G)$, and $\mathbb{F}\text{-cover}_{\mathrm{f}}(G) = 1$ if and only if $G$ itself is in $\mathbb{F}$. By linear programming duality, $\mathbb{F}\text{-cover}_{\mathrm{f}}(G)$ is the maximum possible value of $\sum_{u\in V(G)} g(u)$, where $g : V(G) \to [0,1]$ is a function that satisfies $\sum_{u\in V(H)} g(u) \leq 1$ for all $H \in \mathbb{F}(G)$ (we will call such a function a *fractional* $\mathbb{F}$*-clique*). By complementary slackness, if $f$ is a minimum fractional $\mathbb{F}$-cover and $g$ a maximum fractional $\mathbb{F}$-clique, then $\sum_{u\in V(H)} f(H) = 1$ whenever $g(u) > 0$, and $\sum_{u\in V(H)} g(u) = 1$ whenever $f(H) > 0$.

The *fractional chromatic number* of a graph $G$ is the parameter $\chi_{\mathrm{f}}(G) = \mathbb{I}\text{-cover}_{\mathrm{f}}(G)$, where $\mathbb{I}(G)$ is the family of independent sets of $G$. The reader is referred to [5] for a thorough exposition of the fractional chromatic number and relevant aspects of linear programming. We will consider fractional coverings using the following classes of graphs:

- $\mathbb{C}_k$: the class of $k$-colourable graphs,

- $\mathbb{K}_k$: the class of graphs which do not contain a clique of size $k+1$.

In particular, $\mathbb{I}(G)=\mathbb{C}_1(G)=\mathbb{K}_1(G)$, hence $\chi_{\mathrm{f}}(G)=\mathbb{C}_1\text{-cover}_{\mathrm{f}}(G)=\mathbb{K}_1\text{-cover}_{\mathrm{f}}(G)$. Since $\mathbb{C}_k \subseteq \mathbb{C}_{k+1}$, the sequence $(\mathbb{C}_k\text{-cover}_{\mathrm{f}}(G))_{k\geq 1}$ is non-increasing, and reaches 1 when $k=\chi(G)$. Thus we have $k\cdot\mathbb{C}_k\text{-cover}_{\mathrm{f}}(G)=k$ if and only if $G$ is $k$-colourable. Moreover, since every graph $H$ in $\mathbb{C}_k$ admits a natural covering by $(k-1)$-colourable subgraphs $H_1,\ldots,H_k$ containing every vertex exactly $k-1$ times, we have $\frac{k-1}{k}\cdot\mathbb{C}_{k-1}\text{-cover}_{\mathrm{f}}(G)\leq\mathbb{C}_k\text{-cover}_{\mathrm{f}}(G)$. Thus for any graph $G$ we have

$$\begin{aligned}\chi_{\mathrm{f}}(G)=1\cdot\mathbb{C}_1\text{-cover}_{\mathrm{f}}(G) &\leq 2\cdot\mathbb{C}_2\text{-cover}_{\mathrm{f}}(G)\leq\cdots\\ &\leq \chi(G)\cdot\mathbb{C}_{\chi(G)}\text{-cover}_{\mathrm{f}}(G)=\chi(G).\end{aligned}$$

For every integer $k$, $\mathbb{C}_k\subseteq\mathbb{K}_k$, hence for every graph $G$ we have

$$k\cdot\mathbb{C}_k\text{-cover}_{\mathrm{f}}(G)\geq k\cdot\mathbb{K}_k\text{-cover}_{\mathrm{f}}(G).$$

However the sequence $\{k\cdot\mathbb{K}_k\text{-cover}_{\mathrm{f}}(G)\}_{1\leq k\leq\omega(G)}$ is not necessarily increasing.

## 3. FRACTIONAL RELAXATIONS OF THE ERDŐS-FABER-LOVÁSZ CONJECTURE

Let $G$ be a graph in $\mathbb{EFL}_n$. We will consider the hypotheses $k\cdot\mathbb{C}_k\text{-cover}_{\mathrm{f}}(G)=n$, and $k\cdot\mathbb{K}_k\text{-cover}_{\mathrm{f}}(G)=n$ for $k\in\{1,\ldots,n\}$. Conjecture 1 implies that all of these should be true. Theorem 2 states that for $k=1$, both these hypotheses are indeed true. At the other extreme, we have $n\cdot\mathbb{K}_n\text{-cover}_{\mathrm{f}}(G)=n$ as an application of the de Bruijn-Erdős theorem, while the hypothesis $n\cdot\mathbb{C}_n\text{-cover}_{\mathrm{f}}(G)=n$ implies $\chi(G)=n$. For $k=n-1$ we get the following results.

**Theorem 3.** *Let $G$ be a graph in $\mathbb{EFL}_n$. Then $(n-1)\cdot\mathbb{K}_{n-1}\text{-cover}_{\mathrm{f}}(G)=n$.*

**Theorem 4.** *Let $G$ be a graph in $\mathbb{EFL}_n$. If $(n-1)\cdot\mathbb{C}_{n-1}\text{-cover}_{\mathrm{f}}(G)=n$, then $\chi(G)=n$.*

The proof of both these results relies on the following.

**Lemma 5.** *Let $G$ be a graph containing an $n$-clique $K$, $k\leq n$ an integer such that $k\cdot\mathbb{K}_k\text{-cover}_{\mathrm{f}}(G)=n$ and $f:\mathbb{K}_k(G)\to[0,1]$ a minimum-weight fractional $\mathbb{K}_k$-cover of $G$. Then every $H\in\mathbb{K}_k(G)$ such that $f(H)>0$ intersects $K$ in exactly $k$ elements.*

***Proof.*** Consider the bipartite graph $B$ with parts $A_1=V(K)$ and $A_2=\{H\in\mathbb{K}_k(G):f(H)>0\}$, and with edges $[x,H]$ such that $x\in H$. We weigh the edges of $B$ by putting $w([x,H]) = f(H)$. Evaluating the total weight $w(E(B))$ of $E(B)$ in two ways we get

$$w(E(B)) = \sum\nolimits_{x \in A_1} \sum\nolimits_{V(H) \ni x} f(H) \geq \sum\nolimits_{x \in V(K)} 1 = n,$$

and

$$w(E(B)) = \sum\nolimits_{H \in A_2} \sum\nolimits_{[x,H] \in E(B)} w(H) \leq \sum\nolimits_{H \in A_2} k \cdot w(H) = k \cdot \mathbb{K}_k\text{-cover}_{\mathrm{f}}(G) = n.$$

Therefore all inequalities are tight. In particular, for every $H \in A_2$,

$$\sum\nolimits_{[x,H] \in E(B)} w(H) = k \cdot w(H),$$

hence $H$ intersect $K$ in exactly $k$ elements. ■

***Proof of Theorem* 3.** Let $f : \mathbb{K}_1(G) \to [0,1]$ be a minimum-weight fractional $\mathbb{K}_1$-cover. By Theorem 2, $\mathbb{K}_1\text{-cover}_{\mathrm{f}}(G) = n$. Therefore by Lemma 5 every $I \in \mathbb{K}_1(G)$ such that $f(I) > 0$ intersects every $n$-clique of $G$. We then have $H = G - I \in \mathbb{K}_{n-1}(G)$. We define $f' : \mathbb{K}_{n-1}(G) \to [0,1]$ by

$$f'(H) = \begin{cases} \frac{1}{n-1} f(G - H) & \text{if } G - H \in \mathbb{K}_1(G), \\ 0 & \text{otherwise.} \end{cases}$$

Every vertex $x$ of $G$ is in an $n$-clique $K$ of $G$, hence

$$\sum\nolimits_{V(H) \ni x} f'(H) = \sum\nolimits_{y \in V(K) \setminus \{x\}} \sum\nolimits_{I \ni y} \frac{1}{n-1} f(I) \geq \sum\nolimits_{y \in V(K) \setminus \{x\}} \frac{1}{n-1} \cdot 1 = 1.$$

Therefore $f'$ is a fractional $\mathbb{K}_{n-1}$-cover of $G$ and

$$(n-1) \cdot \sum\nolimits_{H \in \mathbb{K}_{n-1}(G)} f'(H) = (n-1) \cdot \sum\nolimits_{I \in \mathbb{K}_1(G)} \frac{1}{n-1} f(I) = n.$$

Thus $(n-1) \cdot \mathbb{K}_{n-1}\text{-cover}_{\mathrm{f}}(G) = n$. ■

***Proof of Theorem* 4.** Suppose that $(n-1) \cdot \mathbb{C}_{n-1}\text{-cover}_{\mathrm{f}}(G) = n$ and let $f : \mathbb{C}_{n-1}(G) \to [0,1]$ be a minimum fractional $\mathbb{C}_{n-1}$-cover of $G$. We have $(n-1) \cdot \mathbb{K}_{n-1}\text{-cover}_{\mathrm{f}}(G) = n$ and $f$ is also a minimum fractional $\mathbb{K}_{n-1}$-cover of $G$. Let $H$ be an element of $\mathbb{C}_{n-1}(G)$ such that $f(H) > 0$. By Lemma 5, $H$ intersects every $n$-clique of $G$ in exactly $n-1$ elements. Since every edge of $G$ is in an $n$-clique, this implies that the complement $I$ of $H$ in $G$ is an independent set. Therefore we can properly $n$-colour $G$ by properly $(n-1)$-colouring $H$ and using an extra colour on $I$. ■

## 4. CONCLUDING COMMENTS AND PROBLEMS

**Problem 6.** Let $G$ be a graph in $\mathbb{EFL}_n$, and $k \le n$. Is there a graph $H \in \mathbb{C}_k(G)$ such that $|V(H)| \ge \frac{k}{n}|V(G)|$? Is there a graph $H \in \mathbb{K}_k(G)$ such that $|V(H)| \ge \frac{k}{n}|V(G)|$?

Define $\alpha_k(G) = \max\{|V(H)| : H \in \mathbb{C}_k(G)\}$ and $\beta_k(G) = \max\{|V(H)| : H \in \mathbb{K}_k(G)\}$. Then $f : V(G) \to [0,1]$ defined by $f(u) = \frac{1}{\alpha_k(G)}$ for all $u \in V(G)$ is a fractional $\mathbb{C}_k$-clique, hence $\mathbb{C}_k\text{-cover}_{\mathrm{f}}(G) \ge \frac{|V(G)|}{\alpha_k(G)}$, and similarly $\mathbb{K}_k\text{-cover}_{\mathrm{f}}(G) \ge \frac{|V(G)|}{\beta_k(G)}$. For $G \in \mathbb{EFL}_n$, Conjecture 1 implies that $\chi(G) = k \cdot \mathbb{C}_k\text{-cover}_{\mathrm{f}}(G) = k \cdot \mathbb{K}_k\text{-cover}_{\mathrm{f}}(G) = n$, which implies that Problem 6 should have an affirmative answer.

It would also be interesting to give an affirmative answer to Problem 6 by constructive methods. In contrast, the proof of Theorem 2 in [4] is indirect. It leads to a column-generation simplex program that finds an optimal fractional colouring $f : \mathbb{I}(G) \to [0,1]$ of a graph $G \in \mathbb{EFL}_n$ in exponential time. Therefore the proof of Theorem 3, based on Theorem 2, is also nonconstructive.

The derivation of Theorem 3 from Theorem 2 also raises the following.

**Problem 7.** Is it true that for all $n$, all $G \in \mathbb{EFL}_n$ and all $k < n$, we have $k \cdot \mathbb{K}_k\text{-cover}_{\mathrm{f}}(G) = n$ if and only if $(n-k) \cdot \mathbb{K}_{n-k}\text{-cover}_{\mathrm{f}}(G) = n$?

Suppose that $k \cdot \mathbb{K}_k\text{-cover}_{\mathrm{f}}(G) = n$, and let $f : \mathbb{K}_k(G) \to [0,1]$ be a minimum-weight fractional $\mathbb{K}_k$-cover of $G$. By Lemma 5, every $H \in \mathbb{K}_k(G)$ such that $f(H) > 0$ intersects every $n$-clique of $G$ in exactly $k$ vertices. For $k = 1$, this implies $G - H \in \mathbb{K}_{n-1}(G)$. However for larger values of $k$, $G - H$ could contain an $(n-k+1)$-clique that does not extend to an $n$-clique in $G$. We will concentrate on the case $k = 2$. Let $K$ be an $(n-1)$-clique that does not extend to an $n$-clique in $G$. Since $G \in \mathbb{EFL}_n$, $K$ admits the edge-decomposition

$$\{C_i \cap K : |C_i \cap K| \ge 2\},$$

where $C_1, \ldots, C_n$ are the defining cliques of $G$. This decomposition is a "linear space" in the sense of [2]; it has either $n-1$ or $n$ parts. It can be shown that if these parts pairwise intersect, then $G$ is $n$-colourable, hence $2 \cdot \mathbb{K}_2\text{-cover}_{\mathrm{f}}(G) = (n-2) \cdot \mathbb{K}_{n-2}\text{-cover}_{\mathrm{f}}(G) = n$. Therefore, $G$ can witness a negative answer to Problem 7 only if the parts $C_i \cap K$ do not all pairwise intersect. The de Bruijn-Erdős theorem then implies that there are exactly $n$ parts, and the results in [2] then imply that $n = 6$ or $n = m^2 + m + 1$ for some $m \ge 2$. Thus for $k = 2$, Problem 7 has an affirmative answer for infinitely many values of $n$. Overall, the results in [2] seem to make Problem 7 tractable.

We conclude with the following partition problem which would also follow from Conjecture 1.

**Problem 8.** Is it true that for all $n$, all $G \in \mathbb{EFL}_n$ and all $k < n$, there exists $H \in \mathbb{K}_k(G)$ such that $G - H \in \mathbb{K}_{n-k}(G)$?

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