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THE 3-RAINBOW INDEX OF A GRAPH

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Abstract

Let G be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A tree T in G is a rainbow tree if no two edges of T receive the same color. For a vertex subset $S \subseteq V(G)$, a tree that connects S in G is called an S-tree. The minimum number of colors that are needed in an edge-coloring of G such that there is a rainbow S-tree for each k-subset S of $V(G)$ is called the k-rainbow index of G, denoted by $rx_k(G)$. In this paper, we first determine the graphs of size m whose 3-rainbow index equals m, $m-1$, $m-2$ or 2. We also obtain the exact values of $rx_3(G)$ when G is a regular multipartite complete graph or a wheel. Finally, we give a sharp upper bound for $rx_3(G)$ when G is 2-connected and 2-edge connected. Graphs G for which $rx_3(G)$ attains this upper bound are determined.

Keywords: rainbow tree, S-tree, k-rainbow index.

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1. Introduction

We follow the terminology and notation of Bondy and Murty [1]. Let G be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A path of G is a *rainbow path* if

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no two edges of the path are colored the same. The graph G is *rainbow connected* if for every two vertices u and v of G, there is a rainbow path connecting u and v. The minimum number of colors for which there is an edge-coloring of G such that G is rainbow connected is called the *rainbow connection number* of G, denoted by $rc(G)$. Results on the rainbow connections can be found in [2, 3, 5, 6, 7].

These concepts were introduced by Chartrand *et al.* in [3]. In [4], they generalized the concept of rainbow path to rainbow tree. A tree T in G is a *rainbow tree* if no two edges of T receive the same color. For $S \subseteq V(G)$, a *rainbow S-tree* is a rainbow tree that connects the vertices of S. Given a fixed integer k with $2 \leq k \leq n$, the edge-coloring c of G is called a *k-rainbow coloring* if for every k-subset S of $V(G)$, there exists a rainbow S-tree. In this case, G is called *k-rainbow connected.* The minimum number of colors that are needed in a k-rainbow coloring of G is called the *k-rainbow index* of G, denoted by $rx_k(G)$. Clearly, when $k = 2$, $rx_2(G)$ is the rainbow connection number $rc(G)$ of G. For every connected graph G of order n, it is easy to see that $rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$.

The *Steiner distance* $d(S)$ of a subset S of vertices in G is the minimum size of a tree in G that connects S. Such a tree is called a *Steiner S-tree* or simply a *Steiner tree*. The *k-Steiner diameter*, $sdiam_k(G)$, of G is the maximum Steiner distance of S among all k-subsets S of G. Then there is a simple upper bound and a lower bound for $rx_k(G)$.

Observation 1.1 [4]**.** *For every connected graph G of order $n \geq 3$ and each integer k, with $3 \leq k \leq n$, $k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1$.*

It was shown in [4] that trees are contained in a class of graphs whose k-rainbow index attains the upper bound.

Proposition 1.2 [4]**.** *Let T be a tree of order $n \geq 3$. For each integer k, with $3 \leq k \leq n$, $rx_k(T) = n - 1$.*

The authors of [4] also gave the following observation.

Observation 1.3 [4]**.** *Let G be a connected graph of order n containing two bridges e and f. For each integer k with $2 \leq k \leq n$, every k-rainbow coloring of G must assign distinct colors to e and f.*

For $k = 2$, $rx_2(G) = rc(G)$, this case has been studied extensively, see [6, 7]. But for $k \geq 3$, very few results has been obtained. In this paper, we focus on $k = 3$. By Observation 1.1, we have $rx_3(G) \geq 2$. On the other hand, if G is a nontrivial connected graph of size m, then the coloring that assigns distinct colors to the edges of G is a 3-rainbow coloring, hence $rx_3(G) \leq m$. So we want to determine the graphs whose 3-rainbow index equals the values m, $m - 1$, $m - 2$ and 2, respectively. The following results are needed.

Lemma 1.4 [4]**.** *For* $3 \leq n \leq 5$, $rx_3(K_n) = 2$.

Lemma 1.5 [4]**.** *Let* G *be a connected graph of order* $n \geq 6$. *For each integer* k *with* $3 \leq k \leq n$, $rx_k(G) \geq 3$.

Theorem 1.6 [4]**.** *For each integer* k *and* n *with* $3 \leq k \leq n$,

$$rx_k(C_n) = \begin{cases} n-2, & \text{if} \quad k = 3 \text{ and } n \geq 4, \\ n-1, & \text{if} \quad k = n = 3 \text{ or } 4 \leq k \leq n. \end{cases}$$

Theorem 1.7 [4]**.** *If* G *is a unicyclic graph of order* $n \geq 3$ *and girth* $g \geq 3$, *then*

$$rx_k(G) = \begin{cases} n-2, & \text{if} \quad k = 3 \text{ and } g \geq 4, \\ n-1, & \text{if} \quad g = 3 \text{ or } 4 \leq k \leq n. \end{cases}$$

The following observation is easy to verify.

Observation 1.8. *Let* G *be a connected graph and* H *be a connected spanning subgraph of* G. *Then* $rx_3(G) \leq rx_3(H)$.

In Section 2, we determine the graphs whose 3-rainbow index equals the values m, $m-1$, $m-2$ or 2. In Section 3, we determine the 3-rainbow index for the complete bipartite graphs $K_{r,r}$ and complete t-partite graphs $K_{t\times r}$ as well as the wheel W_n. Finally, we give a sharp upper bound of $rx_3(G)$ for 2-connected graphs and 2-edge connected graphs. Moreover, graphs whose 3-rainbow index attains the upper bound are characterized.

2. Graphs with $rx_3(G) = m, m-1, m-2$ or 2

At first, we consider the graphs with $rx_3(G) = 2$. From Lemma 1.5, if $rx_3(G) = 2$, then the order n of G satisfies $3 \leq n \leq 5$.

Theorem 2.1. *Let* G *be a connected graph of order* n. *Then* $rx_3(G) = 2$ *if and only if* $G = K_5$ *or* G *is a* 2*-connected graph of order* 4 *or* G *is of order* 3.

Proof. If $n = 3$, then it is easy to see that $rx_3(G) = 2$.

If $n = 4$, assume that G is not 2-connected, then there is a cut vertex v. It is easy to see that a tree connecting the vertices of $G-v$ has size 3, thus $rx_3(G) \geq 3$, a contradiction.

If $n = 5$, then let $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$. Assume that $rx_3(G) = 2$ but G is not K_5. Let $c : E(G) \to \{1, 2\}$ be a rainbow coloring of G. Since every three vertices belong to a rainbow path of length 2, there is no monochromatic triangle. Now we show that the maximum degree $\Delta(G)$ is 4. If $\Delta(G)$ is 2, then G is a cycle or a path, and it is easy to check that $rx_3(G)$ is 3 or 4, a contradiction.

Assume that $\Delta(G)$ is 3. Let $deg(v_1) = 3$ and $N(v_1) = \{v_2, v_3, v_4\}$. Then at least two edges incident to v_1 have the same color, say $c(v_1v_2) = c(v_1v_3) = 1$. Consider $\{v_1, v_2, v_5\}$, $\{v_1, v_3, v_5\}$, this forces $c(v_2v_5) = c(v_3v_5) = 2$. Consider $\{v_1, v_2, v_3\}$, it implies that $c(v_2v_3) = 2$, but now $\{v_2, v_3, v_5\}$ forms a monochromatic triangle, a contradiction. Thus $\Delta(G) = 4$. Suppose $deg(v_1) = 4$. If there are three edges incident to v_1 colored the same, say $c(v_1v_2) = c(v_1v_3) = c(v_1v_4) = 1$, then consider the three vertices v_2, v_3 and v_4. Since these three vertices must belong to a rainbow path of length 2, without loss of generality, assume that $c(v_2v_3) = 1$ and $c(v_3v_4) = 2$. However then $\{v_1, v_2, v_3\}$ is a monochromatic triangle, which is impossible. Therefore only two edges incident to v_1 are assigned the same color. Since G is not K_5, G is a spanning subgraph of $K_5 - e$. Since $deg(v_1) = 4$, we may assume that G is a spanning subgraph of $K_5 - v_3v_4$. Let $G' = K_5 - v_3v_4$. Consider $\{v_1, v_3, v_4\}$, it implies v_1v_3 and v_1v_4 must have different colors, without loss of generality, assume that $c(v_1v_3) = 1$ and $c(v_1v_4) = 2$. By symmetry, suppose $c(v_1v_2) = 1$ and $c(v_1v_5) = 2$. Then $c(v_2v_3) = 2$, $c(v_4v_5) = 1$. Consider $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_2, v_3, v_5\}$, then $c(v_2v_4) = 1$, $c(v_3v_5) = 2$, $c(v_2v_5) = 1$, but now $\{v_2, v_4, v_5\}$ forms a monochromatic triangle, which is impossible. Hence, $rx_3(G) \geq rx_3(G') \geq 3$, contradicting the assumption. ■

Theorem 2.2. *Let G be a connected graph of size $m \geq 3$. Then*

(1) $rx_3(G) = m$ *if and only if G is a tree.*

(2) $rx_3(G) = m - 1$ *if and only if G is a unicyclic graph with girth* 3.

(3) $rx_3(G) = m - 2$ *if and only if G is a unicyclic graph with girth at least* 4.

Proof. (1) By Proposition 1.2, if G is a tree, then $rx_3(G) = n - 1 = m$. Conversely, if $rx_3(G) = m$ but G is not a tree, then $m \geq n$. By Observation 1.1, $rx_3(G) \leq n - 1 \leq m - 1$, a contradiction.

(2) If G is a unicyclic graph with girth 3, then by Theorem 1.7, $rx_3(G) = n - 1 = m - 1$. Conversely, if $rx_3(G) = m - 1$, then by (1), G must contain cycles. If G contains at least two cycles, then $m \geq n + 1$. By Observation 1.1, $rx_3(G) \leq n - 1 \leq m - 2$, a contradiction. Thus, G contains exactly one cycle. If the cycle of G is of length at least 4, then by Theorem 1.7, $rx_3(G) = n - 2 = m - 2$, a contradiction. Thus, the cycle of G is of length 3, the result holds.

(3) If G is a unicyclic graph with girth at least 4, then by Theorem 1.7, $rx_3(G) = n - 2 = m - 2$. Conversely, if $rx_3(G) = m - 2$ and $m \geq n + 2$, then by Observation 1.1, $rx_3(G) \leq n - 1 \leq m - 3$, a contradiction. Thus, $m \leq n + 1$. If $m = n$, then G is a unicyclic graph. By Theorem 1.7, the girth of G is at least 4. If $m = n + 1$, and there are two edge-disjoint cycles C_1 and C_2 of lengths, respectively g_1 and g_2 such that $g_1 \geq g_2$, then if $g_1 \geq 4$, we assign $g_1 - 2$ colors to C_1, $g_2 - 1$ new colors to C_2 and assign new distinct colors to all the remaining edges, which make G 3-rainbow connected, hence $rx_3(G) \leq m - 3$, a

contradiction. Therefore $g_1 = g_2 = 3$. In this case, we assign to each cycle three colors 1, 2, 3, and assign new colors to all the remaining edges. It follows that, then G is 3-rainbow connected, thus $rx_3(G) \leq m - 3$. If these two cycles are not edge-disjoint, we can also use $m - 3$ colors to make G 3-rainbow connected, a contradiction. ∎

3. THE 3-RAINBOW INDEX OF SOME SPECIAL GRAPHS

In this section, we determine the 3-rainbow index of some special graphs. First, we consider the regular complete bipartite graphs $K_{r,r}$. It is easy to see that when $r = 2$, $rx_3(K_{2,2}) = 2$ and, logically, we can define $rx_3(K_{1,1}) = 0$.

Theorem 3.1. *For each integer r with $r \geq 3$, $rx_3(K_{r,r}) = 3$.*

Proof. Let U and W be the partite sets of $K_{r,r}$, where $|U| = |W| = r$. Suppose that $U = \{u_1, \ldots, u_r\}$ and $W = \{w_1, \ldots, w_r\}$. If $S \subseteq U$ and $|S| = 3$, then every S-tree has size at least 3; hence $rx_3(K_{r,r}) \geq 3$.

Next we show that $rx_3(K_{r,r}) \leq 3$. We define a coloring $c : E(K_{r,r}) \to \{1, 2, 3\}$ as follows.

$$c(u_i w_j) = \begin{cases} 1, & \text{if} \quad 1 \leq i = j \leq r, \\ 2, & \text{if} \quad 1 \leq i < j \leq r, \\ 3, & \text{if} \quad 1 \leq j < i \leq r. \end{cases}$$

Now we show that c is a 3-rainbow coloring of $K_{r,r}$. Let S be a set of three vertices of $K_{r,r}$. We consider two cases.

Case 1. The vertices of S belong to the same partite set of $K_{r,r}$. Without loss of generality, let $S = \{u_i, u_j, u_k\}$, where $i < j < k$. Then $T = \{u_i w_j, u_j w_j, u_k w_j\}$ is a rainbow S-tree.

Case 2. The vertices of S belong to different partite sets of $K_{r,r}$. Without loss of generality, let $S = \{u_i, u_j, w_k\}$, where $i < j$.

Subcase 2.1. Let $k < i < j$. Then $T = \{u_i w_k, u_i w_j, u_j w_j\}$ is a rainbow S-tree.

Subcase 2.2. Let $i \leq k \leq j$. Then $T = \{u_i w_k, u_j w_k\}$ is a rainbow S-tree.

Subcase 2.3. Let $i < j < k$. Then $T = \{u_i w_i, u_j w_i, u_j w_k\}$ is a rainbow S-tree. ∎

With the aid of Theorem 3.1, we are now able to determine the 3-rainbow index of complete t-partite graph $K_{t\times r}$. Note that we always have $t \geq 3$. When $r = 1$, $rx_3(K_{t\times 1}) = rx_3(K_t)$, which was given in [4].

Theorem 3.2. *Let $K_{t\times r}$ be a complete t-partite graph, where $r \geq 2$ and $t \geq 3$. Then $rx_3(K_{t\times r}) = 3$.*

Proof. Let $U_1, U_2, \ldots, U_t$ be the t partite sets of $K_{t\times r}$, where $|U_i| = r$. Suppose that $U_i = \{u_{i1}, \ldots, u_{ir}\}$. If $S \subseteq U_i$ and $|S| = 3$, then every S-tree has size at least 3, hence $rx_3(K_{r,r}) \geq 3$.

Next we show that $rx_3(K_{t\times r}) \leq 3$. We define a coloring c: $E(K_{t\times r}) \to \{1, 2, 3\}$ as follows.

$$c(u_{ai}u_{bj}) = \begin{cases} 1, & \text{if} \quad 1 \leq i = j \leq r, \\ 2, & \text{if} \quad 1 \leq i < j \leq r, \\ 3, & \text{if} \quad 1 \leq j < i \leq r, \end{cases}$$

where $1 \leq a < b \leq t$.

We now show that c is a 3-rainbow coloring of $K_{t\times r}$. Let S be a set of three vertices of $K_{t\times r}$.

Case 1. The vertices of S belong to the same partite set. Without loss of generality, let $S = \{u_{a1}, u_{a2}, u_{a3}\}$. Then $T = \{u_{a1}u_{b2}, u_{a2}u_{b2}, u_{a3}u_{b2}\}$ is a rainbow S-tree.

Case 2. Two vertices of S belong to the same partite set. Without loss of generality, let $S = \{u_{ai}, u_{aj}, u_{bk}\}$. If $k < i < j$, then $T = \{u_{ai}u_{bk}, u_{ai}u_{bj}, u_{aj}u_{bj}\}$ is a rainbow S-tree. If $i \leq k \leq j$, then $T = \{u_{ai}u_{bk}, u_{aj}u_{bk}\}$ is a rainbow S-tree. If $i < j < k$, then $T = \{u_{ai}u_{bi}, u_{aj}u_{bi}, u_{aj}u_{bk}\}$ is a rainbow S-tree.

Case 3. Each vertex of S belongs to a different partite set. Let $S = \{u_{ai}, u_{bj}, u_{ck}\}$, $a < b < c$.

Subcase 3.1. Assume that $i = j = k$. Without loss of generality, let $S = \{u_{a1}, u_{b1}, u_{c1}\}$. Then $T = \{u_{a1}u_{b1}, u_{a1}u_{b2}, u_{b2}u_{c1}\}$ is a rainbow S-tree.

Subcase 3.2. Suppose that $i = j \neq k$. Without loss of generality, let $S = \{u_{a1}, u_{b1}, u_{c2}\}$. Clearly, $T = \{u_{a1}u_{b1}, u_{b1}u_{c2}\}$ is a rainbow S-tree.

Subcase 3.3. Let $i \neq j \neq k$. Without loss of generality, let $S = \{u_{a1}, u_{b2}, u_{c3}\}$. Then $T = \{u_{a1}u_{c1}, u_{c1}u_{b2}, u_{b2}u_{c3}\}$ is a rainbow S-tree. ■

Another well-known class of graphs are wheels. For $n \geq 3$, the *wheel* W_n is a graph constructed by joining a vertex v to every vertex of a cycle C_n : $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$. Given an edge-coloring c of W_n, for two adjacent vertices v_i and v_{i+1}, we define an edge-coloring of the graph by identifying v_i and v_{i+1} to a new vertex v' as follows: set $c(vv') = c(vv_{i+1})$, $c(v_{i-1}v') = c(v_{i-1}v_i)$, $c(v'v_{i+2}) = c(v_{i+1}v_{i+2})$, and keep the coloring for the remaining edges. We call this coloring the *identified-coloring at* v_i *and* v_{i+1}. Next we determine the 3-rainbow index of wheels.

Theorem 3.3. *For $n \geq 3$, the 3-rainbow index of the wheel W_n is*

$$rx_3(W_n) = \begin{cases} 2, & \textit{if} \quad n = 3, \\ 3, & \textit{if} \quad 4 \leq n \leq 6, \\ 4, & \textit{if} \quad 7 \leq n \leq 16, \\ 5, & \textit{if} \quad n \geq 17. \end{cases}$$

Proof. Suppose that W_n consists of a cycle $C_n : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ and another vertex v joined to every vertex of C_n.

Since $W_3 = K_4$, it follows by Lemma 1.4 that $rx_3(W_3) = 2$.

If $n = 6$, then let $S = \{v_1, v_2, v_4\}$. Since every S-tree has size at least 3, $rx_3(W_6) \geq 3$. Next we show that $rx_3(W_6) \leq 3$ by providing a 3-rainbow coloring of W_6 as follows:

$$c(e) = \begin{cases} 1, & \text{if} \quad e \in \{vv_1, vv_4, v_2v_3, v_5v_6\}, \\ 2, & \text{if} \quad e \in \{vv_2, vv_5, v_3v_4, v_1v_6\}, \\ 3, & \text{if} \quad e \in \{vv_3, vv_6, v_4v_5, v_1v_2\}. \end{cases}$$

If $n = 5$, then $|V(W_5)| = 6$ and, by Lemma 1.5, $rx_3(W_5) \geq 3$. Then we show that $rx_3(W_5) \leq 3$. We provide a 3-rainbow coloring of W_5 obtained from the 3-rainbow coloring of W_6 by the identified-coloring at v_5 and v_6.

If $n = 4$, then by Theorem 2.1, $rx_3(W_4) \geq 3$. Then we show that $rx_3(W_4) \leq 3$. We provide a 3-rainbow coloring of W_4 obtained from the 3-rainbow coloring of W_6 by the identified-coloring at v_5 and v_6, v_4 and v_5, respectively.

Claim 1. If $7 \leq n \leq 16$, then $rx_3(W_n) = 4$.

First we show that $rx_3(W_7) \geq 4$. Assume, to the contrary, that $rx_3(W_7) \leq 3$. Let $c : E(W_7) \to \{1, 2, 3\}$ be a 3-rainbow coloring of W_7. Since $deg(v) = 7 > 2 \times 3$, there exists $A \subseteq V(C_n)$ such that $|A| = 3$ and all edges in $\{uv : u \in A\}$ are colored the same. Thus, there must exist at least two vertices $v_i, v_j \in A$ such that $deg_{C_7}(v_i, v_j) \geq 2$ and a vertex $v_k \in C_7$ such that $v_k \notin \{v_{i-1}, v_{i+1}, v_{j-1}, v_{j-1}\}$. Let $S = \{v_i, v_j, v_k\}$. Note that the only S-tree of size 3 is $T = vv_i \cup vv_j \cup vv_k$, but $c(vv_i) = c(vv_j)$, it follows that there is no rainbow S-tree, which is a contradiction. Similarly, we have $rx_3(W_n) \geq 4$ for all $n \geq 8$.

Second, we show that $rx_3(W_{16}) \leq 4$, which we establish by defining a 4-rainbow coloring c of W_{16} as shown in Figure 1. It is easy to check that c is a 4-rainbow coloring of W_{16}. Therefore, $rx_3(W_{16}) = 4$.

When $13 \leq n \leq 15$, we obtain a 4-rainbow coloring of W_{15}, W_{14}, W_{13} from the 4-rainbow coloring c of W_{16} by consecutively using the identified-colorings at v_1 and v_{16}, v_{12} and v_{13}, v_8 and v_9.

When $n = 12$, we define a 4-rainbow coloring of W_{12} as shown in Figure 1.

When $7 \leq n \leq 11$, we obtain a 4-rainbow coloring of W_{11}, W_{10}, W_9, W_8, W_7 from the 4-rainbow coloring c of W_{12} by consecutively using the identified-colorings at v_1 and v_2, v_4 and v_5, v_7 and v_8, v_{10} and v_{11}, v_{11} and v_{12}.

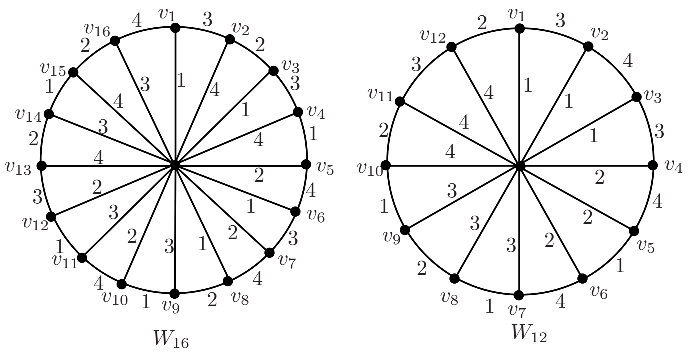

Figure 1. 3-rainbow coloring of W_{16} and W_{12}.

Claim 2. If $n \geq 17$, then $rx_3(W_n) = 5$.

First we show that $rx_3(W_{17}) \geq 5$. Assume, to the contrary, that $rx_3(W_{17}) \leq 4$. Let $c : E(W_{17}) \to \{1, 2, 3, 4\}$ be a 4-rainbow coloring of W_{17}. Since $deg(v) = 17 > 4 \times 4$, there exists $A \subseteq V(C_n)$ such that $|A| = 5$ and all edges in $\{uv : u \in A\}$ are colored the same, say 1. Suppose that $A = \{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}\}$, where $i_1 \leq i_2 \leq i_3 \leq i_4 \leq i_5$. There exists k such that $deg_{C_{17}}(v_{i_k}, v_{i_{k+1}}) \geq 3$, where $1 \leq k \leq 4$. Let $S = \{v_{i_k}, v_{i_{k+1}}, v_{i_{k+3}}\}$. Since $d_{C_{17}}(v_{i_k}, v_{i_{k+3}}) \geq 2$ and $d_{C_{17}}(v_{i_{k+1}}, v_{i_{k+3}}) \geq 2$, the only possible S-tree is the path $P = v_{i_{k+1}} v_{i_{k+2}} v_{i_{k+3}} v_{i_{k+4}} v_{i_{k+5}}$, where addition is performed modulo 5. Thus color 1 must appear in P and every edge of the path must have a distinct color. By symmetry, we consider two cases. First, let $c(v_{i_{k+1}} v_{i_{k+2}}) = 1$. Suppose $c(v_{i_{k+2}} v_{i_{k+3}}) = 2$, $c(v_{i_{k+3}} v_{i_{k+4}}) = 3$. There exists a vertex v_0, where $c(vv_0) = 2$ or 3, such that $d(v_0, A) \geq 3$. It is easy to see that there is no rainbow $\{v_0, v_{i_{k+2}}, v_{i_{k+4}}\}$-tree. In the remaining case, if $c(v_{i_{k+2}} v_{i_{k+3}}) = 1$, then we can also find such a vertex v_0 such that there exists no $\{v_0, v_{i_{k+2}}, v_{i_{k+3}}\}$-tree, which is a contradiction.

To show that $rx_3(W_n) \leq 5$ for $n \geq 17$, we define a 5-rainbow coloring of W_n as follows:

$$c(e) = \begin{cases} j, & \text{if} \quad e = vv_i \ \text{ and } \ i \equiv j (\text{mod } 5),\ 1 \leq j \leq 5, \\ i + 3, & \text{if} \quad e = v_i v_{i+1}. \end{cases}$$

It is easy to see that c is a 5-rainbow coloring of W_n. Therefore, $rx_3(W_n) = 5$ for $n \geq 17$. ■

4. The 3-rainbow Index of 2-connected and 2-edge-connected Graphs

In this section, we give a sharp upper bound of the 3-rainbow index for 2-connected and 2-edge-connected graphs. We start with some lemmas that will be used in the sequel.

Lemma 4.1. *Let G be a connected graph and $\{V_1, V_2, \ldots, V_k\}$ be a partition of $V(G)$. If each V_i induces a connected subgraph H_i of G, then $rx_3(G) \leq k - 1 + \sum_{i=1}^{k} rx_3(H_i)$.*

Proof. Let G' be a graph obtained from G by contracting each H_i to a single vertex. Then G' is a graph of order k, so $rx_3(G') \leq k - 1$. Take an edge-coloring of G' with $k - 1$ colors such that G' is 3-rainbow connected. Now go back to G, and color each edge connecting vertices in distinct H_i with the color of the corresponding edge in G'. For each $i = 1, 2, \ldots, k$, we use $rx_3(H_i)$ new colors to assign the edges of H_i such that H_i is 3-rainbow connected. The resulting edge-coloring makes G 3-rainbow connected. Therefore, $rx_3(G) \leq k-1+\sum_{i=1}^{k} rx_3(H_i)$. ■

To *subdivide* an edge e is to delete e, add a new vertex x, and join x to the ends of e. Any graph derived from a graph G by a sequence of edge subdivisions is called a *subdivision* of G. Given a rainbow coloring of G, if we subdivide an edge $e = uv$ of G by xu and xv, then we can assign xu the same color as e and assign xv a new color, which also make the subdivision of G 3-rainbow connected. Hence, the following lemma holds.

Lemma 4.2. *Let G be a connected graph, and H be a subdivision of G. Then $rx_3(H) \leq rx_3(G) + |V(H)| - |V(G)|$.*

The Θ-*graph* is a graph G consisting of three internally disjoint paths with common end vertices and of lengths a, b, and c, respectively, such that $a \leq b \leq c$. Clearly, $a + b + c = n + 1$ where n is the order of G.

Lemma 4.3. *Let G be a Θ-graph of order n. If $n \geq 7$, then $rx_3(G) \leq n - 3$.*

Proof. Let the three internally disjoint paths be P_1, P_2, P_3 with the common end vertices u and v, and the lengths of P_1, P_2, P_3 are a, b, c, respectively, where $a \leq b \leq c$.

Case 1. $b \geq 3$. Then $c \geq b \geq 3$, $a \geq 1$. First, we consider the graph Θ_1 with $a = 1$, $b = 3$ and $c = 3$. We color uP_1v with color 3, uP_2v with colors $2, 3, 1$, and uP_3v with colors $1, 3, 2$. The resulting coloring makes Θ_1 rainbow connected. Thus, $rx_3(\Theta_1) \leq 3 = |V(\Theta_1)| - 3$. For a general Θ-graph G with $b \geq 3$ and $n \geq 7$, we first observe that it is a subdivision of Θ_1. Hence by Lemma 4.2, $rx_3(G) \leq rx_3(\Theta_1) + |V(G)| - |V(\Theta_1)| \leq |V(G)| - 3$.

Case 2. $a = 1$, $b = 2$. Then since $a + b + c = n + 1 \geq 8$, $c \geq 5$. Consider the graph Θ_2 with $a = 1$, $b = 2$ and $c = 5$. We rainbow color uP_1v with color 4, uP_2v with colors $1, 3$, and uP_3v with colors $2, 3, 4, 2, 1$. Thus, $rx_3(\Theta_2) \leq 4 = |V(\Theta_2)| -$

3. Consider now a general Θ-graph G with $a = 1$, $b = 2$, $c \geq 5$. Clearly, it is a subdivision of Θ_2, hence by Lemma 4.2, $rx_3(G) \leq rx_3(\Theta_2) + |V(G)| - |V(\Theta_2)| \leq |V(G)| - 3$.

Case 3. $a = 2$, $b = 2$, Then since $a + b + c = n + 1 \geq 8$, $c \geq 4$. Consider the graph Θ_3 with $a = 2$, $b = 2$ and $c = 3$. We rainbow color uP_1v with colors $3, 2$, uP_2v with colors $2, 1$, and uP_3v with colors $1, 2, 3$. Thus, $rx_3(\Theta_3) \leq 3 = |V(\Theta_3)| - 3$. Consider now a general Θ-graph G with $a = 2$, $b = 2$, $c \geq 4$. It is a subdivision of Θ_3, hence by Lemma 4.2, $rx_3(G) \leq rx_3(\Theta_3) + |V(G)| - |V(\Theta_3)| \leq |V(G)| - 3$.

Every Θ-graph with $n \geq 7$ is one of the above cases, therefore $rx_3(G) \leq n - 3$. ■

A 3-*sun* is a graph G which is defined from $C_6 = v_1, v_2, \ldots, v_6, v_7 = v_1$ by adding three edges v_2v_4, v_2v_6 and v_4v_6.

Lemma 4.4. *Let G be a 2-connected graph of order 6. If G is a spanning subgraph of a 3-sun, then $rx_3(G) = 4$. Otherwise, $rx_3(G) = 3$.*

Proof. Since G is a 2-connected graph of order 6, G is a graph with a cycle $C_6 = v_1, v_2, \ldots, v_6, v_7 = v_1$ and some additional edges.

If G is a subgraph of a 3-sun, then every tree connecting the three vertices $\{v_1, v_3, v_5\}$ must have size at least 4, which implies that $rx_3(G) \geq 4$. On the other hand, $rx_3(G) \leq rx_3(C_6) \leq 4$. Therefore, $rx_3(G) = 4$.

If there is an edge between the two antipodal vertices of C_6, then by Lemma 4.3, $rx_3(G) = 3$.

If G contains the edges v_1v_3 and v_2v_6, then it contains Θ_3, defined in Lemma 4.3, as a spanning subgraph, thus $rx_3(G) = 3$.

If G contains the edges v_1v_5 and v_2v_4, we give a rainbow 3-coloring c of G: $c(v_1v_2) = c(v_4v_5) = 1$, $c(v_2v_3) = c(v_2v_4) = c(v_1v_5) = c(v_5v_6) = 2$, $c(v_3v_4) = c(v_1v_6) = 3$. ■

Let H be a subgraph of a graph G. An *ear* of H in G is a nontrivial path in G whose ends are in H but whose internal vertices are not. A nested sequence of graphs is a sequence $\{G_0, G_1, \ldots, G_k\}$ of graphs such that $G_i \subset G_{i+1}$, for $0 \leq i < k$. An *ear decomposition* of a 2-connected graph G is a nested sequence $\{G_0, G_1, \ldots, G_k\}$ of 2-connected subgraphs of G such that: (1) G_0 is a cycle; (2) $G_i = G_{i-1} \cup P_i$, where P_i is an ear of G_{i-1} in G, for $1 \leq i \leq k$; (3) $G_k = G$. We call an ear decomposition *nonincreasing* if $\ell(P_1) \geq \ell(P_2) \geq \cdots \geq \ell(P_k)$, where $\ell(P_i)$ denotes the length of P_i.

Theorem 4.5. *Let G be a 2-connected graph of order $n \geq 4$. Then $rx_3(G) \leq n - 2$, with equality if and only if $G = C_n$ or G is a spanning subgraph of 3-sun or G is a spanning subgraph of $K_5 - e$ or G is a spanning subgraph of K_4.*

Proof. Since G is 2-connected, G contains a cycle. Let C be the longest cycle of G. Then $|V(C)| \geq 4$, $rx_3(C) \leq |V(C)| - 2$. Let $H_1 = C$, $H_2, H_3, \ldots, H_{n-|V(C)|+1}$ be subgraphs of G, each is a single vertex. Then by Lemma 4.1, $rx_3(G) \leq n - |V(C)| + rx_3(H_1) \leq n - 2$.

If $G = C$, then by Theorem 1.6, $rx_3(G) = n - 2$.

If $G \neq C$, then G contains a nonincreasing ear decomposition $\{G_0, G_1, \ldots, G_k\}$. Let $H_1 = C \cup P_1$. Then H_1 is a Θ-graph. We choose $H_2, H_3, \ldots, H_{n-|V(H_1)|+1}$ as subgraphs of G with a single vertex each. By Lemma 4.1, $rx_3(G) \leq n - |V(H_1)| + rx_3(H_1)$.

If $|V(H_1)| \geq 7$, then by Lemma 4.3, $rx_3(H_1) \leq |V(H_1)| - 3$, hence $rx_3(G) \leq n - 3$.

If $|V(H_1)| = 6$, we consider three cases.

Case 1. $|V(C)| = 6$. Then $\ell(P_1) = 1$. Hence $\ell(P_1) = \ell(P_2) = \cdots = \ell(P_k) = 1$, G is a graph of order 6. By Lemma 4.4, $rx_3(G) = 4$ if and only if G is a spanning subgraph of a 3-sun.

Case 2. $|V(C)| = 5$. Then $\ell(P_1) = 2$. Let u and v be the end vertices of P_1. If $d_C(u, v) = 1$, then we can find a cycle larger than C, contradicting the choice of C. Otherwise, $d_C(u, v) = 2$ and is the graph Θ_3 defined in Lemma 4.3. Then $rx_3(H_1) = rx_3(\Theta_3) \leq 3 = |V(H_1)| - 3$, thus $rx_3(G) \leq n - 3$.

Case 3. $|V(C)| = 4$. Then $\ell(P_1) = 3$. Let u and v be the end vertices of P_1. Either $d_C(u, v) = 1$ or $d_C(u, v) = 2$, thus we can always find a cycle larger than C, a contradiction.

If $|V(H_1)| = 5$, there are two cases to be considered. If $|V(C)| = 5$, then $\ell(P_1) = 1$, hence G is a graph of order 5. By Theorem 2.1, $rx_3(G) = 3 = n - 2$ except for K_5, whose 3-rainbow index is 2. If $|V(C)| = 4$, then $\ell(P_1) = 2$. Let u and v be the end vertices of P_1. Note that $d_C(u, v) = 2$. If $\ell(P_2) = 1$, then G is a graph of order 5. If $\ell(P_2) \geq 2$, then let u' and v' be the end vertices of P_2. It holds $\{u', v'\} = \{u, v\}$, otherwise, we can find a cycle larger than C. Let $H_1' = H_1 \cup P_2$. Then H_1' is a graph consisting of 4 internally disjoint paths of length 2 with common vertices u and v. We color the edges of the four paths with colors 12, 21, 31, 13, the resulting coloring makes H_1' rainbow connected, thus, $rx_3(H_1') \leq 3 = |V(H_1')| - 3$. Let $H_2', H_3', \ldots, H'_{n-|V(H_1')|+1}$ be subgraphs of G, each is a single vertex. Then by Lemma 4.1, $rx_3(G) \leq n - |V(H_1')| + rx_3(H_1') \leq n - 3$.

If $|V(H_1)| = 4$, then $|V(C)| = 4$, $\ell(P_1) = 1$, G is a graph of order 4, by Theorem 2.1, $rx_3(G) = 2 = n - 2$.

Therefore, $rx_3(G) = n - 2$ if and only if $G = C_n$ or G is a spanning subgraph of 3-sun or G is a spanning subgraph of $K_5 - e$ or G is a spanning subgraph of K_4. ■

Now we turn to 2-edge-connected graphs. We say that an ear is *closed* if its endvertices are identical, otherwise, it is *open*. An open or closed ear is called a *handle*. For a 2-edge-connected graph G, there is a handle-decomposition, that is a sequence $\{G_0, G_1, \ldots, G_k\}$ of graphs such that: (1) G_0 is a cycle; (2) $G_i = G_{i-1} \cup P_i$, where P_i is a handle of G_{i-1} in G, for $1 \leq i \leq k$; (3) $G_k = G$. Similar to Theorem 3.2, we give an upper bound of 2-edge-connected graphs.

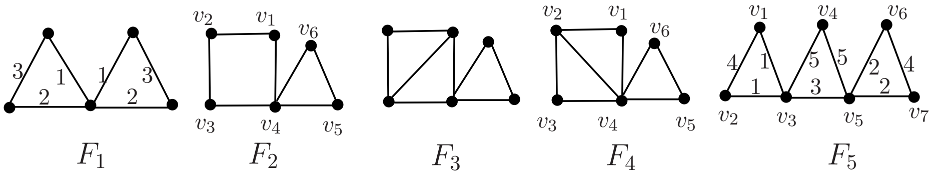

Figure 2. Graphs with $rx_3(G) = n - 2$.

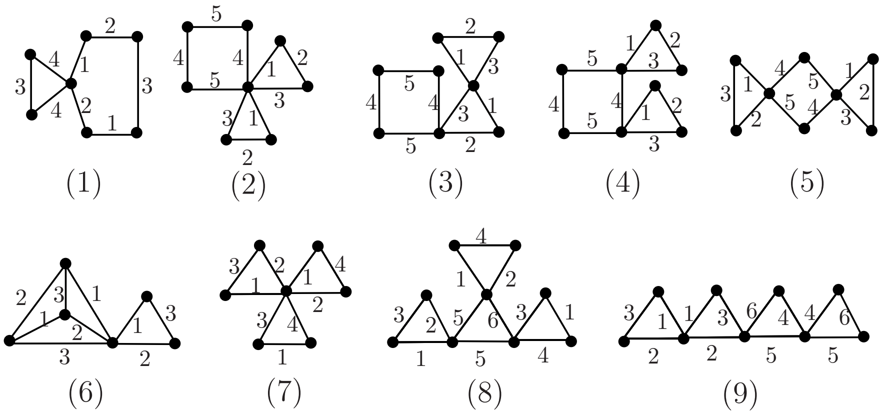

Figure 3. Graphs with $rx_3(G) \leq n - 3$.

Theorem 4.6. *Let G be a 2-edge-connected graph of order $n \geq 4$. Then $rx_3(G) \leq n - 2$, with equality if and only if G is a graph attaining the upper bound in Theorem* 4.5 *or a graph presented in Figure* 2.

Proof. Let C be the largest cycle of G. If $|V(C)| \geq 4$, then $rx_3(C) \leq |V(C)| - 2$. Otherwise, all cycles of G are of length 3. Since $n \geq 4$, there are at least two triangles C_1 and C_2 with a common vertex v. Let $F_1 = C_1 \cup C_2$, we rainbow color F_1 with three colors, see the graph F_1 in Figure 2, thus $rx_3(F_1) \leq 3 = |V(F_1)| - 2$. Let $H_1 = C$ or F_1, $H_2, H_3, \ldots, H_{n-|V(H_1)|+1}$ be subgraphs of G with a single vertex each. Then by Lemma 4.1, $rx_3(G) \leq n - |V(H_1)| + rx_3(H_1) \leq n - 2$.

Now we determine the graphs that obtain the upper bound $n - 2$.

If $G = C$, then by Theorem 1.6, $rx_3(G) = n - 2$.

If $G \neq C$, then G contains a handle-decomposition $\{G_0, G_1, \ldots, G_k\}$. Let $H_1 \subseteq G$, $H_2, H_3, \ldots, H_{n-|V(H_1)|+1}$ be subgraphs of G with a single vertex each. Then by Lemma 4.1, if we show that $rx_3(H_1) \leq |V(H_1)|-3$, then we have $rx_3(G) \leq n-3$.

If $|V(C)| \geq 4$ and P_1 is an open ear, we come back to Theorem 4.5. If $|V(C)| = 3$ and P_1 is an open ear, then a cycle is of length larger than C, a contradiction.

If $|V(C)| \geq 4$ and P_1 is a closed ear, then G_1 is a union of two cycles $C_1 = C$ and $C_2 = P_1$. If both of the cycles are of length at least 4, we rainbow color each cycle C_i with $|V(C_i)| - 2$ colors, which makes G_1 3-rainbow connected. So we assume that C_2 is of length 3. If C_1 is of length 5, we rainbow color G_1 by 4 colors, see Figure 3(1). If C_1 is of length greater than 5, then it is the subdivision of the graph in the case of $|V(C_1)| = 5$. For all the above three cases, we have $rx_3(G_1) \leq |V(G_1)| - 3$. Let $H_1 = G_1$, it follows that $rx_3(G) \leq n - 3$.

So it remains the case that $|V(C_1)| = 4$, $|V(C_2)| = 3$, we denote this graph by F_2, see Figure 2. Then F_2 is a subdivision of F_1, so $rx_3(F_2) \leq 4$. On the other hand, consider $S = \{v_2, v_5, v_6\}$. Every S-tree has size at least 4, hence $rx_3(F_2) = 4 = |V(F_2)| - 2$. Observe that P_2 is a closed ear of length at most 4, then $G_2 = F_2 \cup P_2$. If $\ell(P_2) = 4$, then G_2 contains two cycles of length 4. If $\ell(P_2) = 3$, we rainbow colors G_2 with $|V(G_2)| - 3$ colors, see Figure 3(2–5). For the above two cases, $rx_3(G_2) \leq |V(G_2)| - 3$. Let $H_1 = G_2$, it implies that $rx_3(G) \leq n - 3$. If $\ell(P_2) = 1$, then P_2 must be an edge joining the vertices of C_1, there are two graphs, denoted by F_3 and F_4. Similarly to F_2, we have $rx_3(F_3) = |V(F_3)| - 2$. For F_4, $rx_3(F_4) \leq rx_3(F_2) \leq 4$. On the other hand, suppose $rx_3(F_4) \leq 3$. Consider $\{v_1, v_3, v_5\}$, $\{v_1, v_3, v_6\}$. We have that $c(v_4v_6) = c(v_4v_5)$, which implies that there is no rainbow $\{v_1, v_5, v_6\}$-tree or $\{v_3, v_5, v_6\}$-tree, a contradiction. Hence $rx_3(F_4) = 4 = |V(F_4)| - 2$. Observe that P_3 is of length 1, $G_3 = F_3 \cup P_3$ or $F_4 \cup P_3$, we can rainbow color G_3 by 3 colors, see Figure 3(6). Let $H_1 = G_3$. Then $rx_3(G) \leq n - 3$.

If $|V(C)| = 3$ and P_1 is a closed ear, then $\ell(P_1) = 3$. Thus $G_1 = F_1$, and it is easy to get $rx_3(G_1) = |V(G_1)| - 2$. If P_2 exists, then it must be a closed ear of length 3, and there are two cases for the graph G_2. If G_2 is as in Figure 3(7), then $rx_3(G_2) \leq |V(G_2)|-3$, let $H_1 = G_2$, thus $rx_3(G) \leq n-3$. If G_2 is the graph F_5 in Figure 2, then we prove that its 3-rainbow index is $|V(G_2)| - 2$. Using the graph F_5 in Figure 2, we have that $rx_3(G_2) \leq 5$. If $rx_3(G_2) \leq 4$, then let $c : E(G) \rightarrow \{1, 2, 3, 4\}$ be the 4-rainbow coloring of G_2. Consider $\{v_1, v_4, v_6\}$ and $\{v_1, v_4, v_7\}$, we have $c(v_1v_3) \neq c(v_5v_6)$, $c(v_1v_3) \neq c(v_5v_7)$. If $c(v_5v_6) = c(v_5v_7)$, then suppose that $c(v_5v_6) = 1$, $c(v_1v_3) = 2$. Consider $\{v_1, v_6, v_7\}$, we may assume $c(v_3v_5) = 3$, $c(v_6v_7) = 4$. Consider $\{v_2, v_6, v_7\}$, $\{v_1, v_2, v_6\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_4, v_6\}$, we have $c(v_2v_3) = 2$, $c(v_1v_2) = 4$, $c(v_3v_4) \in \{1, 4\}, c(v_4v_5) \in \{1, 4\}$, but then there is no rainbow tree connecting $\{v_4, v_6, v_7\}$. If $c(v_5v_6) \neq c(v_5v_7)$, then $c(v_1v_3) \neq c(v_2v_3)$. Let $c(v_1v_3) = 1$, $c(v_2v_3) = 2$, $c(v_5v_6) = 3$, $c(v_5v_6) = 4$. Consider $\{v_1, v_4, v_6\}$, then the colors 2 and 4 must appear in the triangle $v_3v_4v_5$. Consider $\{v_2, v_4, v_7\}$, then the colors 1 and 3 must appear in the triangle $v_3v_4v_5$, which is impossible. So we consider P_3 and, if it exists, then it must be a close ear. There are two cases, no matter which case occurs, we can give a rainbow coloring with $|V(G_3)| - 3$ colors, see Figure 3(8–9). Let $H_1 = G_3$. Then $rx_3(G) \leq n - 3$.

Combining all the above cases, $rx_3(G) = n - 2$ if and only if G is a graph attaining the upper bound in Theorem 4.5 or a graph in Figure 2. ■

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