# ON DECOMPOSING REGULAR GRAPHS INTO ISOMORPHIC DOUBLE-STARS 

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#### Abstract

A double-star is a tree with exactly two vertices of degree greater than 1. If $T$ is a double-star where the two vertices of degree greater than one have degrees $k_{1}+1$ and $k_{2}+1$, then $T$ is denoted by $S_{k_{1}, k_{2}}$. In this note, we show that every double-star with $n$ edges decomposes every $2 n$-regular graph. We also show that the double-star $S_{k, k-1}$ decomposes every $2 k$-regular graph that contains a perfect matching.


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## 1. Introduction

By a decomposition of a graph $G$ we mean a sequence $H_{1}, H_{2}, \ldots, H_{k}$ of subgraphs whose edge sets partition the edge set of $G$. If each subgraph $H_{i}$ is isomorphic to a fixed graph $H$, then the decomposition is an $H$-decomposition of $G$ and we say $H$ decomposes $G$. A large amount of research has been done on the topic of graph decompositions over the last five decades (see [1] and [2] for recent surveys). Much investigation has been motivated by the following conjecture of Ringel [10].

Conjecture 1. Every tree $T$ with $n$ edges decomposes the complete graph $K_{2 n+1}$.
A broadening of Ringel's conjecture is due to Graham and Häggkvist (see [5]).
Conjecture 2. Every tree $T$ with $n$ edges decomposes every $2 n$-regular graph $G$.
Despite persistent attacks over the last 40 years, Ringel's conjecture and variations thereof, such as the Graceful Tree Conjecture (see [4]), still stand today. Much less work has been done on the Graham and Häggkvist conjecture however.

Results confirming Conjecture 2, in certain cases, can be found in Snevily's Ph.D. thesis [11]. For example, Snevily shows that every tree $T$ with $n$ edges decomposes every $2 n$-regular graph $G$ provided that the girth of $G$ is larger than the diameter of $T$. He also shows that every tree with $n$ edges decomposes the cartesian product of any $n$ cycles. Other results on decompositions of the cartesian product of graphs into trees can be found in a recent paper by Jao, Kostochka, and West [8].

The graph $K_{1, k}$ is known as a $k$-star and is often denoted by $S_{k}$. A doublestar is a tree with exactly two vertices of degree greater than 1 . The two vertices of degree greater than 1 are called the centers of the double-star and the edge joining them is called the central-edge. If $T$ is a double-star where the two centers have degrees $k_{1}+1$ and $k_{2}+1$, then $T$ is denoted by $S_{k_{1}, k_{2}}$. Note that $S_{k_{1}, k_{2}}$ has $k_{1}+k_{2}+1$ edges and is isomorphic to $S_{k_{2}, k_{1}}$. The double-star $S_{k, k}$ is called symmetric.

Conjecture 2 is simple to verify when $T$ is a star. We will verify it when $T$ is a double-star. We will also show that $S_{k, k-1}$ decomposes every $2 k$-regular graph that contains a perfect matching.

## 2. Main Results

We give some additional definitions before proceeding with our main results. An orientation of a graph $G$ is an assignment of directions to the edges of $G$. An Eulerian orientation of $G$ is an orientation where the indegree at each vertex is equal to the outdegree. It is simple to see that a graph with all even degrees has an Eulerian orientation.

Theorem 3. Every double-star with $n$ edges decomposes every $2 n$-regular graph.
Proof. Let $H$ be the double-star $S_{k_{1}, k_{2}}$ with center vertices $a$ and $b$, where the degree of $a$ is $k_{1}+1$ and the degree of $b$ is $k_{2}+1$. Let $G$ be a $2 n$-regular graph where $n=k_{1}+k_{2}+1$. We will show that $H$ decomposes $G$.

Orient the edges of $H$ so that each leaf has indegree 1. Orient the edge $\{a, b\}$ from $a$ to $b$. Let $F$ be a 2 -factor in $G$. Then $F$ has an Eulerian orientation. Since
$G-E(F)$ is $(2 n-2)$-regular, it has an Eulerian orientation. Consider any cycle $C$ in $F$, and let $D_{C}$ denote the digraph in $G$ consisting of all arcs with tail in $V(C)$. Thus every vertex in $D_{C}$ will have outdegree (in $D_{C}$ ) either $k_{1}+k_{2}+1$ or 0 . Because $\left\{E\left(D_{C}\right): C\right.$ is a cycle in $\left.F\right\}$ partitions $E(G)$, the proof will be complete if we can show that each such subgraph $D_{C}$ has an $H$-decomposition.

Let the cycle $C$ have length $p$ and consist of alternating vertices and arcs labeled $v_{0}, e_{1}, v_{1}, e_{2}, \ldots, v_{p-1}, e_{p}, v_{p}=v_{0}$.

For the first copy $H_{1}$ of $H$ in the decomposition, we use $e_{1}$ as the central arc, and identify $v_{0}$ with $a$ and $v_{1}$ with $b$. Choose $k_{2}$ arcs with tail at $v_{1}$; label as $X$ the set of endvertices of these $k_{2}$ arcs. The remaining $k_{1}$ arcs with tail at $v_{0}$ in $H_{1}$ in this construction will be determined at the end.

We construct the remaining copies $H_{2}, H_{3}, \ldots, H_{p}$ sequentially. After $H_{i-1}$ is determined we construct $H_{i}$ as follows. The central arc of $H_{i}$ is $e_{i}$, with $v_{i-1}$ identified with $a$ from $H$, and $v_{i}$ identified with $b$. The remaining arcs with tail at $v_{i-1}$ are all such arcs of $D_{C}-C$ that were not chosen to be in $H_{i-1}$. From the remaining $k_{1}+k_{2}$ arcs with tail at $v_{i}$, we choose $k_{2}$ arcs so that:
i) no arc is chosen that is adjacent with an arc chosen at this step to have tail $v_{i-1}$ (avoid an immediate triangle), and
ii) we include in the pool all arcs with head a vertex in $X$.

The selection process above can always be implemented because in $H_{i-1}$ we chose all possible arcs with tail at $v_{i-1}$ and head at a vertex in $X$, so no such arc appears in $H_{i}$.

It remains only to complete the construction of $H_{1}$. After $H_{p}$ has been constructed, $k_{1}$ arcs with tail at $v_{0}$ have yet to be assigned; we include these arcs in $H_{1}$. Because of the pattern noted above, none of these arcs has as a head a vertex in $X$. Thus $H_{1}$ also has no triangles and is therefore isomorphic to $H$.

In [5], Häggkvist states that he has proven (but has not published) a result showing that every tree with $n$ edges and diameter $d$ decomposes every $2 n$-regular graph of girth at least $d$. Since the girth of a graph with no multiple edges is at least 3, Häggkvist's unpublished result would cover the result in Theorem 3.

We turn our focus to decompositions of $n$-regular graphs into trees with $n$ edges. If $G$ is $n$-regular and $H$ is a tree with $n$ edges, then $H$ may or may not decompose $G$. In fact, if $n$ is even and $G$ has odd order, then $|E(G)|$ would not be divisible by $n$ and thus $H$ could not decompose $G$. It is also easy to see that $S_{n}$ decomposes an $n$-regular graph $G$ if and only if $G$ is bipartite. Graham and Häggkvist do in fact conjecture that every tree $T$ with $n$ edges decomposes every $n$-regular bipartite graph $G$ (see [5]). This conjecture was verified by Jacobson, Truszczyński, and Tuza [6] for $T$ for the cases when $T$ is a double-star and for when $T=P_{5}$.

In [9], Kotzig conjectured that the symmetric double-star $S_{k, k}$ decomposes a $(2 k+1)$-regular graph $G$ if and only if $G$ contains a perfect matching. Kotzig's conjecture was proved by Jaeger, Payan, and Kouider in [7].

Theorem 4. For $k \geq 1$, let $G$ be a $(2 k+1)$-regular graph. Then $S_{k, k}$ decomposes $G$ if and only if $G$ contains a perfect matching.
It is simple to see why $G$ must contain a perfect matching if $S_{k, k}$ decomposes it. If $G$ has order $2 m$, then the number of $S_{k, k}$ 's in the decomposition is $m$. Since no two central edges in the decomposition can be adjacent, the central edges must form a perfect matching.

Let $G$ be a graph that contains a perfect matching $M$. A tent in $G$ is a pair $\{\{v, x\},\{v, y\}\}$ of adjacent edges such that $\{x, y\}$ is an edge of $M$. The common vertex $v$ is called the top of the tent. Jaeger et al. [7] showed that if $G$ is $(2 k+1)$ regular, then $G-M$ has an Eulerian orientation so that every tent is a directed path.

We use a slight variation of the approach of Jaeger et al. to show that if $G$ is a $2 k$-regular simple graph of even order and with a perfect matching, then $S_{k, k-1}$ decomposes $G$.

Lemma 5. If $G$ is an Eulerian graph that contains a perfect matching $M$, then $G$ has an Eulerian orientation such that every tent is oriented into a directed path.

Proof. We obtain the desired Eulerian orientation as follows. Begin a walk at any vertex $w$, and start with any edge incident with $w$. At each step where there is a choice of edges to continue the walk, if we are at vertex $v$ which is incident with tent edges $\{\{v, x\},\{v, y\}\}$, we choose one of these edges if and only if the other edge was the most recent edge in the walk. This process can only end at start vertex $w$. Orient the edges of the walk according to the direction in which they were traversed. Remove those edges from $G$, and iterate if any edges remain in $G$. It is easy to see this process gives the desired orientation.

Theorem 6. For $k \geq 2$, let $G$ be a $2 k$-regular graph that contains a perfect matching $M$. Then $S_{k, k-1}$ decomposes $G$.

Proof. By Lemma 5, $G$ has an Eulerian orientation such that every tent is a directed path. For $x \in V(G)$, let $I_{x}=\left\{e_{1}, e_{2}, \ldots, e_{k}\right\}$ be the $k$ arcs with terminal vertex $x$ in the orientation of $G$ and let $V_{x}=\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ be the set of initial vertices of these arcs.

If $e=\{x, y\} \in M$, where $e$ is oriented from $x$ to $y$, then $x \in V_{y}, e \in I_{y}$, and $V_{x} \cap V_{y}=\emptyset$ because each tent is oriented into a directed path. It follows that the graph

$$
L_{e}=\left(V_{x} \cup V_{y} \cup\{y\}, I_{x} \cup I_{y}\right)
$$



Figure 1. A 4-regular graph without a perfect matching that is $S_{2,1}$-decomposable.


Figure 2. A 4-regular graph without a perfect matching that is not $S_{2,1}$-decomposable.
is isomorphic to $S_{k, k-1}$. Moreover, since each edge of $G$ has exactly one terminal vertex, which is on exactly one edge of $M,\left\{L_{e}: e \in M\right\}$ forms an $S_{k, k-1^{-}}$ decomposition of $G$. This completes the proof.

If a $2 k$-regular graph does not contain a perfect matching, then it may or may not be $S_{k, k-1}$-decomposable. In Figure 1, we show a 4 -regular graph that does not contain a perfect matching but is $S_{2,1}$-decomposable. Figure 2 shows a 4 -regular graph $G$ that does not contain a perfect matching and is not $S_{2,1}$-decomposable. This graph consists of four vertex-disjoint copies of $K_{5}-e$ with each of the degree 3 vertices in these copies joined to one of two additional vertices. Let $J$ denote one of the four copies of $K_{5}-e$ in $G$. Since $J$ contains 9 edges, three edges from the complement of $J$ are needed to get all the edges of $J$ in an $S_{2,1}$-decomposition of $G$. Since a tree containing edges from more than one $K_{5}-e$ in $G$ must have diameter at least 4 and there are only 8 edges in $G$ that are not in a $K_{5}-e$, there
is no $S_{2,1}$-decomposition of $G$.
For a graph $G$, let ${ }^{2} G$ denote the multigraph obtained from $G$ by replacing every edge in $G$ with two parallel edges. In [3], we show that every double-star with $n$ edges decomposes ${ }^{2} G$ for every $n$-regular graph $G$. We also investigate decompositions of $2 n$-regular multigraphs with edge multiplicity at most 2 into double-stars with $n$ edges.

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