*Discussiones Mathematicae*
*Graph Theory 35 (2015) 191–196*
*doi:10.7151/dmgt.1778*

Note

# ON A SPANNING $k$-TREE IN WHICH SPECIFIED VERTICES HAVE DEGREE LESS THAN $k$

Hajime Matsumura

*College of Education*
*Ibaraki University*
*Ibaraki 310-8512, Japan*

**e-mail:** hajime-m@mx.ibaraki.ac.jp

**Abstract**

A $k$-*tree* is a tree with maximum degree at most $k$. In this paper, we give a degree sum condition for a graph to have a spanning $k$-tree in which specified vertices have degree less than $k$. We denote by $\sigma_k(G)$ the minimum value of the degree sum of $k$ independent vertices in a graph $G$. Let $k \geq 3$ and $s \geq 0$ be integers, and suppose $G$ is a connected graph and $\sigma_k(G) \geq |V(G)|+s-1$. Then for any $s$ specified vertices, $G$ contains a spanning $k$-tree in which every specified vertex has degree less than $k$. The degree condition is sharp.

**Keywords:** spanning tree, degree bounded tree, degree sum condition.

**2010 Mathematics Subject Classification:** 05C05, 05C35.

## 1. Introduction

All graphs considered in this paper are simple and finite. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $x$ of $G$, we denote by $\deg_G(x)$ the degree of $x$ in $G$ and by $N_G(x)$ the set of vertices adjacent to $x$ in $G$. We denote $N_G[x] = N_G(x) \cup \{x\}$, and $V_i(G)$ denotes the set of vertices of $G$ which have degree $i$ in $G$. For a subset $S$ of $V(G)$, $N_G(S) = \bigcup_{x\in S} N_G(x)$. $\alpha(G)$ denotes the independence number of $G$ and we define

$$\sigma_k(G) = \min\left\{\sum\nolimits_{v\in S} \deg_G(x_i) : S \text{ is an independent set of } G \text{ with } |S| = k\right\}$$

for $1 \leq k \leq \alpha(G)$, and $\sigma_k(G) = \infty$ if $\alpha(G) < k$.

The following is a well-known theorem on Hamiltonian cycles and paths by Ore.

**Theorem 1** (Ore [6, 7])**.** *Let $s$ be an integer with $0 \leq s \leq 2$. Suppose $G$ is a graph with $|V(G)| \geq 3$ and $\sigma_2(G) \geq |V(G)| + s - 1$. Then the following hold:*

(1) *if $s = 0$, then $G$ has a Hamiltonian path,*

(2) *if $s = 1$, then $G$ has a Hamiltonian cycle, and*

(3) *if $s = 2$, then $G$ has a Hamiltonian path connecting any two vertices of $G$.*

We can consider a Hamiltonian path as a spanning tree with maximum degree 2. For an integer $k \geq 2$, a tree $T$ is called a *$k$-tree* if $\deg_T(x) \leq k$ for any $x \in V(T)$. As we mention above, a spanning 2-tree is a Hamiltonian path.

In 1975, Win gave a degree sum condition which ensures the existence of a spanning $k$-tree.

**Theorem 2** (Win [8])**.** *Let $k \geq 2$ be an integer and $G$ be a connected graph. If $\sigma_k(G) \geq |V(G)| - 1$, then $G$ has a spanning $k$-tree.*

Note that Theorem 2 implies Theorem 1 (1) for $k = 2$.

In this paper, we consider a spanning $k$-tree with extra degree constraints imposed on a set of vertices. Matsuda and Matsumura considered the case that every specified vertex has degree one. They proved the following theorem.

**Theorem 3** (Matsuda and Matsumura [3])**.** *Let $k$ and $s$ be integers with $k \geq 2$ and $0 \leq s \leq k$. Suppose a graph $G$ is $(s+1)$-connected and satisfies $\sigma_k(G) \geq |V(G)| + (k-1)s - 1$. Then for any $s$ distinct vertices of $G$, $G$ has a spanning $k$-tree such that each of $s$ specified vertices has degree one.*

This is not only a generalization of Theorem 2, but also implies Theorem 1 (3) for $k = s = 2$.

Hereafter, we consider a spanning $k$-tree in which every specified vertex has degree less than $k$. Our main result is the following.

**Theorem 4.** *Let $k \geq 3$ and $s \geq 0$ be integers, and $G$ be a connected graph. If $\sigma_k(G) \geq |V(G)| + s - 1$, then for any $s$ distinct vertices of $G$, $G$ has a spanning $k$-tree such that each specified vertex has degree less than $k$.*

We note that this is also a generalization of Theorem 2 for $k \geq 3$. For $k = 2$, we have to restrict ourselves to $0 \leq s \leq k = 2$ because a spanning 2-tree has just two vertices of degree one. Then we can easily derive the same conclusion by Theorem 1.

Consider a complete bipartite graph $G$ with parts $X$ and $Y$ such that $|X| = s$ and $|Y| = (k-2)s + 2$ and let $S = X$. Then $\sigma_k(G) = |V(G)| + s - 2$. Suppose $G$ has a spanning $k$-tree $T$ with $\deg_T(v) < k$ for every $v \in S$. Then $|V(G)| - 1 = |E(T)| \leq (k-1)s < |V(G)| - 1$, a contradiction. Hence $G$ has no such a tree and the degree sum condition in Theorem 4 is sharp.

*An outdirected tree* $\vec{T}$ is a rooted tree in which all the edges are directed away from the root. Let $V(\vec{T})$ and $A(\vec{T})$ be the vertex set and the arc set of $\vec{T}$, respectively. For a subset $S$ of $V(\vec{T})$, we denote by $N^+_{\vec{T}}(S)$ the set of vertices $w$ of $V(\vec{T})$ for which there is an arc $uw \in A(\vec{T})$ for some $u \in S$. For a tree $T$ and $u, v \in V(T)$, let $P_T(u, v)$ be the unique path in $T$ connecting $u$ and $v$.

## 2. Proof of Theorem 4

If $s = 0$, we have nothing to prove since $G$ has a spanning $k$-tree by Theorem 2. So we may assume that $s \geq 1$. Let $S$ be the set of $s$ specified vertices.

By Theorem 2, $G$ has a spanning $k$-tree. Choose a spanning $k$-tree $T$ of $G$ such that $|V_k(T) \cap S|$ is as small as possible. If $V_k(T) \cap S = \emptyset$, then $T$ is a desired tree. Hence we may assume that $V_k(T) \cap S$ is not empty and let $v$ be a vertex of $S$ which have degree $k$ in $T$.

Let $T_1, \dots, T_k$ be the connected components of $T - \{v\}$. For each $1 \leq i \leq k$, let $t_i$ be the vertex of $T_i$ which is adjacent to $v$ in $T$ and let $u_i$ be a vertex of $T_i$ with $\deg_T(u_i) = 1$.

If $u_i$ and $u_j$ are adjacent in $G$ for some $1 \leq i < j \leq k$, then $T' = T + u_iu_j - vt_i$ is a spanning $k$-tree of $G$ with $|V_k(T') \cap S| < |V_k(T) \cap S|$, a contradiction. Hence $\{u_1, \dots, u_k\}$ is an independent set of $G$.

Let $W_1 = \bigcup_{i=2}^{k} N_G(u_i) \cap V(T_1)$.

**Claim 1.** *$t_1$ is not contained in $W_1$.*

***Proof.*** If $t_1$ is contained in $W_1$, then $t_1$ is adjacent to $u_i$ for some $2 \leq i \leq k$. If we take $T' = T - vt_1 + t_1u_i$, then $|V_k(T') \cap S| < |V_k(T) \cap S|$, a contradiction. □

**Claim 2.** *For each $w \in W_1$, the following statements hold.*
(1) *Either* $\deg_T(w) = k$, *or* $w \in S$ *and* $\deg_T(w) = k - 1$.
(2) $N_G[u_1] \cap (N_T(w) \setminus V(P_T(w, u_1))) = \emptyset$.

***Proof.*** (1) Suppose $\deg_T(w) < k$ for some $w \in W_1$. Since $w$ is adjacent to $u_i$ for some $2 \leq i \leq k$, $T' = T - t_1v + u_iw$ is also a spanning $k$-tree with $\deg_{T'}(v) = k - 1$. If $w \notin S$, then $|V(T') \cap S| < |V(T) \cap S|$, a contradiction. If $w \in S$ and $\deg_T(w) \leq k - 2$, then also $|V(T') \cap S| < |V(T) \cap S|$. This contradicts the choice of $T$.

(2) Suppose there exists $z \in N_T(w) \setminus V(P_T(w, u_1))$ which is adjacent to $u_1$ in $G$ for some $w \in W_1$. Since $w$ is adjacent to $u_i$ in $G$ for some $2 \leq i \leq k$, $T' = T - wz - vt_1 + u_1z + wu_i$ is a spanning $k$-tree with $|V_k(T') \cap S| < |V_k(T) \cap S|$. This contradicts the choice of $T$. □

Let $W_{1,a} = \{w \in W_1 : w \notin S\}$ and $W_{1,b} = \{w \in W_1 : w \in S\}$.

**Claim 3.** $|N_T(W_1) \setminus N_G[u_1]| \geq (k-1)|W_{1,a}| + (k-2)|W_{1,b}|$.

***Proof.*** We may assume that $W_1$ is not empty since otherwise the above inequality obviously holds. Furthermore, since $t_1$ does not belong to $W_1$ by Claim 1, $v$ is not contained in $N_T(W_1)$.

We consider $T_1$ as an outdirected tree with the root $u_1$. For any $w_0 \in W_1$ and $z \in N^+_{T_1}(w_0)$, $z \notin N_G[u_1]$ holds by Claim 2 (2). This implies that $N^+_{T_1}(w_0) \subseteq N_T(W_1) \setminus N_G[u_1]$ for every $w_0 \in W_1$. Moreover, for any two distinct vertices $w_1$ and $w_2$ of $W_1$, $N^+_{T_1}(w_1)$ and $N^+_{T_1}(w_2)$ are disjoint. Consequently,

$$\begin{aligned} |N_T(W_1) \setminus N_G[u_1]| &\geq \left|N^+_{T_1}(W_1)\right| = \sum\nolimits_{w \in W_1} \left|N^+_{T_1}(w)\right| \\ &= (k-1)|W_{1,a}| + (k-2)|W_{1,b}|. \end{aligned}$$

□

**Claim 4.** $\sum_{i=1}^{k} |V(T_1) \cap N_G(u_i)| \leq |V(T_1)| - 1 + |W_{1,b}|$.

***Proof.*** By Claim 3, we obtain

$$\begin{aligned} |V(T_1) \cap N_G(u_1)| &\leq |V(T_1)| - 1 - |N_T(W_1) \setminus N_G[u_1]| \\ &\leq |V(T_1)| - 1 - (k-1)|W_{1,a}| - (k-2)|W_{1,b}|. \end{aligned}$$

By the definition of $W_1$, we have $\sum_{i=2}^{k} |V(T_1) \cap N_G(u_i)| \leq (k-1)|W_1|$. Then

$$\sum\nolimits_{i=1}^{k} |V(T_1) \cap N_G(u_i)| \leq |V(T_1)| - 1 + |W_{1,b}|.$$

□

Similarly, for each $T_j$ we can define $W_j, W_{j,a}, W_{j,b}$ for $2 \leq j \leq k$. As Claim 4 we have

$$\sum\nolimits_{i=1}^{k} |V(T_j) \cap N_G(u_i)| \leq |V(T_j)| - 1 + |W_{j,b}|.$$

Since $\deg_G(u_i) \leq |\{v\}| + \sum_{j=1}^{k} |V(T_j) \cap N_G(u_i)|$ and $\sum_{j=1}^{k} |W_{j,b}| \leq s - 1$,

$$\begin{aligned} \sum\nolimits_{i=1}^{k} d_G(u_i) &\leq k + \sum\nolimits_{i=1}^{k} \sum\nolimits_{j=1}^{k} |V(T_j) \cap N_G(u_i)| \\ &\leq k + \sum\nolimits_{j=1}^{k} (|V(T_j)| - 1 + |W_{j,b}|) \\ &\leq k + |V(G)| - 1 - k + s - 1 \\ &= |V(G)| + s - 2, \end{aligned}$$

a contradiction. This completes the proof of Theorem 4.

## 3. Remarks

For a graph $G$, let $f$ be a mapping from $V(G)$ to positive integers and let $f^{-1}(a) = \{x \in V(G) : f(x) = a\}$ for a positive integer $a$. We call a tree $T$ to be a *$f$-tree* if $\deg_T(v) \leq f(v)$ for every vertex $v$ of $T$. The following sufficient conditions are already known for a graph to have a spanning $f$-tree.

**Theorem 5** (Ellingham *et al.* [1]). *Let $G$ be a connected graph and let $f$ be a mapping from $V(G)$ to positive integers. If $w(G-S) \leq \sum_{x\in S}(f(x)-2)+2$, for all $S \subset V(G)$, then $G$ has a spanning $f$-tree, where $w(G-S)$ denotes the number of components of $G-S$.*

**Theorem 6** (Enomoto and Ozeki [2]). *Let $G$ be an $n$-connected graph and $f$ be a mapping from $V(G)$ to positive integers. Suppose $|f^{-1}(1)| + |f^{-1}(2)| \leq n+1$ and*

$$\alpha(G) \leq \min_R \left\{\sum\nolimits_{x\in R}(f(x)-1) : R \subset V(G), |R| = n\right\} + 1.$$

*Then $G$ has a spanning $f$-tree.*

The above theorems are generalizations of the following classical results on spanning $k$-trees.

**Theorem 7** (Win [9]). *Let $k \geq 3$ be an integer and $G$ be a connected graph. If $w(G-S) \leq (k-2)|S|+2$, for all $S \subset V(G)$, then $G$ has a spanning $k$-tree.*

**Theorem 8** (Neumann-Lara and Rivera-Campo [5]). *Let $k \geq 2$ and $n \geq 2$ be integers and $G$ be an $n$-connected graph. If $\alpha(G) \leq (k-1)n+1$, then $G$ has a spanning $k$-tree.*

It is natural to consider a degree sum condition for a spanning $f$-tree. We pose the following conjecture.

**Conjecture 9.** *Let $G$ be an $n$-connected graph, $f$ be a mapping from $V(G)$ to positive integers and let $k = \max\{f(x) : x \in V(G)\}$. Suppose $|f^{-1}(1)| \leq n$ and*

$$\sigma_k(G) \geq |V(G)| + \sum\nolimits_{x\in V(G)}(k - f(x)) + 1.$$

*Then $G$ has a spanning $f$-tree.*

We note that Theorems 3 and 4 partially confirm this conjecture.

**Acknowledgement**

The author would like to thank Professor Mikio Kano for his valuable comments. The author also would like to thank two anonymous referees for their helpful comments and suggestions.

## References

[1] M.N. Ellingham, Y. Nam and H.-J. Voss, *Connected $(g, f)$-factors*, J. Graph Theory **39** (2002) 62–75.
doi:10.1002/jgt.10019

[2] H. Enomoto and K. Ozeki, *The independence number condition for the existence of a spanning $f$-tree*, J. Graph Theory **65** (2010) 173–184.
doi:10.1002/jgt.20471

[3] H. Matsuda and H.Matsumura, *On a $k$-tree containing specified leaves in a graph*, Graphs Combin. **22** (2006) 371–381.
doi:10.1007/s00373-006-0660-5

[4] H. Matsuda and H. Matsumura, *Degree conditions and degree bounded trees*, Discrete Math. **309** (2009) 3653–3658.
doi:10.1016/j.disc.2007.12.099

[5] V. Neumann-Lara and E. Rivera-Campo, *Spanning trees with bounded degrees*, Combinatorica **11** (1991) 55–61.
doi:10.1007/BF01375473

[6] O. Ore, *Note on Hamilton circuits*, Amer. Math. Monthly **67** (1960) 55.

[7] O. Ore, *Hamilton connected graphs*, J. Math. Pures Appl. **42** (1963) 21–27.
doi:10.2307/2308928

[8] S. Win, *Existenz von gerüsten mit vorgeschriebenem maximalgrad in graphen*, Abh. Math. Seminar Univ. Hamburg **43** (1975) 263–267.
doi:10.1007/BF02995957

[9] S. Win, *On a connection between the existence of $k$-trees and the toughness of a graph*, Graphs Combin. **5** (1989) 201–205.
doi:10.1007/BF01788671

Received 16 October 2013
Revised 31 January 2014
Accepted 31 January 2014