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Note

ON A SPANNING k-TREE IN WHICH SPECIFIED VERTICES HAVE DEGREE LESS THAN k

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Abstract

A k-*tree* is a tree with maximum degree at most k. In this paper, we give a degree sum condition for a graph to have a spanning k-tree in which specified vertices have degree less than k. We denote by $\sigma_k(G)$ the minimum value of the degree sum of k independent vertices in a graph G. Let $k \geq 3$ and $s \geq 0$ be integers, and suppose G is a connected graph and $\sigma_k(G) \geq |V(G)|+s-1$. Then for any s specified vertices, G contains a spanning k-tree in which every specified vertex has degree less than k. The degree condition is sharp.

Keywords: spanning tree, degree bounded tree, degree sum condition.

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1. Introduction

All graphs considered in this paper are simple and finite. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex x of G, we denote by $\deg_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G. We denote $N_G[x] = N_G(x) \cup \{x\}$, and $V_i(G)$ denotes the set of vertices of G which have degree i in G. For a subset S of $V(G)$, $N_G(S) = \bigcup_{x\in S} N_G(x)$. $\alpha(G)$ denotes the independence number of G and we define

$$\sigma_k(G) = \min \left\{ \sum\nolimits_{v\in S} \deg_G(x_i) : S \text{ is an independent set of } G \text{ with } |S| = k \right\}$$

for $1 \leq k \leq \alpha(G)$, and $\sigma_k(G) = \infty$ if $\alpha(G) < k$.

The following is a well-known theorem on Hamiltonian cycles and paths by Ore.

Theorem 1 (Ore [6, 7])**.** *Let s be an integer with $0 \le s \le 2$. Suppose G is a graph with $|V(G)| \ge 3$ and $\sigma_2(G) \ge |V(G)| + s - 1$. Then the following hold:*
(1) *if $s = 0$, then G has a Hamiltonian path,*
(2) *if $s = 1$, then G has a Hamiltonian cycle, and*
(3) *if $s = 2$, then G has a Hamiltonian path connecting any two vertices of G.*

We can consider a Hamiltonian path as a spanning tree with maximum degree 2. For an integer $k \ge 2$, a tree T is called a *k-tree* if $\deg_T(x) \le k$ for any $x \in V(T)$. As we mention above, a spanning 2-tree is a Hamiltonian path.

In 1975, Win gave a degree sum condition which ensures the existence of a spanning k-tree.

Theorem 2 (Win [8])**.** *Let $k \ge 2$ be an integer and G be a connected graph. If $\sigma_k(G) \ge |V(G)| - 1$, then G has a spanning k-tree.*

Note that Theorem 2 implies Theorem 1 (1) for $k = 2$.

In this paper, we consider a spanning k-tree with extra degree constraints imposed on a set of vertices. Matsuda and Matsumura considered the case that every specified vertex has degree one. They proved the following theorem.

Theorem 3 (Matsuda and Matsumura [3])**.** *Let k and s be integers with $k \ge 2$ and $0 \le s \le k$. Suppose a graph G is $(s+1)$-connected and satisfies $\sigma_k(G) \ge |V(G)| + (k-1)s - 1$. Then for any s distinct vertices of G, G has a spanning k-tree such that each of s specified vertices has degree one.*

This is not only a generalization of Theorem 2, but also implies Theorem 1 (3) for $k = s = 2$.

Hereafter, we consider a spanning k-tree in which every specified vertex has degree less than k. Our main result is the following.

Theorem 4. *Let $k \ge 3$ and $s \ge 0$ be integers, and G be a connected graph. If $\sigma_k(G) \ge |V(G)| + s - 1$, then for any s distinct vertices of G, G has a spanning k-tree such that each specified vertex has degree less than k.*

We note that this is also a generalization of Theorem 2 for $k \ge 3$. For $k = 2$, we have to restrict ourselves to $0 \le s \le k = 2$ because a spanning 2-tree has just two vertices of degree one. Then we can easily derive the same conclusion by Theorem 1.

Consider a complete bipartite graph G with parts X and Y such that $|X| = s$ and $|Y| = (k-2)s + 2$ and let $S = X$. Then $\sigma_k(G) = |V(G)| + s - 2$. Suppose G has a spanning k-tree T with $\deg_T(v) < k$ for every $v \in S$. Then $|V(G)| - 1 = |E(T)| \le (k-1)s < |V(G)| - 1$, a contradiction. Hence G has no such a tree and the degree sum condition in Theorem 4 is sharp.

An outdirected tree $\vec{T}$ is a rooted tree in which all the edges are directed away from the root. Let $V(\vec{T})$ and $A(\vec{T})$ be the vertex set and the arc set of $\vec{T}$, respectively. For a subset S of $V(\vec{T})$, we denote by $N^+_{\vec{T}}(S)$ the set of vertices w of $V(\vec{T})$ for which there is an arc $uw \in A(\vec{T})$ for some $u \in S$. For a tree T and $u, v \in V(T)$, let $P_T(u, v)$ be the unique path in T connecting u and v.

2. Proof of Theorem 4

If $s = 0$, we have nothing to prove since G has a spanning k-tree by Theorem 2. So we may assume that $s \geq 1$. Let S be the set of s specified vertices.

By Theorem 2, G has a spanning k-tree. Choose a spanning k-tree T of G such that $|V_k(T) \cap S|$ is as small as possible. If $V_k(T) \cap S = \emptyset$, then T is a desired tree. Hence we may assume that $V_k(T) \cap S$ is not empty and let v be a vertex of S which have degree k in T.

Let $T_1, \ldots, T_k$ be the connected components of $T - \{v\}$. For each $1 \leq i \leq k$, let t_i be the vertex of T_i which is adjacent to v in T and let u_i be a vertex of T_i with $\deg_T(u_i) = 1$.

If u_i and u_j are adjacent in G for some $1 \leq i < j \leq k$, then $T' = T + u_i u_j - v t_i$ is a spanning k-tree of G with $|V_k(T') \cap S| < |V_k(T) \cap S|$, a contradiction. Hence $\{u_1, \ldots, u_k\}$ is an independent set of G.

Let $W_1 = \bigcup_{i=2}^{k} N_G(u_i) \cap V(T_1)$.

Claim 1. *t_1 is not contained in W_1.*

Proof. If t_1 is contained in W_1, then t_1 is adjacent to u_i for some $2 \leq i \leq k$. If we take $T' = T - vt_1 + t_1 u_i$, then $|V_k(T') \cap S| < |V_k(T) \cap S|$, a contradiction. □

Claim 2. *For each $w \in W_1$, the following statements hold.*
(1) *Either* $\deg_T(w) = k$, *or* $w \in S$ *and* $\deg_T(w) = k - 1$.
(2) $N_G[u_1] \cap (N_T(w) \setminus V(P_T(w, u_1))) = \emptyset$.

Proof. (1) Suppose $\deg_T(w) < k$ for some $w \in W_1$. Since w is adjacent to u_i for some $2 \leq i \leq k$, $T' = T - t_1 v + u_i w$ is also a spanning k-tree with $\deg_{T'}(v) = k - 1$. If $w \notin S$, then $|V(T') \cap S| < |V(T) \cap S|$, a contradiction. If $w \in S$ and $\deg_T(w) \leq k - 2$, then also $|V(T') \cap S| < |V(T) \cap S|$. This contradicts the choice of T.

(2) Suppose there exists $z \in N_T(w) \setminus V(P_T(w, u_1))$ which is adjacent to u_1 in G for some $w \in W_1$. Since w is adjacent to u_i in G for some $2 \leq i \leq k$, $T' = T - wz - vt_1 + u_1 z + w u_i$ is a spanning k-tree with $|V_k(T') \cap S| < |V_k(T) \cap S|$. This contradicts the choice of T. □

Let $W_{1,a} = \{w \in W_1 : w \notin S\}$ and $W_{1,b} = \{w \in W_1 : w \in S\}$.

Claim 3. $|N_T(W_1) \setminus N_G[u_1]| \geq (k-1)|W_{1,a}| + (k-2)|W_{1,b}|$.

Proof. We may assume that W_1 is not empty since otherwise the above inequality obviously holds. Furthermore, since t_1 does not belong to W_1 by Claim 1, v is not contained in $N_T(W_1)$.

We consider T_1 as an outdirected tree with the root u_1. For any $w_0 \in W_1$ and $z \in N^+_{T_1}(w_0)$, $z \notin N_G[u_1]$ holds by Claim 2 (2). This implies that $N^+_{T_1}(w_0) \subseteq N_T(W_1) \setminus N_G[u_1]$ for every $w_0 \in W_1$. Moreover, for any two distinct vertices w_1 and w_2 of W_1, $N^+_{T_1}(w_1)$ and $N^+_{T_1}(w_2)$ are disjoint. Consequently,

$$\begin{aligned} |N_T(W_1) \setminus N_G[u_1]| &\geq \left|N^+_{T_1}(W_1)\right| = \sum\nolimits_{w \in W_1} \left|N^+_{T_1}(w)\right| \\ &= (k-1)|W_{1,a}| + (k-2)|W_{1,b}|. \end{aligned}$$

□

Claim 4. $\sum_{i=1}^{k} |V(T_1) \cap N_G(u_i)| \leq |V(T_1)| - 1 + |W_{1,b}|$.

Proof. By Claim 3, we obtain

$$\begin{aligned} |V(T_1) \cap N_G(u_1)| &\leq |V(T_1)| - 1 - |N_T(W_1) \setminus N_G[u_1]| \\ &\leq |V(T_1)| - 1 - (k-1)|W_{1,a}| - (k-2)|W_{1,b}|. \end{aligned}$$

By the definition of W_1, we have $\sum_{i=2}^{k} |V(T_1) \cap N_G(u_i)| \leq (k-1)|W_1|$. Then

$$\sum\nolimits_{i=1}^{k} |V(T_1) \cap N_G(u_i)| \leq |V(T_1)| - 1 + |W_{1,b}|.$$

□

Similarly, for each T_j we can define $W_j, W_{j,a}, W_{j,b}$ for $2 \leq j \leq k$. As Claim 4 we have

$$\sum\nolimits_{i=1}^{k} |V(T_j) \cap N_G(u_i)| \leq |V(T_j)| - 1 + |W_{j,b}|.$$

Since $\deg_G(u_i) \leq |\{v\}| + \sum_{j=1}^{k} |V(T_j) \cap N_G(u_i)|$ and $\sum_{j=1}^{k} |W_{j,b}| \leq s-1$,

$$\begin{aligned} \sum\nolimits_{i=1}^{k} d_G(u_i) &\leq k + \sum\nolimits_{i=1}^{k} \sum\nolimits_{j=1}^{k} |V(T_j) \cap N_G(u_i)| \\ &\leq k + \sum\nolimits_{j=1}^{k} (|V(T_j)| - 1 + |W_{j,b}|) \\ &\leq k + |V(G)| - 1 - k + s - 1 \\ &= |V(G)| + s - 2, \end{aligned}$$

a contradiction. This completes the proof of Theorem 4.

3. Remarks

For a graph G, let f be a mapping from $V(G)$ to positive integers and let $f^{-1}(a) = \{x \in V(G) : f(x) = a\}$ for a positive integer a. We call a tree T to be a *f-tree* if $\deg_T(v) \leq f(v)$ for every vertex v of T. The following sufficient conditions are already known for a graph to have a spanning f-tree.

Theorem 5 (Ellingham *et al.* [1]). *Let G be a connected graph and let f be a mapping from $V(G)$ to positive integers. If $w(G - S) \leq \sum_{x \in S}(f(x) - 2) + 2$, for all $S \subset V(G)$, then G has a spanning f-tree, where $w(G - S)$ denotes the number of components of $G - S$.*

Theorem 6 (Enomoto and Ozeki [2]). *Let G be an n-connected graph and f be a mapping from $V(G)$ to positive integers. Suppose $|f^{-1}(1)| + |f^{-1}(2)| \leq n + 1$ and*

$$\alpha(G) \leq \min_R \left\{ \sum\nolimits_{x \in R}(f(x) - 1) : R \subset V(G), |R| = n \right\} + 1.$$

Then G has a spanning f-tree.

The above theorems are generalizations of the following classical results on spanning k-trees.

Theorem 7 (Win [9]). *Let $k \geq 3$ be an integer and G be a connected graph. If $w(G - S) \leq (k - 2)|S| + 2$, for all $S \subset V(G)$, then G has a spanning k-tree.*

Theorem 8 (Neumann-Lara and Rivera-Campo [5]). *Let $k \geq 2$ and $n \geq 2$ be integers and G be an n-connected graph. If $\alpha(G) \leq (k - 1)n + 1$, then G has a spanning k-tree.*

It is natural to consider a degree sum condition for a spanning f-tree. We pose the following conjecture.

Conjecture 9. *Let G be an n-connected graph, f be a mapping from $V(G)$ to positive integers and let $k = \max\{f(x) : x \in V(G)\}$. Suppose $|f^{-1}(1)| \leq n$ and*

$$\sigma_k(G) \geq |V(G)| + \sum\nolimits_{x \in V(G)}(k - f(x)) + 1.$$

Then G has a spanning f-tree.

We note that Theorems 3 and 4 partially confirm this conjecture.

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