Note

# ON A SPANNING $k$-TREE IN WHICH SPECIFIED VERTICES HAVE DEGREE LESS THAN $k$ 

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#### Abstract

A $k$-tree is a tree with maximum degree at most $k$. In this paper, we give a degree sum condition for a graph to have a spanning $k$-tree in which specified vertices have degree less than $k$. We denote by $\sigma_{k}(G)$ the minimum value of the degree sum of $k$ independent vertices in a graph $G$. Let $k \geq 3$ and $s \geq 0$ be integers, and suppose $G$ is a connected graph and $\sigma_{k}(G) \geq|V(G)|+s-1$. Then for any $s$ specified vertices, $G$ contains a spanning $k$-tree in which every specified vertex has degree less than $k$. The degree condition is sharp.


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## 1. Introduction

All graphs considered in this paper are simple and finite. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $x$ of $G$, we denote by $\operatorname{deg}_{G}(x)$ the degree of $x$ in $G$ and by $N_{G}(x)$ the set of vertices adjacent to $x$ in $G$. We denote $N_{G}[x]=N_{G}(x) \cup\{x\}$, and $V_{i}(G)$ denotes the set of vertices of $G$ which have degree $i$ in $G$. For a subset $S$ of $V(G), N_{G}(S)=\bigcup_{x \in S} N_{G}(x) . \alpha(G)$ denotes the independence number of $G$ and we define

$$
\sigma_{k}(G)=\min \left\{\sum_{v \in S} \operatorname{deg}_{G}\left(x_{i}\right): S \text { is an independent set of } G \text { with }|S|=k\right\}
$$

for $1 \leq k \leq \alpha(G)$, and $\sigma_{k}(G)=\infty$ if $\alpha(G)<k$.
The following is a well-known theorem on Hamiltonian cycles and paths by Ore.

Theorem 1 (Ore [6, 7]). Let $s$ be an integer with $0 \leq s \leq 2$. Suppose $G$ is a graph with $|V(G)| \geq 3$ and $\sigma_{2}(G) \geq|V(G)|+s-1$. Then the following hold:
(1) if $s=0$, then $G$ has a Hamiltonian path,
(2) if $s=1$, then $G$ has a Hamiltonian cycle, and
(3) if $s=2$, then $G$ has a Hamiltonian path connecting any two vertices of $G$.

We can consider a Hamiltonian path as a spanning tree with maximum degree 2 . For an integer $k \geq 2$, a tree $T$ is called a $k$-tree if $\operatorname{deg}_{T}(x) \leq k$ for any $x \in V(T)$. As we mention above, a spanning 2-tree is a Hamiltonian path.

In 1975, Win gave a degree sum condition which ensures the existence of a spanning $k$-tree.

Theorem 2 (Win [8]). Let $k \geq 2$ be an integer and $G$ be a connected graph. If $\sigma_{k}(G) \geq|V(G)|-1$, then $G$ has a spanning $k$-tree.

Note that Theorem 2 implies Theorem 1 (1) for $k=2$.
In this paper, we consider a spanning $k$-tree with extra degree constraints imposed on a set of vertices. Matsuda and Matsumura considered the case that every specified vertex has degree one. They proved the following theorem.

Theorem 3 (Matsuda and Matsumura [3]). Let $k$ and $s$ be integers with $k \geq 2$ and $0 \leq s \leq k$. Suppose a graph $G$ is $(s+1)$-connected and satisfies $\sigma_{k}(G) \geq$ $|V(G)|+(k-1) s-1$. Then for any $s$ distinct vertices of $G, G$ has a spanning $k$-tree such that each of s specified vertices has degree one.

This is not only a generalization of Theorem 2, but also implies Theorem 1 (3) for $k=s=2$.

Hereafter, we consider a spanning $k$-tree in which every specified vertex has degree less than $k$. Our main result is the following.

Theorem 4. Let $k \geq 3$ and $s \geq 0$ be integers, and $G$ be a connected graph. If $\sigma_{k}(G) \geq|V(G)|+s-1$, then for any $s$ distinct vertices of $G, G$ has a spanning $k$-tree such that each specified vertex has degree less than $k$.

We note that this is also a generalization of Theorem 2 for $k \geq 3$. For $k=2$, we have to restrict ourselves to $0 \leq s \leq k=2$ because a spanning 2-tree has just two vertices of degree one. Then we can easily derive the same conclusion by Theorem 1.

Consider a complete bipartite graph $G$ with parts $X$ and $Y$ such that $|X|=s$ and $|Y|=(k-2) s+2$ and let $S=X$. Then $\sigma_{k}(G)=|V(G)|+s-2$. Suppose $G$ has a spanning $k$-tree $T$ with $\operatorname{deg}_{T}(v)<k$ for every $v \in S$. Then $|V(G)|-1=$ $|E(T)| \leq(k-1) s<|V(G)|-1$, a contradiction. Hence $G$ has no such a tree and the degree sum condition in Theorem 4 is sharp.

An outdirected tree $\vec{T}$ is a rooted tree in which all the edges are directed away from the root. Let $V(\vec{T})$ and $A(\vec{T})$ be the vertex set and the arc set of $\vec{T}$, respectively. For a subset $S$ of $V(\vec{T})$, we denote by $N_{T}^{+}(S)$ the set of vertices $w$ of $V(\vec{T})$ for which there is an arc $u w \in A(\vec{T})$ for some $u \in S$. For a tree $T$ and $u, v \in V(T)$, let $P_{T}(u, v)$ be the unique path in $T$ connecting $u$ and $v$.

## 2. Proof of Theorem 4

If $s=0$, we have nothing to prove since $G$ has a spanning $k$-tree by Theorem 2 . So we may assume that $s \geq 1$. Let $S$ be the set of $s$ specified vertices.

By Theorem 2, $G$ has a spanning $k$-tree. Choose a spanning $k$-tree $T$ of $G$ such that $\left|V_{k}(T) \cap S\right|$ is as small as possible. If $V_{k}(T) \cap S=\emptyset$, then $T$ is a desired tree. Hence we may assume that $V_{k}(T) \cap S$ is not empty and let $v$ be a vertex of $S$ which have degree $k$ in $T$.

Let $T_{1}, \ldots, T_{k}$ be the connected components of $T-\{v\}$. For each $1 \leq i \leq k$, let $t_{i}$ be the vertex of $T_{i}$ which is adjacent to $v$ in $T$ and let $u_{i}$ be a vertex of $T_{i}$ with $\operatorname{deg}_{T}\left(u_{i}\right)=1$.

If $u_{i}$ and $u_{j}$ are adjacent in $G$ for some $1 \leq i<j \leq k$, then $T^{\prime}=T+u_{i} u_{j}-v t_{i}$ is a spanning $k$-tree of $G$ with $\left|V_{k}\left(T^{\prime}\right) \cap S\right|<\left|V_{k}(T) \cap S\right|$, a contradiction. Hence $\left\{u_{1}, \ldots, u_{k}\right\}$ is an independent set of $G$.

Let $W_{1}=\bigcup_{i=2}^{k} N_{G}\left(u_{i}\right) \cap V\left(T_{1}\right)$.
Claim 1. $t_{1}$ is not contained in $W_{1}$.
Proof. If $t_{1}$ is contained in $W_{1}$, then $t_{1}$ is adjacent to $u_{i}$ for some $2 \leq i \leq k$. If we take $T^{\prime}=T-v t_{1}+t_{1} u_{i}$, then $\left|V_{k}\left(T^{\prime}\right) \cap S\right|<\left|V_{k}(T) \cap S\right|$, a contradiction.

Claim 2. For each $w \in W_{1}$, the following statements hold.
(1) Either $\operatorname{deg}_{T}(w)=k$, or $w \in S$ and $\operatorname{deg}_{T}(w)=k-1$.
(2) $N_{G}\left[u_{1}\right] \cap\left(N_{T}(w) \backslash V\left(P_{T}\left(w, u_{1}\right)\right)\right)=\emptyset$.

Proof. (1) Suppose $\operatorname{deg}_{T}(w)<k$ for some $w \in W_{1}$. Since $w$ is adjacent to $u_{i}$ for some $2 \leq i \leq k, T^{\prime}=T-t_{1} v+u_{i} w$ is also a spanning $k$-tree with $\operatorname{deg}_{T^{\prime}}(v)=k-1$. If $w \notin S$, then $\left|V\left(T^{\prime}\right) \cap S\right|<|V(T) \cap S|$, a contradiction. If $w \in S$ and $\operatorname{deg}_{T}(w) \leq k-2$, then also $\left|V\left(T^{\prime}\right) \cap S\right|<|V(T) \cap S|$. This contradicts the choice of $T$.
(2) Suppose there exists $z \in N_{T}(w) \backslash V\left(P_{T}\left(w, u_{1}\right)\right)$ which is adjacent to $u_{1}$ in $G$ for some $w \in W_{1}$. Since $w$ is adjacent to $u_{i}$ in $G$ for some $2 \leq i \leq k$, $T^{\prime}=T-w z-v t_{1}+u_{1} z+w u_{i}$ is a spanning $k$-tree with $\left|V_{k}\left(T^{\prime}\right) \cap S\right|<\left|V_{k}(T) \cap S\right|$. This contradicts the choice of $T$.

Let $W_{1, a}=\left\{w \in W_{1}: w \notin S\right\} \quad$ and $\quad W_{1, b}=\left\{w \in W_{1}: w \in S\right\}$.

Claim 3. $\left|N_{T}\left(W_{1}\right) \backslash N_{G}\left[u_{1}\right]\right| \geq(k-1)\left|W_{1, a}\right|+(k-2)\left|W_{1, b}\right|$.
Proof. We may assume that $W_{1}$ is not empty since otherwise the above inequality obviously holds. Furthermore, since $t_{1}$ does not belong to $W_{1}$ by Claim $1, v$ is not contained in $N_{T}\left(W_{1}\right)$.

We consider $T_{1}$ as an outdirected tree with the root $u_{1}$. For any $w_{0} \in W_{1}$ and $z \in N_{T_{1}}^{+}\left(w_{0}\right), z \notin N_{G}\left[u_{1}\right]$ holds by Claim 2 (2). This implies that $N_{T_{1}}^{+}\left(w_{0}\right) \subseteq$ $N_{T}\left(W_{1}\right) \backslash N_{G}\left[u_{1}\right]$ for every $w_{0} \in W_{1}$. Moreover, for any two distinct vertices $w_{1}$ and $w_{2}$ of $W_{1}, N_{T_{1}}^{+}\left(w_{1}\right)$ and $N_{T_{1}}^{+}\left(w_{2}\right)$ are disjoint. Consequently,

$$
\begin{aligned}
\left|N_{T}\left(W_{1}\right) \backslash N_{G}\left[u_{1}\right]\right| & \geq\left|N_{T_{1}}^{+}\left(W_{1}\right)\right|=\sum_{w \in W_{1}}\left|N_{T_{1}}^{+}(w)\right| \\
& =(k-1)\left|W_{1, a}\right|+(k-2)\left|W_{1, b}\right| .
\end{aligned}
$$

Claim 4. $\sum_{i=1}^{k}\left|V\left(T_{1}\right) \cap N_{G}\left(u_{i}\right)\right| \leq\left|V\left(T_{1}\right)\right|-1+\left|W_{1, b}\right|$.
Proof. By Claim 3, we obtain

$$
\begin{aligned}
\left|V\left(T_{1}\right) \cap N_{G}\left(u_{1}\right)\right| & \leq\left|V\left(T_{1}\right)\right|-1-\left|N_{T}\left(W_{1}\right) \backslash N_{G}\left[u_{1}\right]\right| \\
& \leq\left|V\left(T_{1}\right)\right|-1-(k-1)\left|W_{1, a}\right|-(k-2)\left|W_{1, b}\right| .
\end{aligned}
$$

By the definition of $W_{1}$, we have $\sum_{i=2}^{k}\left|V\left(T_{1}\right) \cap N_{G}\left(u_{i}\right)\right| \leq(k-1)\left|W_{1}\right|$. Then

$$
\sum_{i=1}^{k}\left|V\left(T_{1}\right) \cap N_{G}\left(u_{i}\right)\right| \leq\left|V\left(T_{1}\right)\right|-1+\left|W_{1, b}\right| .
$$

Similarly, for each $T_{j}$ we can define $W_{j}, W_{j, a}, W_{j, b}$ for $2 \leq j \leq k$. As Claim 4 we have

$$
\sum_{i=1}^{k}\left|V\left(T_{j}\right) \cap N_{G}\left(u_{i}\right)\right| \leq\left|V\left(T_{j}\right)\right|-1+\left|W_{j, b}\right| .
$$

Since $\operatorname{deg}_{G}\left(u_{i}\right) \leq|\{v\}|+\sum_{j=1}^{k}\left|V\left(T_{j}\right) \cap N_{G}\left(u_{i}\right)\right|$ and $\sum_{j=1}^{k}\left|W_{j, b}\right| \leq s-1$,

$$
\begin{aligned}
\sum_{i=1}^{k} d_{G}\left(u_{i}\right) & \leq k+\sum_{i=1}^{k} \sum_{j=1}^{k}\left|V\left(T_{j}\right) \cap N_{G}\left(u_{i}\right)\right| \\
& \leq k+\sum_{j=1}^{k}\left(\left|V\left(T_{j}\right)\right|-1+\left|W_{j, b}\right|\right) \\
& \leq k+|V(G)|-1-k+s-1 \\
& =|V(G)|+s-2
\end{aligned}
$$

a contradiction. This completes the proof of Theorem 4.

## 3. Remarks

For a graph $G$, let $f$ be a mapping from $V(G)$ to positive integers and let $f^{-1}(a)=$ $\{x \in V(G): f(x)=a\}$ for a positive integer $a$. We call a tree $T$ to be a $f$-tree if $\operatorname{deg}_{T}(v) \leq f(v)$ for every vertex $v$ of $T$. The following sufficient conditions are already known for a graph to have a spanning $f$-tree.

Theorem 5 (Ellingham et al. [1]). Let $G$ be a connected graph and let $f$ be a mapping from $V(G)$ to positive integers. If $w(G-S) \leq \sum_{x \in S}(f(x)-2)+2$, for all $S \subset V(G)$, then $G$ has a spanning $f$-tree, where $w(G-S)$ denotes the number of components of $G-S$.

Theorem 6 (Enomoto and Ozeki [2]). Let $G$ be an n-connected graph and $f$ be a mapping from $V(G)$ to positive integers. Suppose $\left|f^{-1}(1)\right|+\left|f^{-1}(2)\right| \leq n+1$ and

$$
\alpha(G) \leq \min _{R}\left\{\sum_{x \in R}(f(x)-1): R \subset V(G),|R|=n\right\}+1 .
$$

Then $G$ has a spanning $f$-tree.
The above theorems are generalizations of the following classical results on spanning $k$-trees.

Theorem 7 (Win [9]). Let $k \geq 3$ be an integer and $G$ be a connected graph. If $w(G-S) \leq(k-2)|S|+2$, for all $S \subset V(G)$, then $G$ has a spanning $k$-tree.

Theorem 8 (Neumann-Lara and Rivera-Campo [5]). Let $k \geq 2$ and $n \geq 2$ be integers and $G$ be an $n$-connected graph. If $\alpha(G) \leq(k-1) n+1$, then $G$ has a spanning $k$-tree.

It is natural to consider a degree sum condition for a spanning $f$-tree. We pose the following conjecture.

Conjecture 9. Let $G$ be an n-connected graph, $f$ be a mapping from $V(G)$ to positive integers and let $k=\max \{f(x): x \in V(G)\}$. Suppose $\left|f^{-1}(1)\right| \leq n$ and

$$
\sigma_{k}(G) \geq|V(G)|+\sum_{x \in V(G)}(k-f(x))+1 .
$$

Then $G$ has a spanning $f$-tree.
We note that Theorems 3 and 4 partially confirm this conjecture.

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