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# HARARY INDEX OF PRODUCT GRAPHS

K. Pattabiraman and P. Paulraja

*Department of Mathematics*
*Annamalai University*
*Annamalainagar 608 002, India*

**e-mail:** pramank@gmail.com
ppraja56@gmail.com

## Abstract

The *Harary index* is defined as the sum of reciprocals of distances between all pairs of vertices of a connected graph. In this paper, the exact formulae for the Harary indices of tensor product $G \times K_{m_0,m_1,\ldots,m_{r-1}}$ and the strong product $G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$, where $K_{m_0,m_1,\ldots,m_{r-1}}$ is the complete multipartite graph with partite sets of sizes $m_0, m_1, \ldots, m_{r-1}$ are obtained. Also upper bounds for the Harary indices of tensor and strong products of graphs are estabilished. Finally, the exact formula for the Harary index of the wreath product $G \circ G'$ is obtained.

**Keywords:** tensor product, strong product, wreath product, Harary index.

**2010 Mathematics Subject Classification:** 05C12, 05C76.

## 1. Introduction

All graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the distance between $u$ and $v$ in $G$, denoted by $d_G(u,v)$, is the length of a shortest $(u,v)$-path in $G$. For two simple graphs $G$ and $H$ their *tensor product,* denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent whenever $g_1g_2$ is an edge in $G$ and $h_1h_2$ is an edge in $H$. Note that if $G$ and $H$ are connected graphs, then $G \times H$ is connected only if at least one of the graphs is nonbipartite. The *strong product* of graphs $G$ and $H$, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u,v) : u \in V(G), v \in V(H)\}$ and $(u,x)(v,y)$ is an edge whenever $(i)$ $u = v$ and $xy \in E(H)$, or $(ii)$ $uv \in E(G)$ and $x = y$, or $(iii)$ $uv \in E(G)$ and $xy \in E(H)$. Similarly, the *wreath product* of the graphs $G$ and $H$, denoted by $G \circ H$, has vertex set $V(G) \times V(H)$ in which

$(g_1, h_1)(g_2, h_2)$ is an edge whenever $g_1g_2$ is an edge in $G$, or $g_1 = g_2$ and $h_1h_2$ is an edge in $H$. The tensor product of graphs has been extensively studied in relation to the areas such as graph colorings, graph recognition, decompositions of graphs, design theory, see [1, 2, 4, 11, 17].

A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [9]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Harary index; for other related topological indices see [26].

Let $G$ be a connected graph. Then *Harary index* of $G$ is defined as $H(G) = \frac{1}{2}\sum_{u,\, v \,\in V(G)} \frac{1}{d_G(u,v)}$ with the summation going over all pairs of distinct vertices of $G$. The Harary index of a graph $G$ has been introduced by Plavsić *et al.* [22] and independently by Ivanciuć *et al.* [13] in 1993. Its applications and mathematical properties are well studied in [5, 8, 27, 15]. Zhou *et al.* [28] have obtained the lower and upper bounds of the Harary index of a connected graph. Very recently, Xu *et al.* [25] have obtained lower and upper bounds for the Harary index of a connected graph in relation to $\chi(G)$, the chromatic number of $G$ and $\omega(G)$, the clique number of $G$, and characterized the extremal graphs that attain the lower and upper bounds. Also, Feng *et al.* [8] have given a sharp upper bound for the Harary index of a graph based on the matching number, that is, the size of a maximum matching.

The Harary index and its related molecular descriptors have shown some success in structure property correlations [6, 7, 10]. Its modification has also been proposed [15] and its use in combination with other molecular descriptors improves the correlations [24, 23]. There are many topological indices such as Wiener index, hyper-Wiener index, vertex and edge PI indices, vertex and edge Szeged indices; they have been studied for general graphs and also for the product graphs such as tensor product [12, 19, 21], strong product [20], Cartesian product [14]. In the same way we would like to investigate the Harary index of tensor product, strong product and wreath product. We have obtained formulea for the Harary indices of $G \times K_{m_0,\, m_1,\ldots,m_{r-1}}$ and $G \boxtimes K_{m_0,\, m_1,\ldots,m_{r-1}}$. Also we have obtained upper bounds for the Harary indices of the tensor and strong products of graphs. Finally, the exact formula for Harary index of the wreath product of graphs is obtained. Based on the results obtained, exact Harary indices of some classes of graphs are obtained.

If $m_0 = m_1 = \cdots = m_{r-1} = s$ in $K_{m_0,\, m_1,\ldots,m_{r-1}}$ (the complete multipartite graph with partite sets of sizes $m_0,\, m_1, \ldots, m_{r-1}$), then we denote it by $K_{r(s)}$. For $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph of $G$ induced by $S$. A path and cycle on $n$ vertices are denoted by $P_n$ and $C_n$, respectively. We call $C_3$ a triangle. For two subsets $S, T \subset V(G)$, not necessarily disjoint, by $d_G(S,T)$ we mean the sum of the distances in $G$ from each vertex of $S$ to every vertex of $T$, that is, $d_G(S,T) = \sum_{s \in S,\, t \in T} d_G(s,t)$. For subsets $S, T \subset V(G)$, $E(S,T)$ denotes the set of edges of $G$ having one end in $S$ and the other end in $T$. For subsets $S$ and $T$, not necessarily disjoint, by $d_G^H(S,T)$ we mean the sum $d_G^H(S,T) = \sum_{s \in S,\, t \in T} \frac{1}{d_G(s,t)}$. Notation and definitions which are not given here can be found in [3] or [11].

## 2. HARARY INDEX OF TENSOR PRODUCT OF GRAPHS

Let $G$ be a connected graph with $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and let $K_{m_0, m_1, \ldots, m_{r-1}}$, $r \geq 3$, be the complete multiparite graph with partite sets $V_0, V_1, \ldots, V_{r-1}$ with $|V_i| = m_i$, $0 \leq i \leq r-1$. In the graph $G \times K_{m_0, m_1, \ldots, m_{r-1}}$, let $B_{ij} = v_i \times V_j, v_i \in V(G)$ and $0 \leq j \leq r-1$. For our convenience, we write

$$\begin{aligned} V(G) \times V(K_{m_0, m_1, \ldots, m_{r-1}}) &= \bigcup_{i=0}^{n-1} \left\{ v_i \times \bigcup_{j=0}^{r-1} V_j \right\} \\ &= \bigcup_{i=0}^{n-1} \left\{ \{v_i \times V_0\} \cup \{v_i \times V_1\} \cup \cdots \cup \{v_i \times V_{r-1}\} \right\} \\ &= \bigcup_{i=0}^{n-1} \left\{ B_{i0} \cup B_{i1} \cup \cdots \cup B_{i(r-1)} \right\}, \text{ where } B_{ij} = v_i \times V_j \\ &= \bigcup_{\substack{i=0 \\ j=0}}^{\substack{r-1 \\ n-1}} B_{ij}. \end{aligned}$$

Let $\mathscr{B} = \{B_{ij}\}_{\substack{i=0,1,\ldots,n-1 \\ j=0,1,\ldots,r-1}}$. We call $X_i = \bigcup_{j=0}^{r-1} B_{ij}$ a *layer* and $Y_j = \bigcup_{i=0}^{n-1} B_{ij}$ a *column* of $G \times K_{m_0, m_1, \ldots, m_{r-1}}$, see Figures 1 and 2. Clearly, a layer (resp. column) is an independent set in $G \times K_{m_0, m_1, \ldots, m_{r-1}}$; in particular, $B_{ij}$ is an independent set. Further, if $v_i v_k \in E(G)$, then the subgraph $\langle B_{ij} \cup B_{kp} \rangle$ of $G \times K_{m_0, m_1, \ldots, m_{r-1}}$ is isomorphic to $K_{|V_j||V_p|}$ or a totally disconnected graph according to $j \neq p$ or $j = p$. It is used in the proof of the next lemma.

The proof of the following lemma follows easily from the properties, structure of $G \times K_{m_0, m_1, \ldots, m_{r-1}}$ and the paths as shown in Figures 1 and 2.

**Lemma 1.** *Let $G$ be a connected graph on $n \geq 2$ vertices and let $B_{ij}$, $B_{kp} \in \mathscr{B}$ of the graph $G \times K_{m_0, m_1, \ldots, m_{r-1}}$, where $r \geq 3$.*

(i) *For any two distinct vertices in $B_{ij}$, their distance is 2.*

(ii) *Distance between two distinct vertices one from $B_{ij}$ and another from $B_{ip}$, $j \neq p$, is 2.*

(iii) *Distance between two vertices one from $B_{ij}$ and another from $B_{kj}$, $i \neq k$, is 2 or 3 according as $v_i v_k$ lies on a triangle in $G$ or $v_i v_k \in E(G)$ and $v_i v_k$ does not lie on a triangle in $G$.*

(iv) *If $v_i v_k \in E(G)$, then distance between two vertices one in $B_{ij}$ and the another in $B_{kp}, i \neq k, j \neq p$, is 1.*

(v) *If $v_i v_k \notin E(G)$, then distance between the vertices one in $B_{ij}$ and another in $B_{kp}$ is $d_G(v_i, v_k)$.*

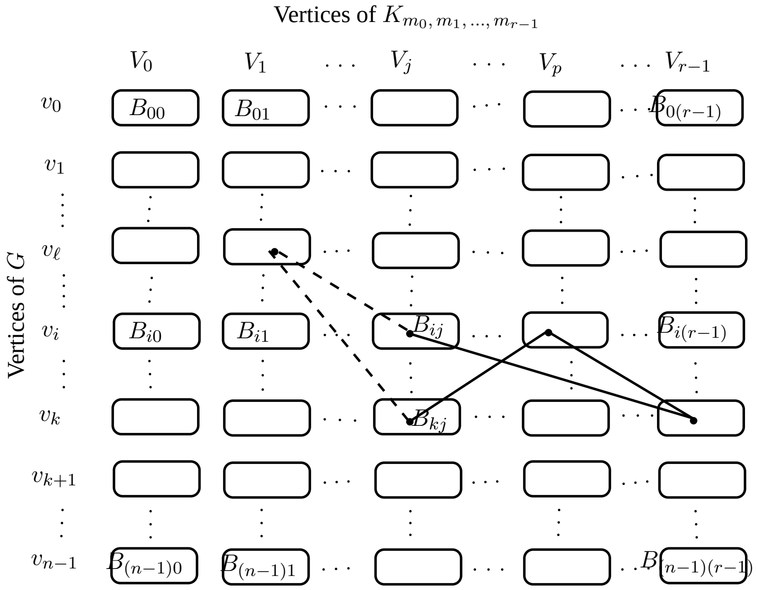

Figure 1. If $v_iv_k$ is on a triangle $v_iv_\ell v_k$ of $G$, then a shortest path of length 2 from a vertex of $B_{ij}$ to a vertex of $B_{kj}$ is shown in broken edges. If $v_iv_k$ is an edge but not on a triangle of $G$, then a shortest path of length 3 from a vertex of $B_{ij}$ to a vertex of $B_{kj}$ is shown in solid edges.

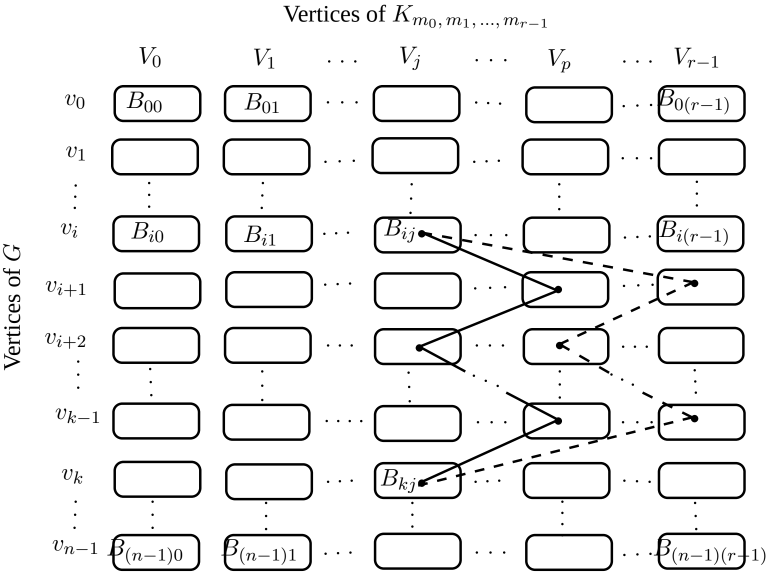

Figure 2. If a $(v_i, v_k)$- shortest path is of even (resp. odd$\geq 3$) length in $G$, then a shortest path from a vertex of $B_{ij}$ to a vertex of $B_{kj}$ is shown in solid edges (resp. broken edges).

The proof of the following lemma follows easily from Lemma 1 and hence it is left to the reader. The lemma is used in the proof of the main theorem of this section.

**Lemma 2.** *Let $G$ be a connected graph on $n \geq 2$ vertices and let $B_{ij},\ B_{kp} \in \mathscr{B}$ of the graph $G' = G \times K_{m_0, m_1, \ldots, m_{r-1}}$, where $r \geq 3$.*

(i) *If $v_i v_k \in E(G)$, then*

$$d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} m_j m_p, & \text{if } j \neq p, \\ \frac{m_j^2}{2}, & \text{if } j = p \text{ and } v_i v_k \text{ is on a triangle of } G, \\ \frac{m_j^2}{3}, & \text{if } j = p \text{ and } v_i v_k \text{ is not on a triangle of } G. \end{cases}$$

(ii) *If $v_i v_k \notin E(G)$, then* $d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} \frac{m_j m_p}{d_G(v_i, v_k)}, & if\ j \neq p, \\ \frac{m_j^2}{d_G(v_i, v_k)}, & if\ j = p. \end{cases}$

(iii) $d^H_{G'}(B_{ij}, B_{ip}) = \begin{cases} \frac{m_j(m_j-1)}{2}, & if\ j = p, \\ \frac{m_j m_p}{2}, & if\ j \neq p. \end{cases}$

**Theorem 3.** *Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges and let $\lambda$ be the number of edges of $G$ which do not lie on any $C_3$ of it. If $n_0$ and $q$ are the numbers of vertices and edges of $K_{m_0, m_1, \ldots, m_{r-1}}$, $r \geq 3$, respectively, then* $H(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = n_0^2\, H(G) + \frac{n\, n_0(n_0-1)}{4} - \left(m + \frac{\lambda}{3}\right) \frac{(n_0^2 - 2q)}{2}$.

***Proof.*** Let $G' = G \times K_{m_0, m_1, \ldots, m_{r-1}}$. Clearly,

$$\begin{aligned} H(G') &= \frac{1}{2} \sum\nolimits_{B_{ij},\, B_{kp} \in \mathscr{B}} d^H_{G'}(B_{ij}, B_{kp}) \\ &= \frac{1}{2} \Bigg( \sum\nolimits_{i=0}^{n-1} \sum\nolimits_{\substack{j,p=0 \\ j \neq p}}^{r-1} d^H_{G'}(B_{ij}, B_{ip}) + \sum\nolimits_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum\nolimits_{j=0}^{r-1} d^H_{G'}(B_{ij}, B_{kj}) \\ &\quad + \sum\nolimits_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum\nolimits_{\substack{j,p=0 \\ j \neq p}}^{r-1} d^H_{G'}(B_{ij}, B_{kp}) + \sum\nolimits_{i=0}^{n-1} \sum\nolimits_{j=0}^{r-1} d^H_{G'}(B_{ij}, B_{ij}) \Bigg) \\ &= \frac{1}{2} \{A_1 + A_2 + A_3 + A_4\}, \end{aligned} \tag{1}$$

where $A_1$ to $A_4$ are the sums of the above terms, in order. We shall calculate $A_1$ to $A_4$ of (1) separately.

$(A_1)$. First we compute $\sum_{i=0}^{n-1} \left( \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} d^H_{G'}(B_{ij}, B_{ip}) \right)$. For this, we compute

$\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} d^H_{G'}(B_{ij}, B_{ip})$.

$$\begin{aligned}
\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} d^H_{G'}(B_{ij}, B_{ip}) &= \sum_{\substack{p=0\\ p\neq 0}}^{r-1} d^H_{G'}(B_{i0}, B_{ip}) + \sum_{\substack{p=0\\ p\neq 1}}^{r-1} d^H_{G'}(B_{i1}, B_{ip}) + \cdots \\
&\quad + \sum_{\substack{p=0\\ p\neq r-1}}^{r-1} d^H_{G'}(B_{i(r-1)}, B_{ip}),\ \text{since } |B_{ip}| = m_p \\
&= \sum_{\substack{p=0\\ p\neq 0}}^{r-1} \frac{m_0 m_p}{2} + \sum_{\substack{p=0\\ p\neq 1}}^{r-1} \frac{m_1 m_p}{2} + \cdots + \sum_{\substack{p=0\\ p\neq r-1}}^{r-1} \frac{m_{r-1} m_p}{2}, \\
&= \sum_{\substack{a,\,p=0\\ a\neq p}}^{r-1} \frac{m_a m_p}{2}.
\end{aligned} \tag{2}$$

Now summing (2) over $i = 0, 1, \ldots, n-1$, we get

$$\begin{aligned}
\sum_{i=0}^{n-1} \sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} d^H_{G'}(B_{ij}, B_{ip}) &= \sum_{i=0}^{n-1} \sum_{\substack{a,\,p=0\\ a\neq p}}^{r-1} \frac{m_a m_p}{2} \\
&= \frac{n}{2}\left(\sum_{\substack{a,\,p=0\\ a\neq p}}^{r-1} m_a m_p\right).
\end{aligned} \tag{3}$$

$(A_2)$. Next we compute $\sum_{j=0}^{r-1}\left(\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1} d^H_{G'}(B_{ij}, B_{kj})\right)$. For this, initialy we calculate $\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1} d^H_{G'}(B_{ij}, B_{kj})$.

Let $E_1 = \{uv \in E(G) \,|\, uv \text{ is on a } C_3 \text{ in } G\}$ and $E_2 = E(G) - E_1$.

$$\begin{aligned}
\sum_{\substack{i,k=0\\ i\neq k}}^{n-1} d^H_{G'}(B_{ij}, B_{kj}) &= \sum_{\substack{i,k=0,\, i\neq k\\ v_iv_k \notin E(G)}}^{n-1} d^H_{G'}(B_{ij}, B_{kj}) + \sum_{\substack{i,k=0,\, i\neq k\\ v_iv_k \in E_1}}^{n-1} d^H_{G'}(B_{ij}, B_{kj}) \\
&\quad + \sum_{\substack{i,k=0,\, i\neq k\\ v_iv_k \in E_2}}^{n-1} d^H_{G'}(B_{ij}, B_{kj}) \\
&= \sum_{\substack{i,k=0,\, i\neq k\\ v_iv_k \notin E(G)}}^{n-1} \frac{m_j^2}{d_G(v_i, v_k)} + \sum_{\substack{i,k=0,\, i\neq k\\ v_iv_k \in E_1}}^{n-1} \frac{m_j^2}{2} \\
&\quad + \sum_{\substack{i,\,k=0,\, i\neq k\\ v_iv_k \in E_2}}^{n-1} \frac{m_j^2}{3},\ \text{ by Lemma 2,} \\
&= \sum_{\substack{i,\,k=0,\, i\neq k\\ v_iv_k \notin E(G)}}^{n-1} \frac{m_j^2}{d_G(v_i, v_k)} + \sum_{\substack{i,k=0,\, i\neq k\\ v_iv_k \in E_1}}^{n-1} \left(\frac{m_j^2}{2} + m_j^2 - m_j^2\right) \\
&\quad + \sum_{\substack{i,k=0,\, i\neq k\\ v_iv_k \in E_2}}^{n-1} \left(\frac{m_j^2}{3} + m_j^2 - m_j^2\right) * \\
&= \left(\sum_{\substack{i,k=0,\, i\neq k\\ v_iv_k \notin E(G)}}^{n-1} \frac{m_j^2}{d_G(v_i, v_k)} + \sum_{\substack{i,k=0,\, i\neq k\\ v_iv_k \in E_1}}^{n-1} \frac{m_j^2}{d_G(v_i, v_k)}\right.
\end{aligned}$$

$$+\sum_{\substack{i,k=0,\, i\neq k\\ v_iv_k\in E_2}}^{n-1}\frac{m_j^2}{d_G(v_i,v_k)}\Bigg)-\sum_{\substack{i,k=,\, i\neq k\\ v_iv_k\in E_1}}^{n-1}\frac{m_j^2}{2}-\sum_{\substack{i,k=0,\, i\neq k\\ v_iv_k\in E_2}}^{n-1}\frac{2m_j^2}{3},$$

since $d_G(v_i,v_k)=1$ if $v_iv_k\in E_1$,

$$=\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\frac{m_j^2}{d_G(v_i,v_k)}-\Bigg(\sum_{\substack{i,k=0,\, i\neq k\\ v_iv_k\in E_1}}^{n-1}\frac{m_j^2}{2}+\sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E_2}}^{n-1}\frac{m_j^2}{2}\Bigg)$$

$$-\sum_{\substack{i,k=0,\, i\neq k\\ v_iv_k\in E_2}}^{n-1}\frac{m_j^2}{6}.$$

Thus

$$\sum_{\substack{i,k=0\\ i\neq k}}^{n-1} d_{G'}^{H}(B_{ij},B_{kj})=2H(G)\,m_j^2-m\,m_j^2-\frac{\lambda\, m_j^2}{3}, \tag{4}$$

where $m$ and $\lambda$ are the number of edges of $G$ and the number of edges of $G$ which do not lie on any $C_3$, respectively. Note that each edge $v_iv_k$ of $G$ is being counted twice in the sum, namely, $v_iv_k$ and $v_kv_i$.

Now summing (4) over $j=0,1,\ldots,r-1$, we get

$$\begin{aligned}\sum_{j=0}^{r-1}\sum_{\substack{i,k=0\\ i\neq k}}^{n-1} d_{G'}^{H}(B_{ij},B_{kj}) =&2H(G)\left(\sum_{j=0}^{r-1}m_j^2\right)\\ &-\left(m+\frac{\lambda}{3}\right)\left(\sum_{j=0}^{r-1}m_j^2\right).\end{aligned} \tag{5}$$

$(A_3)$. Next we compute $\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\left(\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} d_{G'}^{H}(B_{ij},B_{kp})\right)$. For this, first we calculate $\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} d_{G'}^{H}(B_{ij},B_{kp})$.

$$\begin{aligned}\sum_{\substack{j,\,p=0,\\ j\neq p}}^{r-1} d_{G'}^{H}(B_{ij},B_{kp}) =&\sum_{\substack{p=0\\ p\neq 0}}^{r-1} d_{G'}^{H}(B_{i0},B_{kp})+\sum_{\substack{p=0\\ p\neq 1}}^{r-1} d_{G'}^{H}(B_{i1},B_{kp})+\cdots\\ &+\sum_{\substack{p=0\\ p\neq r-1}}^{r-1} d_{G'}^{H}(B_{i(r-1)},B_{kp})\\ =&\sum_{\substack{p=0\\ p\neq 0}}^{r-1}\frac{m_0m_p}{d_G(v_i,v_k)}+\sum_{\substack{p=0\\ p\neq 1}}^{r-1}\frac{m_1m_p}{d_G(v_i,v_k)}+\cdots\\ &+\sum_{\substack{p=0\\ p\neq r-1}}^{r-1}\frac{m_{r-1}m_p}{d_G(v_i,v_k)},\ \text{by Lemma 2,}\\ =&\sum_{\substack{a,\,p=0\\ a\neq p}}^{r-1}\frac{m_am_p}{d_G(v_i,v_k)}.\end{aligned} \tag{6}$$

Using (6) we have

$$\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} d^H_{G'}(B_{ij},B_{kp}) = \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{a,p=0\\ a\neq p}}^{r-1}\frac{m_a m_p}{d_G(v_i,v_k)} = 2H(G)\left(\sum_{\substack{a,p=0\\ a\neq p}}^{r-1} m_a m_p\right). \tag{7}$$

$(A_4)$. Finally, we compute $\sum_{i=0}^{n-1}\left(\sum_{j=0}^{r-1} d^H_{G'}(B_{ij},B_{ij})\right)$.

$$\sum_{j=0}^{r-1} d^H_{G'}(B_{ij},B_{ij}) = \sum_{j=0}^{r-1}\frac{m_j(m_j-1)}{2}, \quad \text{by Lemma 2.} \tag{8}$$

Now

$$\sum_{i=0}^{n-1}\sum_{j=0}^{r-1} d^H_{G'}(B_{ij},B_{ij}) = \sum_{i=0}^{n-1}\left(\sum_{j=0}^{r-1}\frac{m_j(m_j-1)}{2}\right) = \frac{n}{2}\left(\sum_{j=0}^{r-1} m_j(m_j-1)\right). \tag{9}$$

Using (1) and the sums $A_1, A_2, A_3$ and $A_4$ in (3),(5),(7) and (9), respectively, we have

$$\begin{aligned} H(G') =& \frac{1}{2}\Bigg\{\frac{n}{2}\left(\sum_{\substack{a,p=0\\ a\neq p}}^{r-1} m_a m_p\right) + 2H(G)\left(\sum_{j=0}^{r-1} m_j^2\right) \\ & -\left(m+\frac{\lambda}{3}\right)\left(\sum_{j=0}^{r-1} m_j^2\right) + 2H(G)\left(\sum_{\substack{a,p=0\\ a\neq p}}^{r-1} m_a m_p\right) \\ & +\frac{n}{2}\left(\sum_{j=0}^{r-1} m_j(m_j-1)\right)\Bigg\} = H(G)\left(\sum_{j=0}^{r-1} m_j^2 + \sum_{\substack{a,p=0\\ a\neq p}}^{r-1} m_a m_p\right) \\ & +\frac{n}{4}\left(\sum_{\substack{a,p=0\\ a\neq p}}^{r-1} m_a m_p + \sum_{j=0}^{r-1} m_j(m_j-1)\right) \\ & -\frac{1}{2}\left(m+\frac{\lambda}{3}\right)\left(\sum_{j=0}^{r-1} m_j^2\right) \\ =& n_0^2 H(G) + \frac{n\, n_0(n_0-1)}{4} - \left(m+\frac{\lambda}{3}\right)\frac{(n_0^2-2q)}{2}, \end{aligned}$$

where $n_0 = \sum_{i=0}^{r-1} m_i$ and $q$ is the number of edges of $K_{m_0,m_1,\ldots,m_{r-1}}$. ■

**Remark.** In the above theorem if $r = 2$, then $G \times K_{m_0,m_1}$ would be a disconnected whenever $G$ is a bipartite graph. As we deal with only connected graphs, we consider $r \geq 3$.

If $m_i = s,\ 0 \leq i \leq r-1$ in Theorem 3, then we have the following corollary.

**Corollary 4.** *Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges; let $\lambda$ be the number of edges of $G$ which do not lie on any $C_3$ of it. Then* $H(G \times K_{r(s)}) = r^2s^2\, H(G) + \frac{nrs(rs-1)}{4} - \frac{rs^2}{2}\left(m + \frac{\lambda}{3}\right)$, *where* $r \geq 3$.

As $K_r = K_{r(1)}$, from the above corollary we have the following corollary.

**Corollary 5.** *Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges; let $\lambda$ be the number of edges of $G$ which do not lie on any $C_3$ of it. Then* $H(G \times K_r) = r^2 H(G) - \left(m + \frac{\lambda}{3}\right)\frac{r}{2} + \frac{nr(r-1)}{4}$, *where* $r \geq 3$.

**Corollary 6.** *Let $G$ be a connected graph on $n \geq 2$ vertices with $m$ edges. If each edge of $G$ is on a $C_3$, then* $H(G \times K_{r(s)}) = r^2s^2\, H(G) - \frac{mrs^2}{2} + \frac{nsr(sr-1)}{4}$, *where* $r \geq 3$.

For a triangle free graph $G$, $\lambda = m$ and hence we have the following corollary.

**Corollary 7.** *If $G$ is a connected triangle free graph on $n \geq 2$ vertices and $m$ edges, then* $H(G \times K_{r(s)}) = r^2s^2\, H(G) - \frac{2mrs^2}{3} + \frac{nsr(sr-1)}{4}$, *where* $r \geq 3$.

If $s = 1$ in the above corollary, we obtain the following corollary.

**Corollary 8.** *If $G$ is a connected triangle free graph on $n \geq 2$ vertices and $m$ edges, then* $H(G \times K_r) = r^2\, H(G) - \frac{2mr}{3} + \frac{nr(r-1)}{4}$, *where* $r \geq 3$.

One can see that [25], $H(P_n) = n\left(\sum_{i=1}^{n} \frac{1}{i}\right) - n$ and

$$H(C_n) = \begin{cases} n\left(\sum_{i=1}^{\frac{n}{2}} \frac{1}{i}\right) - 1, & \text{if } n \text{ is even,} \\ n\left(\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\right), & \text{if } n \text{ is odd.} \end{cases}$$

By using Corollary 5, $H(P_n)$ and $H(C_n)$, we obtain the exact Harary indices of the following graphs.

**Example 1.**

(i) If $n \geq 2$ and $r \geq 3$, then $H(P_n \times K_r) = nr^2\left(\sum_{i=1}^{n} \frac{1}{i}\right) - \frac{r}{12}(11n + 9rn - 8)$.

(ii) $$H(C_n \times K_r) = \begin{cases} r^2\left\{n\left(\sum_{i=1}^{\frac{n}{2}} \frac{1}{i}\right) - 1\right\} + \frac{nr}{12}(3r - 11), & \text{if } n \text{ is even,} \\ \frac{3r(5r-3)}{4}, & \text{if } n = 3 \\ r^2 n\left(\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\right) + \frac{nr}{12}(3r - 11), & \text{if } n > 3 \text{ is odd.} \end{cases}$$

## 3. AN UPPER BOUND FOR HARARY INDEX OF TENSOR PRODUCT OF GRAPHS

In this section, we establish an upper bound for the Harary index of the tensor product of graphs.

Let $(V_1, V_2, \ldots, V_\chi)$ be a proper $\chi(G)$-colouring of $G$, where $\chi(G)$ is the chromatic number of $G$, such that no $V_i$ can be augmented by adding any vertex of $V_j,\ j \geq i+1$, that is, no vertex of $V_j$ is nonadjacent to all the vertices of $V_i,\ i < j$, in $G$. Without loss of generality we assume that $|V_1| \geq |V_2| \geq \cdots \geq |V_r|$. We call such a $\chi(G)$-colouring a *decreasing* $\chi(G)$*-colouring* of $G$.

**Theorem 9.** *Let $G$ be connected graph with $n \geq 2$ vertices and $m$ edges; let $G'$ be a graph with $\chi(G') = r \geq 3$. If color classes of the decreasing $\chi(G')$-coloring of $G'$ have $m_0, m_1, \ldots, m_{r-1}$ vertices, then $H(G \times G') \leq H(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = n_0^2\, H(G) + \frac{n\, n_0(n_0-1)}{4} - \left(m + \frac{\lambda}{3}\right) \frac{(n_0^2-2q)}{2}$, where $\sum_{i=0}^{r-1} m_i = n_0$ equals the number of vertices of $G'$, $q$ is the number of edges of $K_{m_0, m_1, \ldots, m_{r-1}}$ and $\lambda$ is the number of edges of $G$ which do not lie on a triangle.*

***Proof.*** As $G'$ is a subgraph of $K_{m_0, m_1, \ldots, m_{r-1}}$, $H(G \times G') \leq H(G \times K_{m_0, m_1, \ldots, m_{r-1}})$, since $d_{G\times G'}((x_1, y_1), (x_2, y_2)) \geq d_{G\times K_{m_0, m_1, \ldots, m_{r-1}}}((x_1, y_1), (x_2, y_2))$ for any pair of vertices $(x_1, y_1)$ and $(x_2, y_2)$ of $G \times G'$. Thus, $H(G \times G') \leq H(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = n_0^2\, H(G) + \frac{n\, n_0(n_0-1)}{4} - \left(m + \frac{\lambda}{3}\right) \frac{(n_0^2-2q)}{2}$, by Theorem 3. ■

## 4. HARARY INDEX OF STRONG PRODUCT OF GRAPHS

In this section, we obtain the Harary index of $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$. Let $G$ be a simple connected graph with $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and let $K_{m_0, m_1, \ldots, m_{r-1}},\ r \geq 2$, be the complete multiparite graph with partite sets $V_0, V_1, \ldots, V_{r-1}$ and let $|V_i| = m_i,\ 0 \leq i \leq r-1$. In the graph $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, let $B_{ij} = v_i \times V_j, v_i \in V(G)$ and $0 \leq j \leq r-1$. For our convenience, as in the case of tensor product, the vertex set of $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$ is written as $V(G) \times V(K_{m_0, m_1, \ldots, m_{r-1}}) = \bigcup_{i=0,\, j=0}^{r-1,\, n-1} B_{ij}$. As in the tensor product of graphs, let $\mathscr{B} = \{B_{ij}\}_{\substack{i=0,1,\ldots,n-1 \\ j=0,1,\ldots,r-1}}$. Let $X_i = \bigcup_{j=0}^{r-1} B_{ij}$ and $Y_j = \bigcup_{i=0}^{n-1} B_{ij}$; we call $X_i$ and $Y_j$ as *layer* and *column* of $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, respectively, see Figures 3 and 4. If we denote $V(B_{ij}) = \{x_{i1}, x_{i2}, \ldots, x_{im_j}\}$ and $V(B_{kp}) = \{x_{k1}, x_{k2}, \ldots, x_{k\,m_p}\}$, then $x_{i\ell}$ and $x_{k\ell}, 1 \leq \ell \leq j$, are called the *corresponding vertices* of $B_{ij}$ and $B_{kp}$. Further, if $v_i v_k \in E(G)$, then the induced subgraph $\langle B_{ij} \bigcup B_{kp} \rangle$ of $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$ is isomorphic to $K_{|V_j||V_p|}$ or $m_p$ independent edges joining the corresponding vertices of $B_{ij}$ and $B_{kj}$ according as $j \neq p$ or $j = p$, respectively.

The proof of the following lemma follows easily from the properties and structure of $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, see Figures 3 and 4.

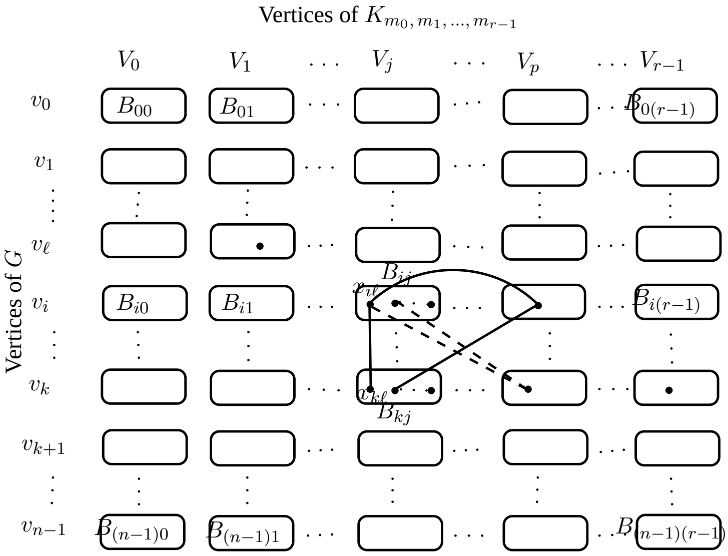

Figure 3. If $v_iv_k \in E(G)$, then shortest paths of length 1 and 2 from $B_{ij}$ to $B_{kj}$ are shown in solid edges, where the vertical line between $B_{ij}$ and $B_{kj}$ denotes the edge joining the corresponding vertices of $B_{ij}$ and $B_{kj}$. The broken edges denote a shortest path of length 2 from a vertex of $B_{ij}$ to a vertex of $B_{ij}$.

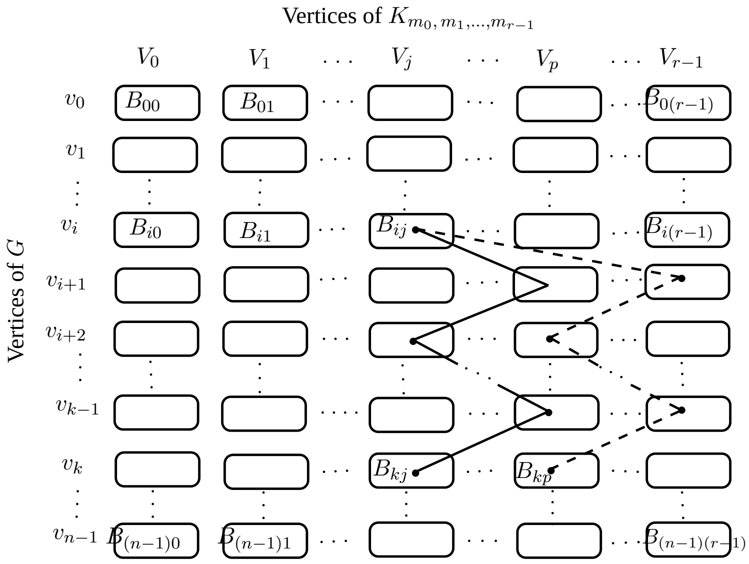

Figure 4. Corresponding to a shortest path of length $k > 1$ in $G$, the shortest path from any vertex of $B_{ij}$ to any vertex of $B_{kj}$ (resp. any vertex of $B_{ij}$ to any vertex of $B_{kp}$, $p \neq j$) of lenght $k$ is shown in solid edges (resp. broken edges).

**Lemma 10.** *Let $G$ be a connected graph and let $B_{ij},\ B_{kp} \in \mathscr{B}$ of the graph $G' = G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, where $r \geq 2$.*

(i) *If $v_iv_k \in E(G)$ and $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kj}$, then*

$$d_{G'}(x_{it}, x_{k\ell}) = \begin{cases} 1, & \text{if } t = \ell, \\ 2, & \text{if } t \neq \ell, \end{cases}$$

*and if $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kp},\ j \neq p$, then $d_{G'}(x_{it}, x_{k\ell}) = 1$.*

(ii) *If $v_iv_k \notin E(G)$, then for any two vertices $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kp}$, $d_{G'}(x_{it}, x_{k\ell}) = d_G(v_i, v_k)$.*

(iii) *For any two distinct vertices in $B_{ij}$, their distance is 2.*

The proof of the following lemma follows easily from Lemma 10. The lemma is used in the proof of the main theorems of this section.

**Lemma 11.** *Let $G$ be a connected graph and let $B_{ij},\ B_{kp} \in \mathscr{B}$ of the graph $G' = G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, where $r \geq 2$.*

(i) *If $v_iv_k \in E(G)$, then* $d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} m_jm_p, & \text{if } j \neq p, \\ \frac{m_j(m_j+1)}{2}, & \text{if } j = p. \end{cases}$

(ii) *If $v_iv_k \notin E(G)$, then* $d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} \frac{m_jm_p}{d_G(v_i,v_k)}, & \text{if } j \neq p, \\ \frac{m_j^2}{d_G(v_i,v_k)}, & \text{if } j = p. \end{cases}$

(iii) $d^H_{G'}(B_{ij}, B_{ip}) = \begin{cases} m_jm_p, & \text{if } j \neq p, \\ \frac{m_j(m_j-1)}{2}, & \text{if } j = p. \end{cases}$

The proof of the following theorem is similar to the proof of Theorem 3. Here the Lemma 11 is used for the computation of $A_1$,$A_2$,$A_3$ and $A_4$, by Theorem 3. Hence the proof of the following Theorem 12 is omitted.

**Theorem 12.** *Let $G$ be a connected graph with $n$ vertices. Then $H(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}) = n_0^2 H(G) - \frac{m}{2}\left(n_0^2 - 2q - n_0\right) + \frac{n}{4}\left(n_0^2 + 2q - n_0\right)$, where $n_0 = \sum_{i=0}^{r-1} m_i$ and $q$ is the number of edges of $K_{m_0, m_1, \ldots, m_{r-1}}$.*

If $m_i = s,\ 0 \leq i \leq r-1$, in Theorem 12, we have the following corollary.

**Corollary 13.** *Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then $H(G \boxtimes K_{r(s)}) = r^2s^2H(G) - \frac{mrs(s-1)}{2} + \frac{nrs(2rs-s-1)}{4}$.*

As $K_r = K_{r(1)}$, the above corollary gives the next one.

**Corollary 14.** *Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then $H(G \boxtimes K_r) = r^2H(G) + \frac{nr(r-1)}{2}$.*

By using Corollary 14, $H(P_n)$ and $H(C_n)$, we obtain the exact Harary indices of the following graphs.

**Example 2.**

(i) If $r \geq 2$, then $H(P_n \boxtimes K_r) = nr^2\Big(\sum_{i=1}^{n}\frac{1}{i}\Big) - \frac{nr(r+1)}{2}$.

(ii) $$H(C_n \boxtimes K_r) = \begin{cases} r^2\Big\{n\Big(\sum_{i=1}^{\frac{n}{2}}\frac{1}{i}\Big) - 1\Big\} + \frac{nr(r-1)}{2}, & \text{if } n \text{ is even,}\\ r^2 n\Big(\sum_{i=1}^{\frac{n-1}{2}}\frac{1}{i}\Big) + \frac{nr(r-1)}{2}, & \text{if } n \text{ is odd.}\end{cases}$$

Next, we obtain an upper bound for the Harary index of the graph $G \boxtimes G'$. The following theorem follows from Theorem 12.

**Theorem 15.** *Let $G$ be connected graph with $n$ vertices and $m$ edges; let $G'$ be a graph with $\chi(G') = r \geq 2$. If the decreasing $\chi(G')$-coloring color classes of $G'$ have $m_0, m_1, \ldots, m_{r-1}$ vertices, then $H(G \boxtimes G') \leq H(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}) = n_0^2\, H(G) - \frac{m}{2}\left(n_0^2 - 2q - n_0\right) + \frac{n}{4}\left(n_0^2 + 2q - n_0\right)$, where $n_0$ is the number of vertices of $G'$ and $q$ is the number of edges of $K_{m_0, m_1, \ldots, m_{r-1}}$.*

***Proof.*** As $G'$ is a subgraph of $K_{m_0, m_1, \ldots, m_{r-1}}$, $H(G \boxtimes G') \leq H(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}})$, since $d_{G\boxtimes G'}((x_1, y_1), (x_2, y_2)) \geq d_{G\boxtimes K_{m_0, m_1, \ldots, m_{r-1}}}((x_1, y_1), (x_2, y_2))$ for any pair of vertices $(x_1, y_1)$ and $(x_2, y_2)$ of $G \boxtimes G'$. Hence, $H(G \boxtimes G') \leq H(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}) = n_0^2\, H(G) - \frac{m}{2}\left(n_0^2 - 2q - n_0\right) + \frac{n}{4}\left(n_0^2 + 2q - n_0\right)$, by Theorem 12. ■

## 5. HARARY INDEX OF THE WREATH PRODUCT OF GRAPHS

In this section, we obtain the Harary index of $G \circ G'$.

**Theorem 16.** *Let $G$ and $G'$ be two connected graphs with $|V(G)| = n$ and $|V(G')| = m$. Then $H(G \circ G') = \frac{n}{4}\left(m^2 + 2\,|E(G')| - m\right) + m^2\, H(G)$.*

***Proof.*** Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and let $V(G') = \{v_1, v_2, \ldots, v_m\}$. Let $x_{ij}$ denote the vertex $(u_i, v_j)$ of $G \circ G'$. By the definition of Harary index

$$\begin{aligned} H(G \circ G') &= \frac{1}{2}\sum_{x_{ij}, x_{k\ell} \in V(G\circ G')} \frac{1}{d_{G\circ G'}(x_{ij}, x_{k\ell})}\\ &= \frac{1}{2}\Bigg(\sum_{i=0}^{n-1}\sum_{\substack{j,\ell=0\\ j\neq \ell}}^{m-1}\frac{1}{d_{G\circ G'}(x_{ij}, x_{i\ell})}\\ &\quad+\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{j=0}^{m-1}\frac{1}{d_{G\circ G'}(x_{ij}, x_{kj})}\\ &\quad+\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,\ell=0\\ j\neq \ell}}^{m-1}\frac{1}{d_{G\circ G'}(x_{ij}, x_{k\ell})}\Bigg) = \frac{1}{2}\{A_1 + A_2 + A_3\}, \end{aligned} \tag{10}$$

where $A_1$ to $A_3$ are the sums of the above terms, in order.

We shall calculate the terms $A_1$ to $A_3$ of above expression separately.

$$\begin{aligned}
A_1 =& \sum_{i=0}^{n-1} \sum_{\substack{j,\ell=0 \\ j\neq\ell}}^{m-1} \frac{1}{d_{G\circ G'}(x_{ij}, x_{i\ell})} \\
=& n\left(\sum_{v_jv_\ell\in E(G')} \frac{1}{d_{G'}(v_j, v_\ell)} + \sum_{v_jv_\ell\notin E(G')} \frac{1}{d_{G'}(v_j, v_\ell)}\right) \\
=& n\left(\sum_{v_j\in V(G')} deg(v_j) + \sum_{v_j\in V(G')} \frac{1}{2}(\,m - deg(v_j) - 1\,)\right),
\end{aligned}$$

(11) since each layer induces a copy of $G'$ and

$$d_{G\circ G'}(x_{ij}, x_{i\ell}) = \begin{cases} 1, & \text{if } \ v_jv_\ell \in E(G'), \\ 2, & \text{if } \ v_jv_\ell \notin E(G'). \end{cases}$$

$$\begin{aligned}
=& n\left(2\left|E(G')\right| + \frac{1}{2}(\,m^2 - 2\left|E(G')\right| - m\,)\right) \\
=& \frac{n}{2}\left(4\left|E(G')\right| + m^2 - 2\left|E(G')\right| - m\right) \\
=& \frac{n}{2}\left(m^2 + 2\left|E(G')\right| - m\right).
\end{aligned}$$

$$\begin{aligned}
A_2 =& \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} \sum_{j=0}^{m-1} \frac{1}{d_{G\circ G'}(x_{ij}, x_{kj})} \\
=& m \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} \frac{1}{d_G(u_i, u_k)} = 2m\, H(G).
\end{aligned} \tag{12}$$

since the distance between a pair of vertices in a column is the same as the distance between the corresponding vertices of any other column.

Similar to the computation of $A_2$, we have

$$A_3 = \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} \sum_{\substack{j,\ell=0 \\ j\neq\ell}}^{m-1} \frac{1}{d_{G\circ G'}(x_{ij}, x_{k\ell})} = 2m(m-1)\, H(G). \tag{13}$$

Using (11), (12) and (13) in (10) we have

$$\begin{aligned}
H(G\circ G') =& \frac{1}{2}\left(\frac{n}{2}\left(m^2 + 2\left|E(G')\right| - m\right) + 2m\, H(G) + 2m(m-1)H(G)\right) \\
=& \frac{n}{4}\left(m^2 + 2\left|E(G')\right| - m\right) + m^2\, H(G).
\end{aligned}$$

■

As an application we present formulae for Harary indices of open and closed fence graphs, $P_n \circ K_2$ and $C_n \circ K_2$, respectively.

**Example 3.**

(i) $H(P_n \circ K_2) = 4n\Big(\sum_{i=1}^{n} \frac{1}{i}\Big) - 3n.$

(ii) $$H(C_n \circ K_2) = \begin{cases} n\Big(1 + 4\sum_{i=1}^{\frac{n}{2}} \frac{1}{i}\Big) - 4, & \text{if } n \text{ is even,} \\ n\Big(1 + 4\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\Big), & \text{if } n \text{ is odd.} \end{cases}$$

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