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α-LABELINGS OF A CLASS OF GENERALIZED PETERSEN GRAPHS

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Abstract

An α-labeling of a bipartite graph Γ of size e is an injective function $f : V(\Gamma) \rightarrow \{0, 1, 2, \ldots, e\}$ such that $\{|f(x) - f(y)| \ : \ [x, y] \in E(\Gamma)\} = \{1, 2, \ldots, e\}$ and with the property that its maximum value on one of the two bipartite sets does not reach its minimum on the other one. We prove that the generalized Petersen graph $P_{8n,3}$ admits an α-labeling for any integer $n \geq 1$ confirming that the conjecture posed by Vietri in [10] is true. In such a way we obtain an infinite class of decompositions of complete graphs into copies of $P_{8n,3}$.

Keywords: generalized Petersen graph, α-labeling, graph decomposition.

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1. Introduction

As usual, we denote by K_v and $K_{m\times n}$ the *complete graph on v vertices* and the *complete m-partite graph with parts of size n*, respectively. Given a subgraph Γ of a graph K, a Γ-*decomposition of* K is a set of graphs, called *blocks*, isomorphic to Γ, whose edges partition the edge-set of K. Such a decomposition is said to be *cyclic* when it is invariant under a cyclic permutation of all the vertices of K. For a survey on the subject see [3].

The problem of establishing the set of values of v for which a Γ-decomposition of K_v exists has been extensively studied and it is in general quite difficult. The concept of a *graceful labeling* of a graph Γ, introduced by Rosa [7], is proved to be an useful tool for determining the existence of cyclic Γ-decompositions of the complete graph. A *graceful labeling* of a graph Γ of size e is an injective function $f : V(\Gamma) \to \{0, 1, 2, \ldots, e\}$ such that

$$\{|f(x) - f(y)| \ : \ [x, y] \in E(\Gamma)\} = \{1, 2, \ldots, e\}.$$

In the case where Γ is bipartite and f has the additional property that its maximum value on one of the two bipartite sets does not reach its minimum on the other one, one says that f is an *α-labeling.* For a very rich survey on graceful labelings we refer to [5]. In [7], Rosa proved the following result.

Theorem 1. *If a graph Γ of size e admits a graceful labeling f, then there exists a cyclic Γ-decomposition of K_{2e+1}. Also, if f is, in addition, an α-labeling, then there exists a cyclic Γ-decomposition of K_{2et+1} for any positive integer t.*

In this paper we shall investigate the existence of α-labelings of a class of generalized Petersen graphs.

Definition. Let n, k be positive integers such that $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. The *generalized Petersen graph* $P_{n,k}$ is the graph whose vertex set is $\{a_i, b_i : \ 1 \leq i \leq n\}$ and whose edge set is $\{[a_i, b_i], [a_i, a_{i+1}], [b_i, b_{i+k}] \ : \ 1 \leq i \leq n\}$, where subscripts are meant modulo n.

In [4], Frucht and Gallian proved that $P_{n,1}$, which can be seen as the prism on $2n$ vertices, is graceful. Moreover when n is even, namely when the graph is bipartite, their labelings are α-labelings. In [6], with the aid of a computer, some $P_{n,k}$'s with $k \geq 2$ and small values of n where shown to be graceful. The only results about infinite classes of $P_{n,k}$'s with $k > 1$ were obtained by Vietri. He proved that $P_{8n,3}$ is graceful for every positive integer n, see [9, 10], and that $P_{8n+4,3}$ is graceful for every positive integer n, see [8]. Also, in [10] Vietri conjectured that there exists an α-labeling for every graph $P_{8n,3}$. Here we prove that Vietri's conjecture is true. As a consequence we obtain a new infinite class of decompositions of the complete graph into generalized Petersen graphs. Even though the literature is quite poor about results on $P_{n,k}$-decompositions of the complete graph, we point out that Adams and Bryant in [1] determined the spectrum of values of v for which a $P_{5,2}$-decomposition of K_v exists and that Bonisoli, Buratti and Rinaldi in [2] obtained some results about sharply vertex-transitive $P_{n,k}$-decomposition of K_v.

The results contained in this paper were already briefly presented in [A. Benini and A. Pasotti, *Decompositions into generalized Petersen graphs via graceful labeling*, Electron. Notes Discrete Math. **40** (2013) 295–298].

2. On α-labelings of $P_{8n,3}$

In this section we prove the existence of an α-labeling of $P_{8n,3}$ by a direct construction. The basic idea is to see the graph as a disjoint union of suitable subgraphs as skillfully done by Vietri in [8, 10].

Vietri's decomposition. Using the notation given in Definition 1, any generalized Petersen graph of the form $P_{8n,3}$ can be decomposed into the cycle $C_{12n} = (b_1\ a_1\ a_2\ a_3\ b_3\ b_{8n}\ b_{8n-3}\ a_{8n-3}\ a_{8n-2}\ a_{8n-1}\ b_{8n-1}\ b_{8n-4}\ b_{8n-7}\ a_{8n-7}\ \cdots\ a_{11}\ b_{11}\ b_8\ b_5\ a_5\ a_6\ a_7\ b_7\ b_4)$ together with a family of stars with 3 rays, whose endvertices belong to C_{12n}. We point out that it results $V(C_{12}) = \{a_i\ :\ 1 \leq i \leq 8n-1,\ i \not\equiv 0 \pmod 4\} \cup \{b_i\ :\ 1 \leq i \leq 8n,\ i \not\equiv 2 \pmod 4\}$. The stars completing the graph can be divided into two classes: stars of class 1, of center b_{4i-2} and endvertices $b_{4i-5}, a_{4i-2}, b_{4i+1}$ for $1 \leq i \leq 2n$, and stars of class 2, of center a_{4i} and endvertices $a_{4i-1}, b_{4i}, a_{4i+1}$ for $1 \leq i \leq 2n$.

Using the previous decomposition, we are able to prove Vietri's conjecture.

Theorem 2. *For any positive integer $n \geq 1$, $P_{8n,3}$ admits an α-labeling.*

Proof. We distinguish two cases depending on the parity of n.

Case 1: n even. We consider the Vietri's decomposition of $P_{8n,3}$ and we start labeling the vertices of C_{12n} as follows:

$$(\overset{b_1}{0}, \overset{a_1}{24n-1}, \overset{a_2}{2}, \overset{a_3}{24n-3}, \overset{b_3}{4}, \overset{b_{8n}}{24n-5}, \overset{b_{8n-3}}{6}, \overset{a_{8n-3}}{24n-7}, \ldots, \overset{b_{4n+7}}{6n-2}, \overset{b_{4n+4}}{18n+1}, \overset{b_{4n+1}}{6n},$$

$$\overset{a_{4n+1}}{18n-3}, \overset{a_{4n+2}}{6n+2}, \overset{a_{4n+3}}{18n-5}, \overset{b_{4n+3}}{6n+4}, \ldots, \overset{a_6}{12n-4}, \overset{a_7}{12n+1}, \overset{b_7}{12n-2}, \overset{b_4}{12n-1}).$$

In formal terms, we have the following labels for the vertices of C_{12}:

$$a_i = \begin{cases} 24n-1 & \text{for } i = 1, \\ 12n+6k-3 & \text{for } i = 4k+1 \text{ and } 1 \leq k \leq n, \\ 12n+6k-1 & \text{for } i = 4k+1 \text{ and } n+1 \leq k \leq 2n-1, \\ 2 & \text{for } i = 2, \\ 12n-6k+2 & \text{for } i = 4k+2 \text{ and } 1 \leq k \leq 2n-1, \\ 24n-3 & \text{for } i = 3, \\ 12n+6k-5 & \text{for } i = 4k+3 \text{ and } 1 \leq k \leq n, \\ 12n+6k-3 & \text{for } i = 4k+3 \text{ and } n+1 \leq k \leq 2n-1. \end{cases}$$

$$b_i = \begin{cases} 12n+6k-7 & \text{for } i=4k \text{ and } 1 \le k \le n, \\ 12n+6k-5 & \text{for } i=4k \text{ and } n+1 \le k \le 2n, \\ 0 & \text{for } i=1, \\ 12n-6k & \text{for } i=4k+1 \text{ and } 1 \le k \le 2n-1, \\ 4 & \text{for } i=3, \\ 12n-6k+4 & \text{for } i=4k+3 \text{ and } 1 \le k \le 2n-1. \end{cases}$$

It is easy to see that the absolute values of the differences between the labels of adjacent vertices give all the odd integers from 1 to $24n-1$.
In particular, setting $y_i = 6(4n-i)$, the labels of the endvertices of the completing stars have the following form:

$$\text{stars of class 1} \begin{cases} S_x &= \{6x, 6x+8, 6x+16\} \qquad \text{for } 0 \le x \le 2n-3, \\ S_{2n-2} &= \{12n-12, 12n-4, 4\}, \\ S_{2n-1} &= \{12n-6, 2, 10\}. \end{cases}$$

$$\text{stars of class 2} \begin{cases} S'_i &= \{y_i-1, y_i-5, y_i-9\} \qquad \text{for } 0 \le i \le n-2, \\ S'_{n-1} &= \{18n+5, 18n+1, 18n-5\}, \\ S'_i &= \{y_i-3, y_i-7, y_i-11\} \qquad \text{for } n \le i \le 2n-2, \\ S'_{2n-1} &= \{12n+3, 12n-1, 24n-3\}. \end{cases}$$

Now, we want to label the centers of the completing stars in such a way that all the even integers from 2 to $24n$ have to appear as absolute values of the differences between the labels of the centers and the related endvertices.

For any star S_x of class 1 we define its center c_x as follows:

$$c_x = \begin{cases} 24n-6x & \text{for } x \text{ even, } 0 \le x \le n-2, \\ 24n-6x-8 & \text{for } x \text{ odd, } 1 \le x \le n-3, \\ 24n-6x-10 & \text{for } x \text{ even, } n \le x \le 2n-4, \\ 24n-6x-18 & \text{for } x \text{ odd, } n-1 \le x \le 2n-3, \\ 12n+2 & \text{for } x = 2n-2, \\ 24n-2 & \text{for } x = 2n-1. \end{cases}$$

For the stars S'_i of class 2 we define the center $c'_i = 1+6i$ for any i.

From the stars S_x of class 1, setting $\lambda_x = 24n-12x$, we obtain the following $6n$ differences:

$$\begin{array}{ll} \{\lambda_x, \lambda_x-8, \lambda_x-16\} & \text{for } x \text{ even, } 0 \le x \le n-2, \\ \{\lambda_x-8, \lambda_x-16, \lambda_x-24\} & \text{for } x \text{ odd, } 1 \le x \le n-3, \\ \{\lambda_x-10, \lambda_x-18, \lambda_x-26\} & \text{for } x \text{ even, } n \le x \le 2n-4, \\ \{\lambda_x-18, \lambda_x-26, \lambda_x-34\} & \text{for } x \text{ odd, } n-1 \le x \le 2n-3, \\ \{6, 14, 12n-2\} & \text{for } x = 2n-2, \\ \{12n+4, 24n-4, 24n-12\} & \text{for } x = 2n-1. \end{array}$$

From the stars S_i' of class 2, setting $\mu_i = 6(4n-2i)$, we have the $6n$ differences:

$$\begin{array}{ll} \{\mu_i - 2, \mu_i - 6, \mu_i - 10\} & \text{for } \ 0 \le i \le n-2, \\ \{12n, 12n+6, 12n+10\} & \text{for } \ i = n-1, \\ \{\mu_i - 4, \mu_i - 8, \mu_i - 12\} & \text{for } \ n \le i \le 2n-2, \\ \{4, 8, 12n+2\} & \text{for } \ i = 2n-1. \end{array}$$

It is a simple routine to verify that the absolute values of the $12n$ differences so obtained are all the even integers from 2 to $24n$.

Call $f : V(P_{8n,3}) \to \{i \in \mathbb{N} \ : \ 0 \le i \le 24n\}$ the function defined by the above labels. Now we check that f is an injective function by writing explicitly the labels of the vertices of $P_{8n,3}$. Denoting by A and B the two bipartite sets of the cycle, we have

$$f(A) = \{2i \in \mathbb{N} \ : \ 0 \le i \le 6n-1\}$$

and

$$f(B) = \{2i+1 \in \mathbb{N} \ : \ 6n-1 \le i \le 12n-1\} \setminus \{18n-1\}.$$

Also, denoting by C and D the centers of the stars of class 1 and class 2, respectively, we have

$$\begin{aligned} f(C) = & \left\{12n + 12i \in \mathbb{N} \ : \ 0 \le i \le \frac{n}{2} - 1\right\} \\ & \cup \left\{12n + 2 + 12i \in \mathbb{N} \ : \ 0 \le i \le \frac{n}{2} - 1\right\} \\ & \cup \left\{6n + 10 + 12i \in \mathbb{N} \ : \ n \le i \le \frac{3n}{2} - 1\right\} \\ & \cup \left\{6n + 12i \in \mathbb{N} \ : \ n+1 \le i \le \frac{3n}{2}\right\} \end{aligned}$$

and

$$f(D) = \{1 + 6i \in \mathbb{N} \ : \ 0 \le i \le 2n-1\}.$$

Since $f(A), f(B), f(C)$ and $f(D)$ are disjoint sets, the function f is injective and this implies that $P_{8n,3}$ is graceful.

Finally, it is easy to see that the two bipartite sets of $P_{8n,3}$ are $A \cup D$ and $B \cup C$, and that $f(A \cup D) \subseteq \{i \in \mathbb{N} \ : \ 0 \le i \le 12n-2\}$ and $f(B \cup C) \subseteq \{i \in \mathbb{N} \ : \ 12n-1 \le i \le 24n\}$. Then $\max_{A \cup D} f < \min_{B \cup C} f$, so f is an α-labeling of $P_{8n,3}$.

Case 2: n odd. We consider again the Vietri's decomposition and we label the vertices of C_{12n} as follows:

$$(\overset{b_1}{0}, \overset{a_1}{24n}, \overset{a_2}{2}, \overset{a_3}{24n-2}, \overset{b_3}{4}, \overset{b_{8n}}{24n-4}, \overset{b_{8n-3}}{6}, \overset{a_{8n-3}}{24n-6}, \ldots, \overset{b_{4n+7}}{6n-2}, \overset{b_{4n+4}}{18n+2}, \overset{b_{4n+1}}{6n+3},$$

$$\overset{a_{4n+1}}{18n}, \overset{a_{4n+2}}{6n+5}, \overset{a_{4n+3}}{18n-2}, \overset{b_{4n+3}}{6n+7}, \ldots, \overset{a_6}{12n-1}, \overset{a_7}{12n+4}, \overset{b_7}{12n+1}, \overset{b_4}{12n+2}).$$

In formal terms, we have the following labels for the vertices of C_{12n}:

$$a_i = \begin{cases} 24n & \text{for } i = 1, \\ 12n + 6k & \text{for } i = 4k+1 \text{ and } 1 \le k \le 2n-1, \\ 2 & \text{for } i = 2, \\ 12n - 6k + 5 & \text{for } i = 4k+2 \text{ and } 1 \le k \le n, \\ 12n - 6k + 2 & \text{for } i = 4k+2 \text{ and } n+1 \le k \le 2n-1, \\ 24n - 2 & \text{for } i = 3, \\ 12n + 6k - 2 & \text{for } i = 4k+3 \text{ and } 1 \le k \le 2n-1. \end{cases}$$

$$b_i = \begin{cases} 12n + 6k - 4 & \text{for } i = 4k \text{ and } 1 \le k \le 2n, \\ 0 & \text{for } i = 1, \\ 12n - 6k + 3 & \text{for } i = 4k+1 \text{ and } 1 \le k \le n, \\ 12n - 6k & \text{for } i = 4k+1 \text{ and } n+1 \le k \le 2n-1, \\ 4 & \text{for } i = 3, \\ 12n - 6k + 7 & \text{for } i = 4k+3 \text{ and } 1 \le k \le n, \\ 12n - 6k + 4 & \text{for } i = 4k+3 \text{ and } n+1 \le k \le 2n-1. \end{cases}$$

One can easily see that the absolute values of the differences between the labels of adjacent vertices give all the odd integers from 1 to $12n-1$ together with all the even integers from $12n+2$ to $24n$. If $n = 1$, then the labels of the endvertices of the remaining stars are $\{0, 4, 11\}$, $\{2, 9, 13\}$, $\{24, 20, 16\}$ and $\{22, 18, 14\}$. One can directly check that if we label the centers of these stars with $21, 15, 1, 10$ respectively, we obtain an α-labeling of $P_{8,3}$.

Let now $n \ge 3$. Setting again $y_i = 6(4n - i)$, it is easy to see that the labels of the endvertices of the remaining stars have the following form:

$$\text{stars of class 1} \begin{cases} S_x & = \{6x, 6x+8, 6x+16\} & \text{for } 0 \le x \le n-3, \\ S_{n-2} & = \{6n-12, 6n-4, 6n+7\}, \\ S_{n-1} & = \{6n-6, 6n+5, 6n+13\}, \\ S_x & = \{6x+3, 6x+11, 6x+19\} & \text{for } n \le x \le 2n-3, \\ S_{2n-2} & = \{12n-9, 12n-1, 4\}, \\ S_{2n-1} & = \{2, 10, 12n-3\}. \end{cases}$$

$$\text{stars of class 2} \begin{cases} S'_i & = \{y_i, y_i - 4, y_i - 8\} & \text{for } 0 \le i \le 2n-2, \\ S'_{2n-1} & = \{24n-2, 12n+2, 12n+6\}. \end{cases}$$

Now we are going to label the centers of these stars in such a way that all the even integers from 2 to $12n$ and all the odd integers from $12n+1$ to $24n-1$ appear as absolute values of the differences between the labels of the centers and the related endvertices. If $n=3$, then we label the centers of the stars of class 1 and class 2, respectively, as follows:

$c_0=69$ $c_1=53$ $c_2=55$ $c_3=39$ $c_4=41$ $c_5=67$
$c'_0=1$ $c'_1=7$ $c'_2=11$ $c'_3=24$ $c'_4=28$ $c'_5=34$

A direct calculation shows that we obtain an α-labeling of $P_{24,3}$.

From now on let $n\geq 5$. For any star S_x of class 1 we define its center c_x as follows:

$$c_x=\begin{cases} 24n-6x-3 & \text{for } x \text{ even},\ 0\leq x\leq n-3,\\ 24n-6x-11 & \text{for } x \text{ odd},\ 1\leq x\leq n-4,\\ 18n-1 & \text{for } x=n-2,\\ 18n+1 & \text{for } x=n-1,\\ 24n-6x-13 & \text{for } x \text{ odd},\ n\leq x\leq 2n-5,\\ 24n-6x-5 & \text{for } x \text{ even},\ n+1\leq x\leq 2n-4,\\ 12n+3 & \text{for } x=2n-3,\\ 12n+5 & \text{for } x=2n-2,\\ 24n-5 & \text{for } x=2n-1. \end{cases}$$

For the stars S'_i of class 2 we define the centers c'_i's as follows:

$$c'_i=\begin{cases} 1+6i & \text{for } 0\leq i\leq n-2,\\ 6n-7 & \text{for } i=n-1,\\ 6(i+1) & \text{for } n\leq i\leq 2n-3,\\ 12n-8 & \text{for } i=2n-2,\\ 12n-2 & \text{for } i=2n-1. \end{cases}$$

From the stars S_x of class 1, setting $\lambda_x=24n-12x$, we obtain the following $6n$ differences:

$$\begin{array}{ll} \{\lambda_x-3,\lambda_x-11,\lambda_x-19\} & \text{for } x \text{ even},\ 0\leq x\leq n-3,\\ \{\lambda_x-11,\lambda_x-19,\lambda_x-27\} & \text{for } x \text{ odd},\ 1\leq x\leq n-4,\\ \{12n+11,12n+3,12n-8\} & \text{for } x=n-2,\\ \{12n+7,12n-4,12n-12\} & \text{for } x=n-1,\\ \{\lambda_x-16,\lambda_x-24,\lambda_x-32\} & \text{for } x \text{ odd},\ n\leq x\leq 2n-5,\\ \{\lambda_x-8,\lambda_x-16,\lambda_x-24\} & \text{for } x \text{ even},\ n+1\leq x\leq 2n-4,\\ \{18,10,2\} & \text{for } x=2n-3,\\ \{14,6,12n+1\} & \text{for } x=2n-2,\\ \{24n-7,24n-15,12n-2\} & \text{for } x=2n-1. \end{array}$$

From the stars S'_i of class 2, setting again $\mu_i=6(4n-2i)$, we have the $6n$ differences:

$$\begin{array}{ll}
\{\mu_i - 1, \mu_i - 5, \mu_i - 9\} & \text{for } \ 0 \leq i \leq n-2, \\
\{12n+13, 12n+9, 12n+5\} & \text{for } \ i = n-1, \\
\{\mu_i - 6, \mu_i - 10, \mu_i - 14\} & \text{for } \ n \leq i \leq 2n-3, \\
\{20, 16, 12\} & \text{for } \ i = 2n-2, \\
\{12n, 4, 8\} & \text{for } \ i = 2n-1.
\end{array}$$

It is not hard to check that the absolute values of the $12n$ differences so obtained are all the even integers from 2 to $12n$ together with all the odd integers from $12n+1$ to $24n-1$.

Let $f : V(P_{8n,3}) \to \{i \in \mathbb{N} \ : \ 0 \leq i \leq 24n\}$ be the function defined by the above labels. We have to check that f is an injective function, so we list all the labels of the vertices. Denoting by A and B the two bipartite sets of C_{12n}, we have

$$f(A) = \{2i \in \mathbb{N} \ : \ 0 \leq i \leq 3n-1\} \cup \{2i+1 \in \mathbb{N} \ : \ 3n+1 \leq i \leq 6n\}$$

and

$$f(B) = \{2i \in \mathbb{N} \ : \ 6n+1 \leq i \leq 12n\}.$$

Also, denoting by C and D the centers of the stars of class 1 and class 2, respectively, we have:

$$\begin{aligned}
f(C) = & \left\{5 + 12i \in \mathbb{N} \ : \ n+1 \leq i \leq \frac{3n-3}{2}\right\} \\
& \cup \left\{7 + 12i \in \mathbb{N} \ : \ n+1 \leq i \leq \frac{3n-3}{2}\right\} \\
& \cup \left\{6n + 1 + 12i \in \mathbb{N} \ : \ n+1 \leq i \leq \frac{3n-3}{2}\right\} \\
& \cup \left\{6n + 3 + 12i \in \mathbb{N} \ : \ n+1 \leq i \leq \frac{3n-1}{2}\right\} \\
& \cup \ \{18n-1, 18n+1, 12n+3, 12n+5, 24n-5\}
\end{aligned}$$

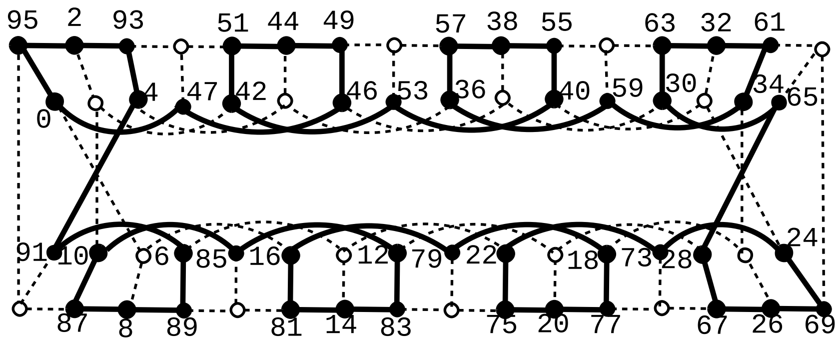
and

$$\begin{aligned}
f(D = & \{1 + 6i \in \mathbb{N} \ : \ 0 \leq i \leq n-2\} \\
& \cup \{6i \in \mathbb{N} \ : \ n+1 \leq i \leq 2n-2\} \\
& \cup \{6n-7, 12n-8, 12n-2\}.
\end{aligned}$$

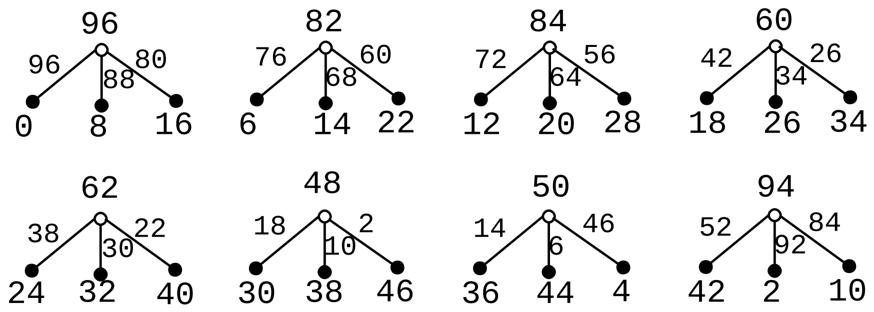
Since $f(A), f(B), f(C)$ and $f(D)$ are disjoint sets, f is injective and so we have proved that $P_{8n,3}$ is graceful.

To conclude, the two bipartite sets of $P_{8n,3}$ are $A \cup D$ and $B \cup C$, and $f(A \cup D) \subseteq \{i \in \mathbb{N} \ : \ 0 \leq i \leq 12n+1\}$ and $f(B \cup C) \subseteq \{i \in \mathbb{N} \ : \ 12n+2 \leq i \leq 24n\}$. Hence $\max_{A \cup D} f < \min_{B \cup C} f$, so f is an α-labeling of $P_{8n,3}$. ■

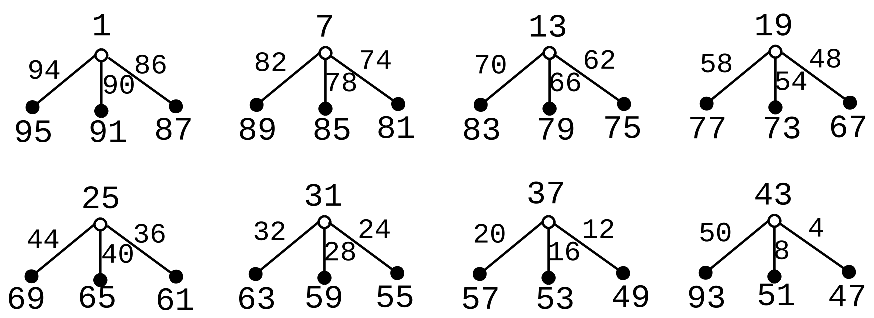
Example 3. Here we show the α-labeling of $P_{32,3}$ obtained through the construction given in the proof of Theorem 2. In the next figure below we have the labels of the vertices of the cycle C_{48}:



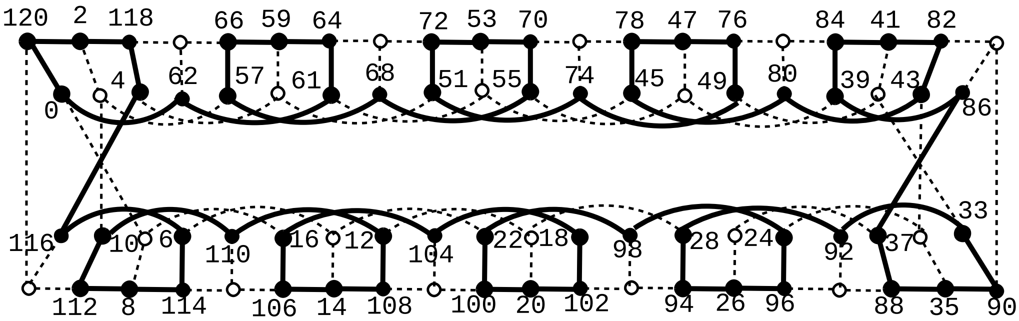
Now we consider the labels of the completing stars whose differences are all the even integers from 2 to 96. Stars of class 1:



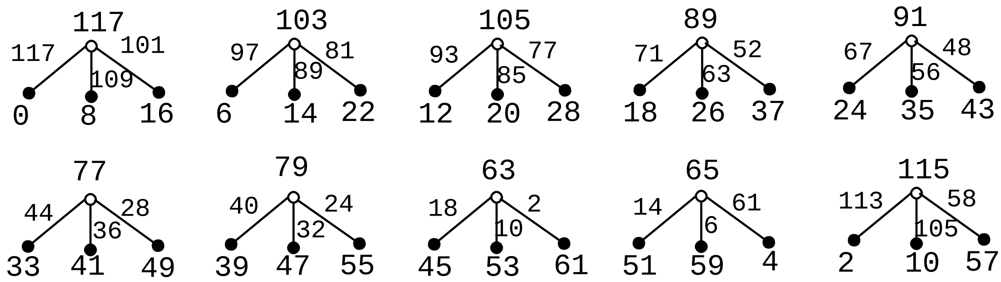
and stars of class 2:



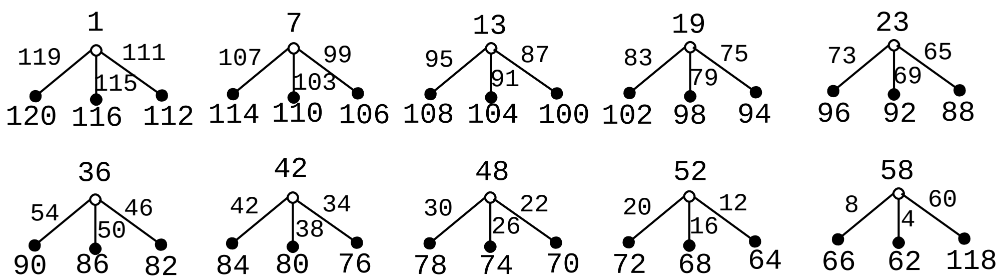
Example 4. We show the α-labeling of $P_{40,3}$ obtained through the construction given in the proof of Theorem 2. In the figure below we have the labels of the vertices of the cycle C_{60}.



The cycle so labeled gives as differences all the odd integers from 1 to 59 together with all the even integers from 62 to 120. Now we consider the completing stars. Stars of class 1:



and stars of class 2:



The differences appearing in the stars are exactly all the even integers from 2 to 60 together with all the odd integers from 61 to 119. So all the integers from 1 to 120 appear exactly once as a difference of adjacent vertices of $P_{40,3}$.

As an immediate consequence of Theorems 1 and 2, we have

Theorem 5. *There exists a cyclic $P_{8n,3}$-decomposition of K_{24nt+1} for any positive integer t.*

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