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PACKING PARAMETERS IN GRAPHS

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Abstract

In a graph $G = (V, E)$, a non-empty set $S \subseteq V$ is said to be an open packing set if no two vertices of S have a common neighbour in G. An open packing set which is not a proper subset of any open packing set is called a maximal open packing set. The minimum and maximum cardinalities of a maximal open packing set are respectively called the lower open packing number and the open packing number and are denoted by ρ_L^o and ρ^o. In this paper, we present some bounds on these parameters.

Keywords: packing number, open packing number.

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1. Introduction

By a graph $G = (V, E)$, we mean a connected, finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to Chartand and Lesniak [2].

The *open neighbourhood* of a vertex v is $N_G(v)=\{u \in V\colon\ uv \in E\}$, while its *closed neighbourhood* is $N_G[v]=N_G(v) \cup \{v\}$. For a set $S \subseteq V$, $N_G(S) = \bigcup_{v\in S} N_G(v)$ and $N_G[S]=N_G(S) \cup S$. If the graph is clear from the context, then we omit the subscript on these neighbourhood names. For a set of $S \subseteq V$, the subgraph induced by S is denoted by $\langle S\rangle_G$ or simply $\langle S\rangle$. A *clique* in a graph G is a complete subgraph of G. The maximum order of a clique in G is called the *clique number* and is denoted by $\omega(G)$ and a clique of order $\omega(G)$ is called a *maximum clique*. If G is a graph, then G^+ is the graph obtained from G by attaching a pendant edge at every vertex of G. A subset D of vertices is said to be a *dominating set* of G if every vertex x in V either belongs to D or is adjacent to a vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set D of a graph G is called a *total dominating set* if $\langle D \rangle$ has no isolates and the minimum cardinality of a total dominating set is the *total domination number* of G, denoted by $\gamma_t(G)$.

A set S of vertices in a graph G is called an *independent set* of G if no two vertices of S are adjacent in G. The *independence number* $\beta_0(G)$ of a graph G is defined to be the maximum cardinality of an independent set of G and the minimum cardinality of a maximal independent set of G is called the *independence domination number* of G, denoted by $i(G)$. A 2-*packing* of a graph G is a set of vertices whose closed neighbourhoods are pairwise disjoint in G. Equivalently, a 2-*packing* of a graph G is a set of vertices whose elements are pairwise at at least 3 apart in G. The *lower packing number* of G, denoted $\rho_L(G)$, is the minimum cardinality of a maximal 2-packing of G while the distance *packing number* of G, denoted $\rho(G)$, is the maximum cardinality of a maximal 2-packing of G. A set S of vertices of G is an *open packing* of G if the open neighbourhoods of the vertices of S are pairwise disjoint in G. The *lower open packing number* of G, denoted $\rho_L^o(G)$, is the minimum cardinality of a maximal open packing of G while the *open packing number* of G, denoted $\rho^o(G)$, is the maximum cardinality among all open packings of G. An open packing set of cardinality ρ_L^o and ρ^o are respectively called the ρ_L^o-*set* and ρ^o-*set* of G. The packing number of a graph has been studied in [1, 3, 6, 7] and the open packing number of a graph has been studied in [5]. This paper further studies these packing parameters.

2. Graphs of Diameter Two

Obviously, if G is a graph of diameter 2, then $\rho(G) = \rho_L(G) = 1$. The values of ρ^o and ρ_L^o of a graph of diameter 2 are determined in this section.

Lemma 2.1. *Let G be a graph of order at least 3. Then $\rho^o(G) = 1$ if and only if $diam(G) \leq 2$ and every edge of G lies on a triangle.*

Proof. Suppose $\rho^o(G) = 1$. Now, if there exist two vertices x and y with $d(x, y) \geq 3$, then $\{x, y\}$ is an open packing set so that $\rho^o(G) \geq 2$, which is a contradiction. Hence distance between any two vertices in G is at most 2 so that $diam(G) \leq 2$. Now, let $e = uv$ be any edge of G. Then the vertices u and v have a common neighbour, say w, for otherwise the set consisting of the vertices u and v would be an open packing set and consequently $\rho^o(G) \geq 2$, contradicting the assumption, and thus (u, v, w, u) is a triangle containing the edge e.

Conversely, suppose $diam(G) \leq 2$ and every edge of G lies on a triangle. If $diam(G) = 1$, then G is complete, and thus $\rho^o(G) = 1$. Suppose $diam(G) = 2$.

Let u and v be any two vertices of G. Now, if u and v are adjacent, then the edge uv belongs to some triangle in G, hence u and v have a common neighbour. Even in the case that u and v are not adjacent, they have a common neighbour as $diam(G) = 2$. ■

Corollary 2.2. *Let G be a graph with $diam(G) = 2$. Then $\rho^o(G) = \rho(G)$ if and only if every edge of G lies on a triangle.*

Proof. If every edge of G lies on a triangle, it follows from Lemma 2.1, that $\rho^o(G) = 1$. Also, it is obvious that $\rho(G) = 1$ as $diam(G) = 2$. ■

Lemma 2.3. *Let G be a graph of order $n \geq 3$. Then $\rho^o_L(G) = 1$ if and only if there exists a vertex v of eccentricity at most two such that $\langle N(v)\rangle$ has no isolates.*

Proof. Suppose $\rho^o_L(G) = 1$. Let $\{v\}$ be a maximal open packing set of G. If deg$v = n - 1$, then $e(v) = 1$. Suppose deg$v \leq n - 2$. Then every non-neighbour of v must be adjacent to a neighbour of v. That is, every vertex of G other than v is at a distance of at most two from v which in turn implies that $e(v) = 2$. Further, if $\langle N(v)\rangle$ has an isolated vertex, say u, then $\{u, v\}$ is an open packing set, which is a contradiction to the maximality of $\{v\}$. Thus $\langle N(v)\rangle$ has no isolates. Conversely, suppose there is a vertex v of eccentricity at most two such that $\langle N(v)\rangle$ has no isolates. Since $e(v) \leq 2$, the non-neighbours of v (if any) must have a neighbour in $N(v)$. Also, since $\langle N(v)\rangle$ has no isolates v and each vertex in $N(v)$ have a common neighbour. Thus $\{v\}$ forms a maximal open packing set of G and so $\rho^o_L(G) = 1$. ■

With the aid of the above results one can determine the values of ρ^o and ρ^o_L for a graph of diameter 2 as follows.

Theorem 2.4. *If G is a graph of diameter 2, then $\rho^o_L(G) \leq \rho^o(G) \leq 2$. Further,*

(i) *$\rho^o(G) = 1$ if and only if every edge of G lies in a triangle.*

(ii) *$\rho^o_L(G) = 2$ if and only if $\langle N(v)\rangle$ has an isolated vertex for every $v \in V(G)$.*

Proof. Since $diam(G) = 2$, any two non-adjacent vertices in G have a common neighbour which in turn implies that for any open packing set S of G, the induced subgraph $\langle S\rangle$ is complete. Further, no two vertices of the open packing set S have a common neighbour it follows that each component of $\langle S\rangle$ is either K_1 or K_2 and consequently $\rho^o(G) \leq 2$. The inequality $\rho^o_L(G) \leq \rho^o(G) \leq 2$ immediately follows from the definitions of $\rho^o_L(G)$ and $\rho^o(G)$. Now, (i) is a direct consequence of Lemma 2.1. Further, suppose $\rho^o_L(G) = 2$ and there is a vertex $v \in V(G)$ such that $\langle N(v)\rangle$ has no isolates. Since $diam(G) = 2$, it follows that $e(v) = 1$ or 2. Now, by Lemma 2.3, we have $\rho^o_L(G) = 1$, which is a contradiction and hence the result follows. Conversely, assume that for each $v \in V(G)$, $\langle N(v)\rangle$ has an isolate.

Since no vertex of G has eccentricity three or more, it follows by Lemma 2.3, we have $\rho^o_L(G) \neq 1$ and consequently $\rho^o_L(G) = 2$. ■

Corollary 2.5. *If G is a connected graph on n vertices with $\Delta(G) = n-1$, then $\rho^o(G) \leq 2$. Further, $\rho^o(G) = 2$ if and only if $\delta(G) = 1$ and $\rho^o_L(G) = 2$ if and only if G is a star.*

Proof. Let v be a vertex with $deg\ v = n-1$. Obviously, $diam(G) \leq 2$ and hence by Theorem 2.5, we have $\rho^o(G) \leq 2$. Now, suppose $\rho^o(G) = 2$. Then by Lemma 2.1, there is an edge e not lying on any triangle in G. As $\Delta(G) = n-1$, the edge e must be incident at v, say $e = uv$. Certainly $deg\ u = 1$ so that $\delta(G) = 1$. Conversely, suppose $\delta(G) = 1$. Then any pendant vertex along with v forms an open packing set and so $\rho^o(G) = 2$.

Suppose $\rho^o_L(G) = 2$. We need to prove that no two neighbours of v are adjacent. If not, let u and w be two adjacent neighbours of v. Then $\{u\}$ and $\{w\}$ are maximal open packing sets of G so that $\rho^o_L(G) = 1$, which is a contradiction. Thus G is a star. Also, obviously for a star, we have $\rho^o_L(G) = 2$. ■

3. Bounds

In this section we obtain some bounds for the open packing number of a graph in terms of order, packing number, total domination number and the clique number of a graph.

Theorem 3.1. *If G is a connected graph of order $n \geq 2$, then $\rho^o(G) \leq \frac{n}{\delta(G)}$.*

Proof. Let S be any open packing set of G. Since every vertex in G has at least δ neighbours and no vertex in S is adjacent to two or more vertices in S, it follows that every vertex in S must be adjacent to at least $(\delta - 1)$ vertices in $V - S$. Further, no two vertices in S can have a common neighbour in G, we have $|V - S| \geq |S|\,(\delta - 1)$ and hence the result follows as S is arbitrary. ■

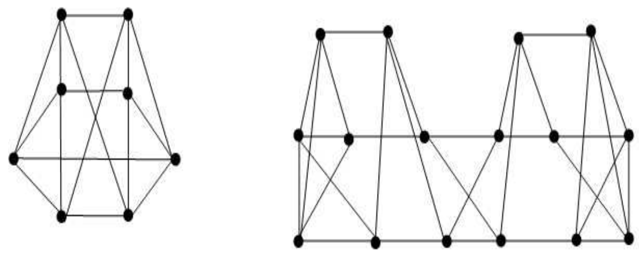
Theorem 3.2. *If G is a graph with $\rho^o(G) = \frac{n}{\delta(G)}$, then $\frac{n}{\delta(G)}$ is an even integer.*

Proof. Suppose $\rho^o(G) = \frac{n}{\delta(G)}$. Let S be a ρ^o-set of G. Now, we claim that $\langle S \rangle$ has no isolates. Suppose $\langle S \rangle$ has r of isolates. Then $|V - S| \geq r\delta + (\rho^o - r)(\delta - 1)$ and hence by our assumption, we get $r \leq 0$. Certainly, $\langle S \rangle = \bigcup K_2$ as S is a ρ^o-set of G and hence the result follows. ■

In the following theorem, we characterize the r-regular graphs attaining the above bound. For this purpose, given positive integers r and n with $\frac{n}{r}$ an even integer and $n \geq 3$, we define $\tau_{r,n}$ to be the family of r-regular graphs of order n which are constructed as follows. Consider an $(r-1)$-regular graph H on $n - \frac{n}{r}$ vertices.

Also, consider $\frac{n}{2r}$ copies of K_2, say u_1v_1,u_2v_2,u_3v_3,…,u_kv_k, where $k = \frac{n}{2r}$. Now, join each of the vertices of $U = \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k\}$ to any $r-1$ vertices of H such that $N(x) \cap N(y) = \emptyset$ for any two distinct vertices x and y of U.

Note that, as at least one of the integers $r-1$ and $n-\frac{n}{r}$ is even, the existence of an $(r-1)$-regular graph H on $n-\frac{n}{r}$ vertices is guaranteed. Therefore the family $\tau_{r,n}$ is always non-empty for any pair r and n of positive integers with $\frac{n}{r}$ an even integer and $n \geq 3$. For example, a graph in $\tau_{4,8}$ and a graph from $\tau_{4,16}$ are given in the following Figure 3.1.



(a) (b)

Figure 3.1. (a) A graph in $\tau_{4,8}$.
(b) A graph in $\tau_{4,16}$.

Theorem 3.3. *Let G be an r-regular graph on n vertices such that $\frac{n}{r}$ is an integer. Then $\rho^o(G) = \frac{n}{r}$ if and only if $G \in \tau_{r,n}$.*

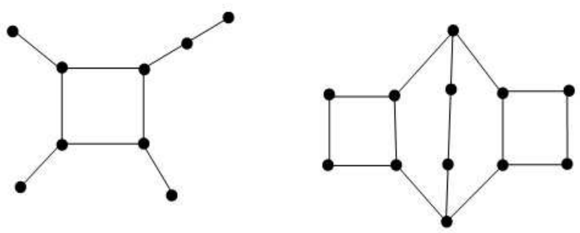
Proof. Suppose $\rho^o(G) = \frac{n}{r}$. By Theorem 3.2, $\frac{n}{r}$ is an even integer. Now, let S be any ρ^o-set of G. If there exists an isolated vertex x in $\langle S\rangle$, then all the neighbours of x will be in $V-S$. Further, every vertex in S has at most one neighbour in S so that $|V-S| \geq r+(r-1)(\rho^o-1)$. Hence $\rho^o = |S| \leq n-r-(r-1)(\rho^o-1)$ which implies that $\rho^o \leq \frac{n-1}{r}$, producing a contradiction. Thus $\langle S\rangle$ has no isolates and so $\langle S\rangle$=$(\frac{n}{2r})K_2$ because each component of $\langle S\rangle$ is either K_1 or K_2. This implies that the $r-1$ neighbours of each vertex in S are in $V-S$. Further, since $N(u) \cap N(v) = \emptyset$ for every pair of distinct vertices u and v of S, it follows that $N(S) \cap (V-S) = V-S$ as $|V-S| = n-\frac{n}{r}$ and therefore every vertex in $V-S$ has exactly one neighbour in S as S is an open packing set of G. Thus $\langle V-S\rangle$ is an $(r-1)$-regular graph on $n-\frac{n}{r}$ vertices. Hence $G \in \tau_{r,n}$.
Conversely, if $G \in \tau_{r,n}$, then the vertices lying on the $\frac{n}{2r}$ copies of K_2 of G form a maximal open packing set of G so that $\rho^o(G) \geq \frac{n}{r}$. Now, it follows from Theorem 3.1 that $\rho^o(G) = \frac{n}{r}$. ■

Now, we present an upper bound for the open packing number ρ^o in terms of the total domination number γ_t.

Theorem 3.4. *Let G be a graph of order n with $\delta(G) \geq 2$. Then $\rho^o(G) \leq n-\gamma_t(G)$, where $\gamma_t(G)$ is the total domination number of G.*

Proof. Let S be any open packing set of G. Now, we claim that $V - S$ is dominating set of G. Since each component of $\langle S \rangle$ is either K_1 or K_2, it follows that every vertex $v \in S$ must be adjacent to at least one vertex in $V - S$ as $\delta(G) \geq 2$. Further, every vertex in $V - S$ is adjacent to at most one vertex in S and so $\langle V - S \rangle$ has no isolates. Hence $V - S$ is a total dominating set of G so that $\rho^o(G) \leq n - \gamma_t(G)$. ∎

The bound for ρ^o provided in Theorem 3.4 is not true for graphs G with $\delta(G) = 1$. For example, $\delta(G) = 1$ for the graph G of Figure 3.2.(a), where $\gamma_t(G) = 5$ and $\rho^o(G) = 5 > n - \gamma_t(G)$. Also, the bound given in Theorem 3.4 is sharp. For example, the graph G of Figure 3.2.(b) is of minimum degree 2 with $\gamma_t(G) = 6$ and $\rho^o(G) = 6 = n - \gamma_t(G)$.



(a) (b)

Figure 3.2. (a) A graph G with $\delta(G) = 1$ and $\rho^o(G) > n - \gamma_t(G)$.
(b) A graph G with $\delta(G) = 2$ and $\rho^o(G) = n - \gamma_t(G)$.

The following theorem provides an upper bound for the open packing number ρ^o in terms of the clique number ω of a graph.

Theorem 3.5. *For any connected graph G of order $n \geq 3$, $\rho^o(G) \leq n - \omega(G) + 1$ with equality if and only if G is either K_n or a graph obtained from K_{n-1} by adding a vertex and joining it to exactly one vertex of K_{n-1}.*

Proof. As an open packing set of G contains at most one vertex of a clique it follows that $\rho^o(G) \leq n - \omega(G) + 1$. Further, suppose $\rho^o(G) = n - \omega(G) + 1$. Let H be a maximum clique and S be a ρ^o-set of G. Certainly, S contains exactly one vertex of H, say u, and of course contains all the vertices of G lying outside H. Hence the vertex u has at most one neighbour in $V(G) - V(H)$ and every other vertex of H has no neighbour in $V(G) - V(H)$. Thus G is either complete or a graph obtained from a complete graph by adding a vertex and joining it to exactly one vertex of the complete graph. ∎

4. Realization Theorems

In this section we present some relationship among the packing parameters along with realization theorems. It has been proved in [4] that the inequalities $\rho(G) \leq$

$\rho^o(G) \leq 2\rho(G)$ hold true for any graph G. Let us recall the proof for the sake of completeness. It immediately follow from the definitions of 2-packing and open packing that $\rho(G) \leq \rho^o(G)$. Further, as each component of $\langle S \rangle$, where S is an open packing set, is either K_1 or K_2, the set obtained by choosing exactly one vertex from each component of $\langle S \rangle$ is a 2-packing set of G so that $\rho^o(G) \leq 2\rho(G)$. In the following, we prove that ρ^o can assume any value between ρ and 2ρ.

Theorem 4.1. *For any two positive integers a and b with $a \leq b \leq 2a$, there exists a graph G for which $\rho(G) = a$ and $\rho^o(G) = b$.*

Proof. Suppose a and b are two positive integers with $a \leq b \leq 2a$. We construct a graph G with $\rho(G) = a$ and $\rho^o(G) = b$ as follows.

Case 1. $a = b$. If $a = 1$, let G be a complete graph. Assume that $a \geq 2$. Now, let $G = K_a^+$. Let S be the set of all pendant vertices of G. Then, obviously S is a 2-packing set of maximum cardinality so that $\rho(G) = |S| = a$. Also, S is an open packing set of G and so $\rho^o(G) \geq |S| = a$. Now, suppose D is any maximal open packing set of G. Then D can have at most one non-pendant vertex of G. Also, if D contains a non-pendant vertex v of G, then $D = \{v, v'\}$, where v' is the pendant vertex of G, forms a maximal open packing set of G. Clearly, $|D| \leq a$ and consequently $\rho^o(G) \leq a$. Thus $\rho^o(G) = a$.

Case 2. $b > a$. Let $b = a + r$, where $1 \leq r \leq a$. We now construct a graph G with $\rho(G) = a$ and $\rho^o(G) = b$ as follows.

Draw a path $P = (v_{11}, v_{12}, v_{21}, v_{22}, \ldots, v_{a1}, v_{a2})$ on $2a$ vertices. Now introduce a vertices $w_1, w_2, \ldots, w_a$ and join w_i to the vertices v_{i1} and v_{i2} for all $i = 1, 2, \ldots, a$. Moreover, at each of the vertices $w_1, w_2, \ldots, w_r$ attach exactly one pendant edge, say $w_i w_i'$, $1 \leq i \leq r$ (by attaching a pendant edge $w_i w_i'$ at w_i, we mean that a new vertex w_i' is introduced and is joined with w_i by an edge). For $a = 3$ and $b = 5$ the graph is illustrated in Figure 4.1.

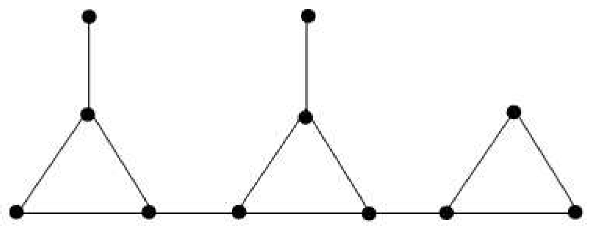


Figure 4.1. A graph G with $\rho(G) = 3$ and $\rho^o(G) = 5$.

We need to prove that $\rho(G) = a$ and $\rho^o(G) = b$. Obviously, $S_1 = \{w_i : 1 \leq i \leq a\}$ is a 2-packing set of G so that $\rho(G) \geq a$. Also, the set $\{w_i, w_j' : 1 \leq i \leq a, 1 \leq j \leq r\}$ forms a maximal open packing set of G. Certainly, $\rho^o(G) \geq a + r = b$.

Further, a maximum 2-packing set of G consists of exactly one vertex from each triangle and hence $\rho(G) \leq a$ which implies that $\rho(G) = a$. Now, let D be any maximal open packing set of G. If D contains one of the vertices on P, say v_{11}, then D consists of the vertices $w_i, w_j^{'}$ where $2 \leq i \leq a$, $2 \leq j \leq r$ so that $|D| \leq a+r-1$. Further, if D contains no vertex on P, then D consists of all the vertices w_i and its neighbours $w_j^{'}$, where $1 \leq i \leq a$, $1 \leq j \leq r$ so that $|D| \leq a+r$. Thus $\rho^o(G) \leq a + r = b$. ∎

Obviously, $\rho_L^o(G) \leq \rho^o(G)$ for any graph G. Further, the difference between these parameters is arbitrarily large. For example, for the graph G obtained from a complete graph K_r where $r \geq 3$, by attaching exactly one pendant edge of any $r-1$ vertices of K_r, we have $\rho_L^o(G) = 1$ and $\rho^o(G) = r-1$ so that $\rho^o(G) - \rho_L^o(G) = r-2$. In fact these parameters can assume arbitrary values as shown below.

Theorem 4.2. *For any two positive integers a and b with $a \leq b$, there exists a graph G such that $\rho_L^o(G) = a$ and $\rho^o(G) = b$.*

Proof. Suppose that a and b are two positive integers with $a \leq b$. We construct the required graph G in the following cases.

Case 1. $a+1 \leq b \leq 2a$. We construct a graph G with $\rho_L^o(G) = a$ and $\rho^o(G) = b$ as follows. Consider a complete graph K_b on b vertices with $V(K_b) = \{v_1, v_2, \ldots, v_b\}$. Attach at each vertex v_i of K_b, a pendant edge $v_iv_i^{'}$, where $1 \leq i \leq b$. Further, attach a triangle at each of the vertices $v_1^{'}, v_2^{'}, \ldots, v_{a-1}^{'}$, say $C_i = (v_i^{'}, w_{i1}, w_{i2}, v_i^{'})$. Let G be the resultant graph. The graph G when $a = 4$ and $b = 8$ is given in Figure 4.2.

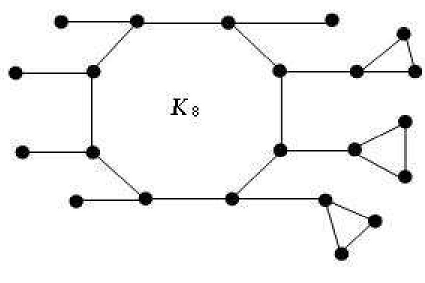


Figure 4.2. A graph G with $\rho_L^o(G) = 4$ and $\rho^o(G) = 8$.

We need to show that $\rho_L^o(G) = a$ and $\rho^o(G) = b$. Obviously, the set $S_1 = \{w_{i1} : 2 \leq i \leq a-1\} \cup \{v_1, v_1^{'}\}$ is a maximal open packing set of G so that $\rho_L^o(G) \leq |S_1| = a$. Now, it is not difficult to see that any maximal open packing set of G contains at least a vertices so that $\rho_L^o(G) \geq a$. Further, $S_2 = \{w_{i1} : 1 \leq i \leq a-1\} \cup \{v_j^{'} : a \leq j \leq b\}$ is a maximal open packing set so that $\rho^o(G) \geq |S_2| = b$. Now, let D be a maximal open packing set of G. Then at most one vertex on K_b will be in D. If D contains one of the vertices v_i, where $1 \leq i \leq a-1$, say v_1, then D consists of the vertex $v_1^{'}$ together with the vertex w_{i1} or w_{i2} for all i, where $2 \leq i \leq a-1$ so that $|D| = a$. On the other hand, if D contains one of the vertices v_j, where $a \leq j \leq b$, say v_a, then D consists of the vertex $v_a^{'}$ along the vertex w_{i1} or w_{i2}, for all $i = 1, 2, \ldots, a-1$ so that $|D| \leq a+1$. Further, if D contains no vertex of K_b, then D consists of all the pendant vertices and exactly one vertex from each triangle C_i, where $1 \leq i \leq a-1$, so that $|D| \leq b$. Thus $\rho^o(G) \leq b$.

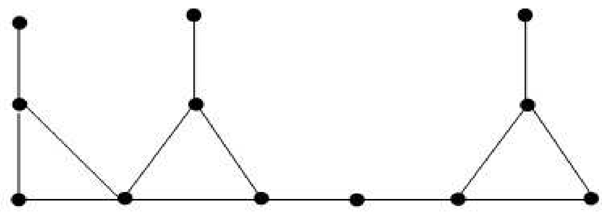


Figure 4.3. A graph G with $\rho_L^o(G) = 2$ and $\rho^o(G) = 5$.

Case 2. $a = b$. In this case, a required graph G is constructed as follows. If $a = 1$, let G be a complete graph on $n \geq 3$ vertices. Assume that $a \geq 2$. Now, consider a complete graph K_a on a vertices. Attach a pendant edge at each of any $a-1$ vertices of K_a and then attach a triangle at each of the newly introduced pendant vertices. Moreover, attach a triangle in the remaining vertex of K_a, where no pendant edge is attached. Let G be the resultant graph. For $a = b = 6$ the graph G is illustrated in Figure 4.4. By a similar argument as given in Case 1, one can prove that $\rho_L^o(G) = \rho^o(G) = a$.

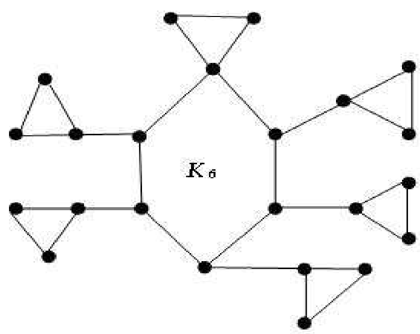


Figure 4.4. A graph G with $\rho_L^o(G) = \rho^o(G) = 6$.

Case 3. $b > 2a$. Let $b = 2a + r$, where $r \geq 1$. We now construct a graph G with $\rho_L^o(G) = a$ and $\rho^o(G) = b$ as follows. Consider a path $P = \{v_{11}, v_{12}, u_1, v_{21}, v_{22}, u_2, \ldots, v_{(a-1)1}, v_{(a-1)2}, u_{a-1}, v_{a1}, v_{a2}\}$ on $3a-1$ vertices. Now, introduce a vertices $w_1, w_2, \ldots, w_a$ and join each w_i, where $1 \leq i \leq a$ to the vertices v_{i1} and v_{i2}. Also, attach exactly one pendant edge at each w_i, where $1 \leq i \leq a$ and let the corresponding pendant vertices be $w_i^{'}$. Moreover, attach r number of triangles $C_1, C_2, \ldots, C_r$ at the vertex v_{11}, say $C_i = (v_{11}, x_{i1}, x_{i2}, v_{11})$, where $1 \leq i \leq r$. Finally, attach exactly one pendant edge at each of the vertices x_{i1}, where $1 \leq i \leq r$ and let $x_{i1}^{'}$ be the corresponding pendant vertices. For $a = 2$ and $b = 5$ the graph is illustrated in Figure 4.3.

Obviously, the sets $S_1 = \{w_i, w_i^{'} : 1 \leq i \leq a\} \cup \{x_{j1}^{'} : 1 \leq j \leq r\}$ and $S_2 = \{v_{i1} : 1 \leq i \leq a\}$ are maximal open packing sets of G so that $\rho_L^o(G) \leq a$ and $\rho^o(G) \geq b$. Now, let D be any maximal open packing set of G. For each i, where $2 \leq i \leq a$, let $H_i = \left\langle \{v_{i1}, v_{i2}, w_i, w_i^{'}\} \right\rangle$ and let $H_1 = \left\langle \{v_{11}, v_{12}, w_1, w_1^{'}\} \cup \{x_{j1}, x_{j2}, x_{j1}^{'} : 1 \leq i \leq r\} \right\rangle$. Then D must contain at least one vertex from each H_i, where $1 \leq i \leq a$. Also, D can have at most two vertices from each H_i, where $2 \leq i \leq a$, and $r + 2$ vertices from H_1. Further, when D has the vertex w_i, it can not have the vertex u_i. These observations together prove that $a \leq |D| \leq b$ which yields $\rho_L^o(G) = a$ and $\rho^o(G) = b$. ■

5. CONCLUSION AND SCOPE

Theory of domination is an important as well as fastest growing area in Graph theory. The bibliography in domination maintained by Haynes *et al.* [4] currently has over 1200 entries. The notion of packing is closely related to domination and consequently the study of packing parameters is of a great importance. The study has been already initiated in [1, 3, 5, 6, 7] and this paper further extends this study. More specifically, we determine the values of open packing parameters ρ^o and ρ_L^o for the graphs of diameter two. Also, several bounds for these parameters in terms of order, degree, total domination number and clique number have been obtained. Moreover, some relationships among packing parameters along with realization theorems are presented. Even if this paper is a little extension of the study of the packing parameters, there is a wide scope of further research on these parameters and here we list some of them.

1. Find a characterization of connected graphs of order $n \geq 2$ with $\rho^o(G) = \frac{n}{\delta(G)}$.
2. Find a characterization of graphs of order n with $\delta(G) \geq 2$ for which $\rho^o(G) + \gamma_t(G) = n$.
3. Characterize the connected graphs of order $n \geq 3$ for which $\rho^o(G) = n -$

$\omega(G)$, where $\omega(G)$ denotes the clique number of G.

4. It has been observed that the value of ρ^o is ranging from ρ and 2ρ and we have proved that ρ^o assumes any value in this range. Hence the problem of characterizing the extremal graphs is worthy trying.
5. Characterize the graphs G for which $\rho^o(G) = \rho^o_L(G)$.

The effect of the removal of a vertex or an edge on any graph theoretic parameter is of practical importance. As far as the parameter ρ^o is concerned, removal of a vertex from a graph G may increase or decrease the value of $\rho^o(G)$ or may remain unchanged. Hence the vertex set $V(G)$ can be split into the sets $V^o(G)$, $V^+(G)$ and $V^-(G)$, where

$$V^o(G) = \{v \in V : \rho^o(G - v) = \rho^o(G)\},$$
$$V^+(G) = \{v \in V : \rho^o(G - v) > \rho^o(G)\},$$
$$V^-(G) = \{v \in V : \rho^o(G - v) < \rho^o(G)\}.$$

Also, the removal of an edge from a graph G may increase the value of $\rho^o(G)$ or may remain unchanged and hence the edge set $E(G)$ can be split into the sets $E^o(G)$ and $E^+(G)$. Now, we can start investigating the properties of these sets. Similar study can be initiated for the remaining packing parameters as well.

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References

[1] N. Biggs, *Perfect codes in graphs*, J. Combin. Theory (B) **15** (1973) 289–296. doi:10.1016/0095-8956(73)90042-7

[2] G. Chartrand and L. Lesniak, Graphs and Digraphs, Fourth Edition (CRC Press, Boca Raton, 2005).

[3] L. Clark, *Perfect domination in random graphs*, J. Combin. Math. Combin. Comput. **14** (1993) 173–182.

[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1988).

[5] M.A. Henning, *Packing in trees*, Discrete Math. **186** (1998) 145–155. doi:10.1016/S0012-365X(97)00228-8

[6] A. Meir and J.W. Moon, *Relations between packing and covering numbers of a tree*, Pacific J. Math. **61** (1975) 225–233.
doi:10.2140/pjm.1975.61.225

[7] J. Topp and L. Volkmann, *On packing and covering number of graphs*, Discrete Math. **96** (1991) 229–238.
doi:10.1016/0012-365X(91)90316-T

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