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PRODUCTS OF GEODESIC GRAPHS AND THE GEODETIC NUMBER OF PRODUCTS

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Abstract

Given a connected graph and a vertex $x \in V(G)$, the geodesic graph $P_x(G)$ has the same vertex set as G with edges uv iff either v is on an $x - u$ geodesic path or u is on an $x - v$ geodesic path. A characterization is given of those graphs all of whose geodesic graphs are complete bipartite. It is also shown that the geodetic number of the Cartesian product of $K_{m,n}$ with itself, where $m, n \geq 4$, is equal to the minimum of m, n and eight.

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1. Introduction

Given a connected graph G and a vertex $x \in V(G)$, the *geodesic graph* $P_x(G)$ [1] is the graph with the same vertex set as G and edges $uv \in E(P_x(G))$ iff either v is on an $x - u$ geodesic path or u is on an $x - v$ geodesic path. (A *geodesic path* is a shortest path between vertices.) In this paper, we prove that for connected graphs G and H with $g \in V(G)$ and $h \in V(H)$, $P_g(G) \,\square\, P_h(H) = P_{(g,h)}(G \,\square\, H)$, where $G \,\square\, H$ is the Cartesian product of graphs [2]. It is shown in [1] that G

is bipartite if and only if $P_x(G) = G$ for every $x \in V(G)$. We characterize those graphs G for which $P_x(G)$ is complete bipartite for every $x \in V(G)$.

For vertices u and v in a connected graph G, $I[u, v]$ is the set of all vertices on some geodesic between u and v. S is a geodetic set if $\bigcup\{I[u, v] : u, v \in S\} = V(G)$. The *geodetic number* of a connected graph G, $g(G)$, is the size of a minimal geodetic set for G. The geodetic number was introduced in [5] and has been widely studied. See, for example, [3, 4, 7, 8], and [9] for results related to the geodetic number of graph products. It is known [6] that the geodetic number of a complete bipartite graph $K_{m,n}$ is four if both m and n are at least four and then it follows from [8] that if $m, n, r, s \geq 4$, then $g(K_{m,n}\Box K_{r,s}) \geq 4$. In [4] the authors introduce linear geodetic sets (see section 3 below) and use them to show that if $m, n, r, s \geq 4$, then $g(K_{m,n}\Box K_{r,s}) \leq 8$. We show that if $m, n \geq 4$, then $g(K_{m,n}\Box K_{m,n}) = \min\{m, n, 8\}$.

2. The Geodesic Graph

In this section we first prove that for connected graphs G and H with $g \in V(G)$ and $h \in V(H)$, $P_g(G)\Box P_h(H) = P_{(g,h)}(G\Box H)$. Recall that the *Cartesian product of graphs G and H, $G\Box H$*, has vertex set $V(G) \times V(H)$; (a, b) is adjacent to (c, d) in the product iff either $a = c$ and b is adjacent to d, or $b = d$ and a is adjacent to c.

We will use the following two lemmas. The first appears in [8] and is called "folklore" in [4].

Lemma 1. *Let G and H be connected graphs. Given any two vertices (u, x) and (v, y) in $G \Box H$, we have $d_{G \Box H}((u, x), (v, y)) = d_G(u, v) + d_H(x, y)$. Furthermore, if P is a $(u, x) - (v, y)$ geodesic in $G \Box H$ and P_1 and P_2 are the projections of $V(P)$ onto G and H, respectively, then P_1 induces a $u - v$ geodesic in G and P_2 induces an $x - y$ geodesic in H.*

Lemma 2 [1]**.** *If $e = uv \in E(G)$ is not an edge in $P_x(G)$, then e is an edge in an odd cycle of G. If $d_G(x, u) = d_G(x, v)$, then uv is not an edge in $P_x(G)$.*

Theorem 3. *Let G and H be connected graphs with $g \in V(G)$ and $h \in V(H)$. Then $P_g(G)\Box P_h(H) = P_{(g,h)}(G\Box H)$.*

Proof. We first prove that $P_g(G)\Box P_h(H) \subseteq P_{(g,h)}(G\Box H)$. The vertex sets are equal so we consider edges. Let e be an edge in $G\Box H$ which is not in $P_{(g,h)}(G\Box H)$. Either $e = (u, y)(v, y)$ or $e = (x, u)(x, v)$. We assume the former. Then by Lemma 2, e belongs to an odd cycle in $G\Box H$ and e lies on neither a $(g, h) - (u, y)$ geodesic nor a $(g, h) - (v, y)$ geodesic in $G\Box H$. By Lemma 1, the projections of these geodesics induce $g - u$ and $g - v$ geodesics in G and an $h - y$ geodesic in

H such that $d_{G\Box H}((g,h),(u,y)) = d_G(g,u) + d_H(h,y) = d_{G\Box H}((g,h),(v,y)) = d_G(g,v) + d_H(h,y)$. It follows that $d_G(g,u) = d_G(g,v)$, hence by Lemma 2, $uv \notin E(P_g(G))$. Then $e = (u,y)(v,y) \notin E(P_g(G)\Box P_h(H))$.

For the reverse inclusion, we consider an edge e in $G\Box H$ which is not in $P_g(G)\Box P_h(H)$. By symmetry, we can assume $e = (u,y)(v,y)$, where $y \in V(H)$ and $uv \notin E(P_g(G))$. Then, u does not lie on a $g-v$ geodesic and v does not lie on a $g-u$ geodesic, from which it follows that $d_G(g,u) = d_G(g,v)$. Then from Lemma 1, $d_{G\Box H}((g,h),(u,y)) = d_{G\Box\ H}((g,h),(v,y))$ and hence, from Lemma 2, $e \notin E(P_{(g,h)}(G\Box H))$, which completes the proof. ■

For a subset S of the vertex set $V(G)$ of the graph G, $G[S]$ will denote the induced subgraph of G. For our main theorem below we will need to develop conditions under which $P_x(G[S]) = P_x(G)[S]$. That is, we need conditions under which the geodesic graph of the induced subgraph is equal to the induced subgraph of the geodesic graph of G. Notice that since $G[S]$ is not necessarily connected, $P_x(G[S])$ may not even be defined. When $G[S]$ is not connected, we understand $P_x(G[S])$ to be the geodesic graph of the component of $G[S]$ containing x. We begin with an example to show that these two graphs are not, in general, equal.

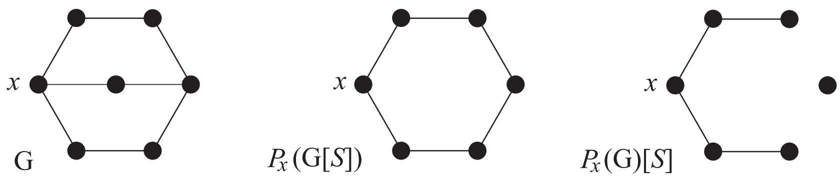

Figure 1

Example 4. It is not necessarily true that $P_x(G[S]) = P_x(G)[S]$. In Figure 1, x is indicated in the diagram and S is the subset of $V(G)$ consisting of the vertices in the outer cycle, C_6.

Lemma 5. *Let $S \subseteq V(G)$ and let $x \in S$. If either $P_x(G[S])$ or $P_x(G)[S]$ is a complete bipartite graph, then they are equal.*

Proof. We assume, first, that $P_x(G[S])$ is complete bipartite. Clearly the vertex sets of $P_x(G[S])$ and $P_x(G)[S]$ are equal, so we show they have the same edge sets. Let $uv \in E(P_x(G[S]))$. We can assume v is on an $x-u$ geodesic in $G[S]$. Since $P_x(G[S])$ is complete bipartite, we have $d_{G[S]}(u,x) = 2$ and $d_{G[S]}(v,x) = 1$. Since u, v, and x are all in S, and v is adjacent to x in S, $uv \in E(P_x(G)[S])$.

To prove the reverse inclusion, let uv be an edge in $P_x(G)[S]$. If uv is not an edge in the complete bipartite graph $P_x(G[S])$, then uv must be an edge in an odd cycle in $G[S]$ and both u and v must be in the same partition of vertices of $P_x(G[S])$. If w is an antipodal point from uv on the odd cycle in $G[S]$, then $d_{G[S]}(x,u) = d_{G[S]}(x,w) + d_{G[S]}(w,u) = d_{G[S]}(x,w) + d_{G[S]}(w,v) = d_{G[S]}(x,v)$. We have two cases.

Case 1 (3-cycle): $d_{G[S]}(x,u) = d_{G[S]}(x,v) = 1$ (in $G[S]$) so that both u and v are adjacent to x in $G[S]$ and so also in G. But by Lemma 2, this implies that uv cannot be an edge in $P_x(G)$. Then since u and v are in S, uv cannot be an edge in $P_x(G)[S]$, which contradicts our assumption.

Case 2 (5-cycle): $d_{G[S]}(x,u) = d_{G[S]}(x,v) = 2$ in $G[S]$ and u, v, and x are all in the same partition. But since uv is an edge in $P_x(G)[S]$, either $d_{G[S]}(x,u)$ or $d_{G[S]}(x,v)$ is less than 2, which contradicts our assumption.

This concludes the proof that if $P_x(G[S])$ is complete bipartite, then the two sets are equal. Now we prove that if $P_x(G)[S]$ is complete bipartite, then again the sets are equal. Assume $P_x(G)[S]$ is complete bipartite and let uv be an edge in $P_x(G)[S]$. Note that $\{x,u,v\} \subseteq S$. We show $uv \in E(P_x(G[S])$. Since uv is an edge in $P_x(G)[S]$, it is an edge in $P_x(G)$, so we may assume that uv lies on an $x - v$ geodesic path in G. Since $P_x(G)[S]$ is complete bipartite, $d_{G[S]}(x,v) = d_{G[S]}(x,u) + 1$ is either equal to one or two. In the first case, $x = u$ and the result immediately follows. In the second case, u is adjacent to x in $G[S]$. Notice that v cannot be adjacent to x in $P_x(G)[S]$ since they are in the same partite set, so $d_{G[S]}(x,u) \neq d_{G[S]}(x,v)$ and hence $uv \in E(P_x(G[S]))$.

Again, for the reverse inclusion, assume $uv \in E(P_x(G[S]))$. Then u and v are in S and $d_{G[S]}(x,u) \neq d_{G[S]}(x,v)$. Now, if $uv \notin E(P_x(G)[S])$, then since u and v are in $S, uv \notin E(P_x(G))$, which would imply that $d_G(x,u) = d_G(x,v)$. Since $P_x(G)[S]$ is complete bipartite, this distance is either one or two. If one, then both u and v are adjacent to x in G, and also in $G[S]$, which contradicts our assumption that $d_{G[S]}(x,v) \neq d_{G[S]}(x,v)$. If two, then x, u, and v are in the same partition of $P_x(G)[S]$. By our assumption, $d_{G[S]}(x,u) \neq d_{G[S]}(x,v)$, so that one of u or v must be adjacent to x in $G[S]$ and so in G. But this contradicts our finding that $d_G(x,u) = d_G(x,v)$. ■

Definition 1. Let G be a graph with A, $B \subseteq V(G)$. If $A \cap B = \emptyset$, the *join of the induced subgraphs of* A *and* B, $G[A] \vee G[B]$, has vertex set $A \cup B$ and edge set $E(G[A]) \cup E(G[B]) \cup \{(a,b) : a \in A \text{ and } b \in B\}$.

Lemma 6. *If a graph G that may be expressed as the join of non-empty induced subgraphs $G[A] \vee G[B]$ has complete bipartite geodesic graphs on all of its vertices and $P_a(G)[A]$ is an independent graph for some $a \in A$, then $G[A]$ is an independent graph as well, and similarly for B.*

Proof. Clearly a is an isolated vertex (degree zero) in $G[A]$. Suppose there is an edge uv in $G[A]$. Then in $P_u(G)$, u is adjacent to v as well as to every vertex in B. The vertex a is also adjacent to every vertex in B but is adjacent to neither

u nor v. But then $P_u(G)$ cannot be complete bipartite. From this contradiction, we conclude that $E(G[A]) = \emptyset$. ■

Theorem 7. *The following are equivalent for a connected graph G.*

(1) *G has complete bipartite geodesic graphs on each of its vertices;*

(2) *G may be expressed as the join of non-empty induced subgraphs $G[A] \vee G[B]$ where for each $a \in A$, $P_a(G)[A]$ is either an independent graph or a complete bipartite graph and similarly for each $b \in B$;*

(3) *$G = G[A] \vee G[B]$ for $A \cup B = V(G)$, where each of $G[A]$ and $G[B]$ either is an independent graph or has complete bipartite geodesic graphs on each of its respective vertices.*

Proof. To prove (2) implies (1), assume $G = G[A] \vee G[B]$, where $A \cup B = V(G)$. If for some $a \in A$, $P_a(G)[A]$ is an independent graph, then by Lemma 6, $E(G[A]) = \emptyset$ so that $P_x(G)$ is complete bipartite for every $x \in A$. A similar argument holds for $G[B]$. We can assume, then, that $P_a(G)[A]$ and $P_b(G)[B]$ are complete bipartite for each $a \in A$ and $b \in B$, and further that every $a \in A$ is adjacent to every $b \in B$. We claim that for $x \in V(G)$, $P_x(G)$ is complete bipartite.

Let $a \in A$. Let A_1 and A_2 be the sets of vertices of distance one or two, respectively, in G from a. Since no two vertices equidistant from a may be adjacent in $P_a(G)$, $P_a(G)$ contains no edge in $E(G[B])$, no two vertices in A_1 are adjacent in $P_a(G)$, and no two vertices in A_2 are adjacent in $P_a(G)$. Since $P_a(G)[A]$ is complete bipartite, every vertex in A_1 is adjacent to every vertex in A_2. Also, since all vertices in A_1 and in B are adjacent to a, no two vertices in $A_1 \cup B$ are adjacent in $P_a(G)$. Thus $P_a(G)$ is complete bipartite across the partition $A_1 \cup B$ and $A_2 \cup \{a\}$. Notice either A_2 or both A_1 and A_2 may be empty, but that $P_a(G)$ remains complete bipartite in those cases. We have shown that $P_a(G)$ is complete bipartite for each $a \in A$. Similarly, $P_b(G)$ is complete bipartite for each $b \in B$. The result follows.

We now prove (3) implies (2). That $P_x(G[A]) = P_x(G)[A]$ when $P_x(G[A])$ is complete bipartite follows from Lemma 5; and that $P_x(G[A])$ is independent when $G[A]$ is independent (the converse of Lemma 6) is clear.

Now to prove (1) implies (2), let G have a complete bipartite geodesic graph at every vertex. Fix x in $V(G)$ and consider the partition of vertices of the geodesic graph $P_x(G)$ into its partite sets A and B; then $G[A] \vee G[B] = G$. We claim that for each $b \in B$, $P_b(G)[B]$ is either an independent graph or a complete bipartite graph and a similar claim holds for each $a \in A$.

Let $b \in B$. We assume the geodesic graph $P_b(G)$ has partite sets B_1 and B_2 so that $P_b(G)$ is the graph $(B_1 \cup B_2, \{uv : u \in B_1, v \in B_2\})$. The subgraph induced by $B \subseteq B_1 \cup B_2$ is $P_b(G)[B] = (B, \{uv : u \in B_1 \cap B, v \in B_2 \cap B\})$. Two cases follow:

Case 1: B contains vertices from both B_1 and B_2 so that $P_b(G)[B]$ is a complete bipartite graph.

Case 2: $B \subseteq B_1$ or $B \subseteq B_2$. Then $P_b(G)[B]$ is an independent graph.

We conclude by proving that (2) implies (3). We assume $G = G[A] \vee G[B]$, where for each $a \in A$, $P_a(G)[A]$ is either an independent graph or a complete bipartite graph and the same is true for each $b \in B$. Let $a \in A$ and assume $P_a(G)[A]$ is a complete bipartite graph. Then by Lemma 5, $P_a(G[A])$ is also a complete bipartite graph. If $P_a(G)[A]$ is an independent graph, then by Lemma 6, $G[A]$ is also an independent graph. Since the proof for $b \in B$ is similar, the result follows. ■

3. The Geodetic Number of Products of Graphs.

In this section we examine the geodetic number of the product of complete bipartite graphs. First, two observations:

1. [2] The Cartesian product of graphs is bipartite if and only if each factor is bipartite.

2. (Folklore) If G and H are non-trivial connected graphs, then $G\Box H = K_{m,n}$ if and only if $G = H = K_2$.

Knowing that the Cartesian product of complete bipartite graphs is not in general complete bipartite, we now consider the geodetic number of the product of complete bipartite graphs. We begin with known results.

Theorem 8 [8]**.** *For graphs G and H,* $\max\{g(G), g(H)\} \leq g(G\Box H)$.

Theorem 9 [6]**.** *If* $m, n \geq 4$, *then* $g(K_{m,n}) = 4$.

Definition 2 [4]**.** Let $S = \{x_1, x_{2,}, \ldots, x_k\}$ be a geodetic set of the graph G. S is called a *linear geodetic set* if for any $x \in V(G)$, there exists an index $i, 1 \leq i \leq k$, such that $x \in I[x_i,\ x_{i+1}]$.

Remark 10. If $m, n, r, s \geq 4$, then as is pointed out in the comments after Theorem 2.3 of [4], $K_{m,n}$ and $K_{r,s}$ have linear geodetic sets and so it follows directly from that theorem that $g(K_{m,n}\Box K_{r,s}) \leq 8$. Putting these results together, we have that if $m, n, r, s \geq 4$, then $4 \leq g(K_{m,n}\Box K_{r,s}) \leq 8$.

Theorem 11. *If* $m, n \geq 4$, *then* $g(K_{m,n}\Box K_{m,n}) = \min\{m, n, 8\}$.

Proof. We first show that $g(K_{m,n}\Box K_{m,n}) \leq \min\{m,n,8\}$. If $\min\{m,n,8\} = 8$, then this inequality holds by the previous Remark, so we assume that $\min\{m,n,8\} = n$ and construct a geodetic set with n elements.

Assume we have a bipartition of $K_{m,n} : \{L_1, L_2, \ldots, L_m\}$ and $\{R_1, R_2, \ldots, R_n\}$. Let $S = \{(R_1,R_1),(R_2,R_2),\ldots,(R_n,R_n)\}$. We claim that S is a geodetic set in $K_{m,n}\Box K_{m,n}$. Let X be an arbitrary element of $K_{m,n}\Box K_{m,n}$ not in S. We show that X is on a geodesic from S.

Case 1: If $X = (L_i, L_j)$, then $(R_1,R_1) \to (R_1,L_j) \to (L_i,L_j) \to (R_2,L_j) \to (R_2,R_2)$ is a geodesic in S containing X.

Case 2: If $X = (R_i,R_j)$, where $i \neq j$, then $(R_i,R_i) \to (R_i,L_1) \to (R_i,R_j) \to (L_2,R_j) \to (R_j,R_j)$ is a geodesic in S containing X. If $i = j$, then $X \in S$.

Case 3: If $X = (L_i,R_j)$, pick $k \neq j,\ k \leq n$. Then $(R_j,R_j) \to (L_i,R_j) \to (R_k,R_j) \to (R_k,L_i) \to (R_k,R_k)$ is a geodesic in S containing X.

Case 4: If $X = (R_j,L_i)$, then X is on a geodesic from S as in Case 3.

Thus S is a geodetic set with n elements and $g(K_{m,n}\Box K_{m,n}) \leq n = \min\{m,n,8\}$.

To complete the proof it is sufficient to show that $g(K_{m,n}\Box K_{m,n}) \neq \min\{m,n,8\}-1$. We label $N = \min\{m,n,8\}$, assume $n \leq m$, assume $N > 4$, and assume we have the same bipartition of $K_{m,n}$ as we had in the first part of this proof. Let T be a geodetic set in $K_{m,n}\Box K_{m,n}$ of size $N-1$. Then, regardless of whether $N = n$ or $N = 8$, there is some $L_i \in K_{m,n}$ such that $(L_i,Y) \notin T$ for any $Y \in K_{m,n}$. Otherwise, there would be at least N pair of points in T. Similarly, there are $R_j,\ L_k,\ R_p$ in $K_{m,n}$ so that $(R_j,Y) \notin T$ for any $Y \in K_{m,n}$, $(X,L_k) \notin T$ for any $X \in K_{m,n}$ and $(X,R_p) \notin T$ for any $X \in K_{m,n}$. We will use these particular points L_i, R_j, L_k, R_p below.

Points in T are of the form (L,L), (R,R), (L,R), or (R,L), where L is a member of the first set of the bipartition and R a member of the second set. Geodesics between these points may be: from (L,L) to (L,L) of length two or four depending on whether two of the Ls are the same and similarly from (R,R) to (R,R); from (L,L) to (R,R) of length two; from (L,R) to (L,R) of length two or four depending on whether Ls or Rs are the same and similarly from (R,L) to (R,L); from (L,R) to (L,L) (or from (L,R) to (R,R)) of length one or three and simliarly from (R,L) to either (R,R) or (L,L); from (L,R) to (R,L) of length two.

Now consider the point (L_i,L_k) as defined above. Because of the conditions on L_i and L_k and the length of the possible geodesics containing this point, (L_i,L_k) can only be on geodesics of the form (R,R) to (R,R) (with the Rs different), so there must be two points of the form (R,R) in T. Similarly, considering the point (R_j,R_p), there must be two points of the form (L,L) in T. The point (L_i,R_p) can only be on geodesics of the form (R,L) to (R,L), with the Rs different and the Ls different and so there must be two points of the form (R,L) in

T. Similarly, the point (R_j, L_k) can only be on a geodesic of the form (L, R) to (L, R), so there must be two points of the form (L, R) in T. There are then eight points in T. But T has $N-1$ points and $N-1 = \min\{m, n, 8\} - 1 \leq 8 - 1 = 7$. Thus, there is no geodesic set of size $N-1$. ■

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