Discussiones Mathematicae
Graph Theory 35 (2015) 55–72
doi:10.7151/dmgt.1773

2-TONE COLORINGS IN GRAPH PRODUCTS

Jennifer Loe

Oklahoma Christian University
2501 E. Memorial Rd.
Edmond, OK, 73013, USA

e-mail: jennifer.loe@eagles.oc.edu

Danielle Middelbrooks

Spelman College
350 Spelman Ln.
Atlanta, GA 30314, USA

e-mail: dmiddle1@scmail.spelman.edu

Ashley Morris

Savannah State University
3219 College St.
Savannah, GA 31404, USA

e-mail: amorri13@student.savannahstate.edu

and

Kirsti Wash

Clemson University
Box 340975
Clemson, SC 29634, USA

e-mail: kirstiw@clemson.edu

Abstract

A variation of graph coloring known as a t-tone k-coloring assigns a set of t colors to each vertex of a graph from the set $\{1, \ldots, k\}$, where the sets of colors assigned to any two vertices distance d apart share fewer than d colors in common. The minimum integer k such that a graph G has a t-tone k-coloring is known as the t-tone chromatic number. We study the 2-tone chromatic number in three different graph products. In particular, given graphs G and H, we bound the 2-tone chromatic number for the direct product $G \times H$, the Cartesian product $G \square H$, and the strong product $G \boxtimes H$.

Keywords: t-tone coloring, Cartesian product, direct product, strong product.

2010 Mathematics Subject Classification: 05C15, 05C76.

1. Introduction

Many variations of classic graph k-colorings abound, whereby we take a k-coloring of a graph to mean an assignment of an element from $\{1, \ldots, k\}$, called a color, to each of the vertices of the graph. Chartrand was the first to introduce a t-tone k-coloring [4], which is an assignment of t elements from the set $\{1, \ldots, k\}$ to each vertex such that the sets of colors assigned to any two distinct vertices within distance d share fewer than d colors. This t-tone k-coloring variation can be viewed as a generalization of classic graph coloring since a 1-tone k-coloring of a graph is simply a k-coloring.

For the purpose of this paper, we consider only simple, undirected graphs G with vertex set $V(G)$ and edge set $E(G)$. For each vertex $v \in V(G)$, $\deg_G(v)$ denotes the number of vertices adjacent to v and the maximum degree of G is defined to be $\Delta(G) = \max_{v \in V(G)} \deg_G(v)$. The distance between two vertices u and v of $V(G)$ is the size of the shortest length path between u and v, and is denoted by $d_G(u, v)$. When the context is clear, we use the shorthand notation $d(u, v)$.

As stated above, a proper k-coloring of a graph G is an assignment of an element from $\{1, \ldots, k\}$, called a *color*, to each vertex in $V(G)$ such that no two adjacent vertices are assigned the same color. The chromatic number of G, denoted $\chi(G)$, is the minimum number k such that G has a proper k-coloring. We use K_n to denote the complete graph on n vertices. Given a graph G, a clique is any complete subgraph of G, and the clique number of G, denoted $\omega(G)$, is the cardinality of the maximum clique of G. For positive integers t and k where $t \leq k$, we let $[k]$ represent the set $\{1, \ldots, k\}$ and denote the family of t-element subsets of $[k]$ by $\mathcal{P}_t([k])$. The following is a formal definition of the t-tone chromatic number of a graph.

Definition. Let G be a graph, and let t and k be positive integers such that $t \leq k$. A *t-tone k-coloring of G* is a function $f : V(G) \to \mathcal{P}_t([k])$ such that $|f(u) \cap f(v)| < d_G(u, v)$ for all distinct vertices u and v. A graph that has a t-tone k-coloring is said to be *t-tone k-colorable*. The *t-tone chromatic number of G*, denoted $\tau_t(G)$, is the minimum integer k such that G is t-tone k-colorable.

Figure 1 depicts a 2-tone 5-coloring of P_5 which is, indeed, minimum. Given a t-tone k-coloring f of G, we call $f(v)$ the *label* of v and the elements of $[k]$ *colors*.

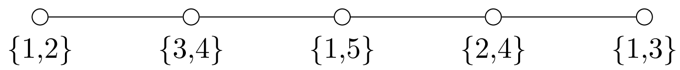

Figure 1. A 2-tone coloring of P_5.

Given a graph G, a proper distance (d, k)-coloring of G is a map $f : V(G) \to [k]$ such that for any two distinct vertices u and v of $V(G)$ with $d_G(u, v) \leq d$, we have $f(u) \neq f(v)$. A t-tone k-coloring can also be viewed as a generalization of a (d, k)-coloring in that both incorporate similar conditions based on the distance between vertices. Applications of $(2, k)$-colorings include channel assignment, or broadcast scheduling, for packet radio networks [7] and facility location problems [8].

Recall that the square of G, denoted G^2, is the graph with $V(G^2) = V(G)$ and edge set $E(G^2) = \{\ uv : \ d_G(u, v) \leq 2\ \}$. We call the reader's attention to the fact that any proper k-coloring of G^2 is a proper distance $(2, k)$-coloring of G, and vice versa. Therefore, we use $\chi(G^2)$ to denote the smallest integer k such that G has a proper distance $(2, k)$-coloring. Fonger *et al.* [4] (p. 11) noted that the relationship between $\chi(G^2)$ and $\tau_2(G)$ can at times seem counterintuitive. For instance, one can show that $\chi(P_5^2) = 3 < 5 = \tau_2(P_5)$, but that $\tau_2(G) < \chi(G^2)$ when G is the Petersen graph. However, the following was shown to be true for any graph G.

Theorem 1 [4]**.** *Given any graph G, $\tau_2(G) \leq \chi(G) + \chi(G^2)$.*

Although a better general upper bound for $\tau_2(G)$ exists, the relationship between $\chi(G^2)$ and $\tau_2(G)$ found in Theorem 1 will be useful for our results. In 2011, Bickle and Phillips [1] gave general bounds for the t-tone chromatic number of a graph G in terms of $\Delta(G)$. Shortly thereafter, Cranston, Kim, and Kinnersley [2] (p. 3) gave the following upper bound, which we will refer to in subsequent sections.

Theorem 2 [2]**.** *For any graph G, $\tau_2(G) \leq \lceil (2 + \sqrt{2})\Delta(G) \rceil$.*

In addition to the above, Bal *et al.* recently studied the t-tone chromatic number of random graphs [3]. We now shift our focus to t-tone colorings in graph products. Recall the definition of the direct product of two graphs.

Definition. Given two graphs G and H, the *direct product* of G and H, denoted $G \times H$, is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$, and whose edge set is

$$E(G \times H) = \{(x_1, y_1)(x_2, y_2) : x_1x_2 \in E(G) \text{ and } y_1y_2 \in E(H)\}.$$

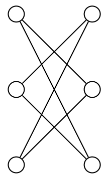

Figure 2. $K_2 \times K_3$.

Figure 2 depicts the direct product of K_2 and K_3. In Section 2, we use similar proof techniques to those used in [4](p. 6) to determine the exact value of $\tau_2(K_m \times K_n)$, and we give general upper and lower bounds for $\tau_2(G \times H)$.

Next, recall the definition of the Cartesian product.

Definition. The *Cartesian product* of graphs G and H, denoted $G \Box H$, is the graph whose vertex set is $V(G) \times V(H)$, whereby two vertices (u_1, u_2) and (v_1, v_2) are adjacent if $u_1 v_1 \in E(G)$ and $u_2 = v_2$, or $u_1 = v_1$ and $u_2 v_2 \in E(H)$.

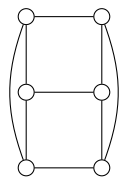

Figure 3. $K_2 \Box K_3$.

Figure 3 depicts the Cartesian product of K_2 and K_3. As mentioned in [5], a straight-forward argument shows that $\tau_t(G \Box H) \leq \tau_t(G)\tau_t(H)$, but that this bound can be improved. Focusing only on the case when $t = 2$, in Section 3 we give an upper bound based on the value of $\max\{\chi(G^2), \chi(H^2)\}$.

Finally, recall the definition of the strong product of graphs.

Definition. The *strong product* of graphs G and H, denoted $G \boxtimes H$, is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edge set is given by $E(G \boxtimes H) = E(G \Box H) \cup E(G \times H)$.

Figure 4 depicts the strong product of K_2 and K_3. In Section 4, we show that $\tau_2(G \boxtimes H) \leq \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\}$ using similar techniques and results for the direct product and the Cartesian product.

2. Direct Product

In this section, we focus on the direct product of two graphs, whose definition we restate for ease of reference. The direct product of two graphs G and H

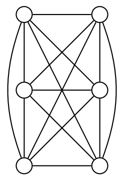

Figure 4. $K_2 \boxtimes K_3$.

is denoted $G \times H$ with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = \{(x_1, y_1)(x_2, y_2) : \ x_1x_2 \in E(G) \ \text{ and } \ y_1y_2 \in E(H)\}$.

Throughout this section, when we consider the direct product $G \times H$, where $|V(G)| = m$ and $|V(H)| = n$, we will represent the vertices of G as $x_1, \ldots, x_m$ and the vertices of H as $y_1, \ldots, y_n$. Using this notation, for each $i \in [m]$ we define the *column* C_i as the set of all vertices with first coordinate x_i. In particular, for $i \in [m]$, the i^{th} column is given by $C_i = \{(x_i, y_j) : j \in [n]\}$. Similarly, for $j \in [n]$, the j^{th} *row* is the set $R_j = \{(x_i, y_j) : i \in [m]\}$.

In order to find an upper and lower bound of the 2-tone chromatic number of the direct product of any two graphs G and H, we first consider the direct product of two complete graphs. By definition of the direct product, we know for $m, n \in \mathbb{N}$ such that $2 \leq m \leq n$,

$$V(K_m \times K_n) = \{(x_i, y_k) : i \in [m] \text{ and } k \in [n]\},$$

and

$$E(K_m \times K_n) = \{(x_i, y_k)(x_j, y_\ell) : \ i \neq j \text{ and } k \neq \ell\}.$$

The following is a direct consequence of the distance formula for the direct product found in [6](p. 54)

Proposition 3. *Let $m, n \in \mathbb{N}$ such that $m \geq 2$ and $n \geq 3$. If u and v are any two distinct vertices of $V(K_m \times K_n)$ that are contained within the same column, then $d(u, v) = 2$.*

Recall from Section 1 that given a graph G and a t-tone k-coloring f of G, we call $f(v)$ the label of v and the elements of $[k]$ colors. Additionally, for any set of vertices $A \subseteq V(G)$, we define the *set of colors contained in the labels associated with A* to be

$$c(A) = \{c \in [k] : \ c \in f(v) \text{ for some } v \in A\}.$$

Theorem 4. *If $m, n \in \mathbb{N}$, where $2 \leq m \leq n$ and $t = \frac{1+\sqrt{1+8n}}{2}$, then*

$$\tau_2(K_m \times K_n) = \min\left\{m\lceil t\rceil, m + n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor - 1)}{2}\right\}.$$

Proof. First consider the case where $m = n = 2$. Since $K_2 \times K_2 \cong 2K_2$,

$$\tau_2(K_2 \times K_2) = \tau_2(K_2) = 4.$$

One can easily verify that this is the minimum of the three functions.

Now consider all other cases where $m \geq 2, n \geq m$ and $n \neq 2$. Let f be a minimum 2-tone k-coloring of $K_m \times K_n$. For each $1 \leq i \leq m$, define A_i as the set of all vertices $v \in C_i$ such that for each $a \in f(v)$, there exists a vertex $w \in C_i$ with $w \neq v$ and $a \in f(w)$. Note the following property of this subset $A_i \subseteq C_i$. Fix $i \in [m]$ and let $v \in A_i$ as described above; that is, for $a \in f(v)$ there exists $w \in C_i$ with $a \in f(w)$. This implies that for all $1 \leq j \leq m$ such that $j \neq i$ and for any $u \in C_j$, $a \notin f(u)$ since u is adjacent to at least one of v or w. Therefore, the set of colors contained in the labels associated with A_i is disjoint from the set of colors contained in the labels associated with C_j; that is, $c(A_i) \cap c(C_j) = \emptyset$.

For each $1 \leq i \leq m$, let $s_i = |c(A_i)|$. Let $s_\ell = \min_{1 \leq i \leq m} s_i$ for some $\ell \in [m]$. Thus, the number of distinct colors contained in the labels associated with $\cup_{i=1}^{m} A_i$ is at least ms_ℓ. By definition, for each $(x_\ell, y_j) \in C_\ell \backslash A_\ell$, there exists a color $a \in f((x_\ell, y_j))$ such that a is not contained in any other label associated with C_ℓ. Furthermore, if for some $1 \leq i \leq m$ where $i \neq \ell$ we have $a \in c(A_i)$, then there would exist $v \in A_i$ and $w \in C_i$ such that $a \in f(v) \cap f(w)$. However, this would contradict the fact that f is a proper 2-tone coloring since one of v or w is adjacent to (x_ℓ, y_j). Thus, for each $1 \leq i \leq m$, $a \notin c(A_i)$. It follows that $k \geq ms_\ell + |C_\ell \backslash A_\ell|$. We now determine the minimum k based on the value of $|C_\ell \backslash A_\ell|$. We do this by considering the following three cases.

Case 1. Assume that $|C_\ell \backslash A_\ell| = n$. Thus, $A_\ell = \emptyset$ and $s_\ell = 0$. Let $Q = \{C_i : \ i \in [m] \text{ and } s_i = 0\}$ and $T = \{C_i : i \in [m] \text{ and } s_i > 0\}$, where $|Q| = q$ and $|T| = t$. Note that $q + t = m$. Since $s_\ell = 0$, we know $t < m$ or equivalently $t + 1 \leq m$. For indexing purposes, we shall write $Q = \{C_{\alpha(1)}, \ldots, C_{\alpha(q)}\}$, where $\alpha(i) \in [m]$ for $1 \leq i \leq q$. Since $q = m - t \leq m \leq n$, there exists a set $W = \{v_{\alpha(1)}, \ldots, v_{\alpha(q)}\}$ such that $v_{\alpha(i)} \in C_{\alpha(i)}$ for each $\alpha(i)$, and if $\alpha(i) \neq \alpha(j)$, then $v_{\alpha(i)}$ and $v_{\alpha(j)}$ are in different rows. Notice that the induced subgraph of W is a clique so that $|f(v_{\alpha(i)}) \cap f(v_{\alpha(j)})| = 0$ when $\alpha(i) \neq \alpha(j)$. Define, $B = \{(x_i, y_j) \in C_i : \ C_i \in Q \text{ and } R_j \cap W = \emptyset\}$.

Note that there exist at least $n - q$ colors that are contained in $c(B)$ which are not contained in $c(W)$. Thus, $|c(B) \cup c(W)| \geq 2q + n - q = n + q$. Finally, for each column $C_i \in T$, we know that $s_i \geq 2$. Thus,

$$\begin{aligned} k &\geq n + q + 2t \\ &= m + n + t \\ &\geq m + n. \end{aligned}$$

Case 2. Assume that $|C_\ell \backslash A_\ell| = 0$. It follows that $A_\ell = C_\ell$ and $|A_\ell| = n$. Furthermore, since s_ℓ represents the number of distinct colors contained in the labels associated with A_ℓ, we know that $s_\ell \geq 2$. Since any two distinct vertices $u, v \in A_\ell$ satisfy $d(u,v) = 2$, we know that $\binom{s_\ell}{2} \geq n$. Using the quadratic formula, this implies that $s_\ell \geq \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$. Consequently, $k \geq m \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$.

Case 3. Assume that $n > |C_\ell \backslash A_\ell| > 0$. If $\binom{s_\ell}{2} > n$, then clearly $k \geq m \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$. So assume $\binom{s_\ell}{2} \leq n$, or equivalently $2 \leq s_\ell \leq \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor$. We have $n = |A_\ell| + |C_\ell \backslash A_\ell|$, which implies $|C_\ell \backslash A_\ell| = n - |A_\ell|$. As in Case 2, we know $\binom{s_\ell}{2} \geq |A_\ell|$. Thus, $n - \binom{s_\ell}{2} \leq n - |A_\ell|$, which implies $n - \binom{s_\ell}{2} \leq |C_\ell \backslash A_\ell|$.

Therefore,

$$ms_\ell + |C_\ell \backslash A_\ell| \geq ms_\ell + n - \binom{s_\ell}{2} = ms_\ell + n - \frac{s_\ell(s_\ell - 1)}{2}.$$

So we consider the function $g(s) = ms + n - \frac{s(s-1)}{2}$ over the interval $2 \leq s \leq \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor$. One can easily verify that $g'(s) = m - s + \frac{1}{2}$ and $g''(s) = -1$. Thus, g is concave down for all values of $2 \leq s \leq \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor$, and over this interval g has a local maximum when $s = m + \frac{1}{2}$. Therefore, the local minimums for g occur when $s = 2$ and $s = \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor$. Letting $t = \frac{1+\sqrt{1+8n}}{2}$, it follows that

$$\begin{aligned} k &\geq ms_\ell + |C_\ell \backslash A_\ell| \\ &\geq \min_s ms + n - \frac{s(s-1)}{2} \\ &\quad \text{such that } 2 \leq s \leq \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor \\ &\geq \min \left\{ 2m + n - 1, m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2} \right\}. \end{aligned}$$

Since $2m + n - 1 > m + n$, we may conclude that

$$k \geq \min \left\{ m + n, m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2} \right\}.$$

Note that these cases sometimes overlap. For example, $\binom{s}{2} = n$ implies that $\lfloor t \rfloor = \lceil t \rceil = t$ and $n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2} = 0$, resulting in $m\lceil t \rceil = m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2}$. In any case, we have

$$\tau_2(K_m \times K_n) \geq \min \left\{ m\lceil t \rceil, m + n, m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2} \right\}.$$

It remains to be shown that

$$\tau_2(K_m \times K_n) \leq \min \left\{ m\lceil t \rceil, m + n, m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2} \right\}.$$

Given $m, n \in \mathbb{N}$ where $2 \leq m \leq n$ and $t = \frac{1+\sqrt{1+8n}}{2}$, we construct different 2-tone colorings, which depend on the value of $\min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}\right\}$.

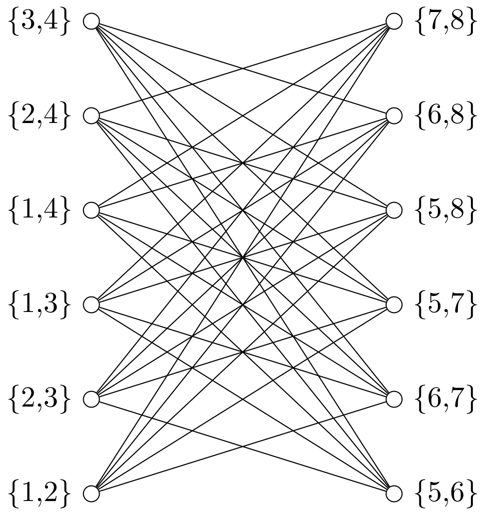

Figure 5. $K_2 \times K_6$.

Case 1. First, assume that

$$\min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}\right\} = m\lceil t\rceil.$$

Choose m pairwise disjoint sets each containing $\lceil t\rceil$ distinct colors, and denote each set of colors S_i for $1 \leq i \leq m$. Since $\binom{\lceil t\rceil}{2} \geq n$, for each $1 \leq i \leq m$ there exist n distinct combinations containing two colors from the set S_i. Thus, we may define $f : V(K_m \times K_n) \to \mathcal{P}_2\left(\left[\lceil t\rceil\right]\right)$ to be any mapping such that for each $1 \leq i \leq m$ the restriction of f to the set of vertices in C_i is an injective mapping to the set of combinations containing two colors from the set S_i. Figure 5 illustrates a labeling of $V(K_2 \times K_6)$ assigned by f. To see that f is a proper 2-tone coloring of $K_m \times K_n$, let u and v be distinct vertices of $V(K_m \times K_n)$. If u and v are not contained in the same column, then $f(u) \cap f(v) = \emptyset$. So assume $u, v \in C_i$ for some $i \in [m]$. We know by Proposition 3 that $d(u, v) = 2$. So we must show that $|f(u) \cap f(v)| \leq 1$. However, this follows from the fact that f does not assign any label to more than one vertex of C_i. Therefore, f is a proper 2-tone coloring.

Case 2. Next, assume that $\min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}\right\} = m+n$. Let f_1 be a proper coloring of K_m and f_2 be a proper coloring of K_n defined as follows:

$$\begin{aligned} f_1 : V(K_m) &\to \{1, \ldots, m\} \\ x_i &\mapsto i \\ f_2 : V(K_n) &\to \{m+1, \ldots, m+n\} \\ y_j &\mapsto m+j. \end{aligned}$$

Define the following function on $V(K_m \times K_n)$:

$$\begin{aligned} g : V(K_m \times K_n) &\to \mathcal{P}_2([m+n]) \\ (x_i, y_j) &\mapsto \{f_1(x_i), f_2(y_j)\}. \end{aligned}$$

Figure 6 illustrates a labeling of $V(K_2 \times K_3)$ assigned by g.

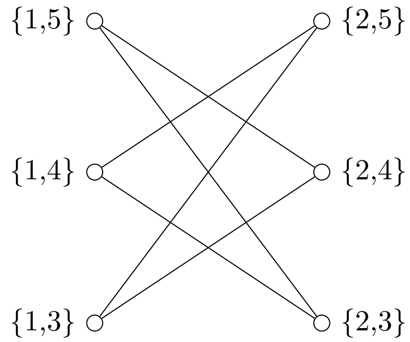

Figure 6. $K_2 \times K_3$.

We claim g is a proper 2-tone coloring of $K_m \times K_n$. Clearly, $|g((x,y))| = 2$ for all $(x,y) \in V(K_m \times K_n)$. Let (x_i, y_k) and (x_j, y_ℓ) be two distinct vertices of $V(K_m \times K_n)$, where $1 \le i,j \le m$ and $1 \le k, \ell \le n$. Then $g((x_i, y_k)) = \{i, k+m\}$ and $g((x_j, y_\ell)) = \{j, \ell + m\}$. If (x_i, y_k) and (x_j, y_ℓ) are adjacent, then $i \neq j$ and $k \neq \ell$. Thus, $|g((x_i, y_k)) \cap g((x_j, y_\ell))| = 0$. If $d((x_i, y_k), (x_j, y_\ell)) = 2$, then either $i \neq j$ or $k \neq \ell$. In any case, $|g((x_i, y_k)) \cap g((x_j, y_\ell))| \le 1$. Therefore, g is a proper 2-tone coloring of $K_m \times K_n$, and we may conclude that $\tau_2(K_m \times K_n) \le m + n$.

Case 3. Assume that $\min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}\right\} = m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}$. Note that $t = \frac{1+\sqrt{1+8n}}{2}$ is the only positive solution to $\binom{t}{2} = n$. Therefore, $\lfloor t\rfloor$ satisfies $\binom{\lfloor t\rfloor}{2} \le n$. Let $s = \binom{\lfloor t\rfloor}{2}$ and consider the subgraph H of $K_m \times K_n$ induced by the set $\{(x_i, y_j) : \ i \in [m], \ j \in [s]\}$. Thus, $H \cong K_m \times K_s$. As in Case 2, choose m pairwise disjoint sets of $\lfloor t\rfloor$ distinct colors and denote each set S_i for each $i \in [m]$. Define $f_1 : V(H) \to \mathcal{P}_2\left(\left[\lfloor t\rfloor\right]\right)$ to be any mapping such that for each $i \in [m]$, the restriction of f_1 to the set of vertices of C_i is an injective mapping to the set of combinations containing two colors from the set S_i. A similar argument as in Case 2 can be used to show that f_1 is a proper 2-tone coloring of H.

Next, choose $n-s$ distinct colors each of which are not contained in the set $\cup_{i=1}^{m} S_i$, and label these colors $\{t_{s+1}, \dots, t_n\}$. Additionally, for each $i \in [m]$, choose one color from the set S_i and call it c_i. Notice that $V(K_m \times K_n) \backslash V(H) = \{(x_i, y_j) :\ i \in [m], s+1 \leq j \leq n\}$. Define

$$\begin{aligned} f_2 : V(K_m \times K_n) \backslash V(H) &\to \mathcal{P}_2([m+n-s]) \\ (x_i, y_j) &\mapsto \{c_i, t_j\}. \end{aligned}$$

We claim that f_2 is a proper 2-tone coloring of $(K_m \times K_n) \backslash H$. To see this, let (x_i, y_k) and (x_j, y_ℓ) be two distinct vertices of $V(K_m \times K_n) \backslash V(H)$ for some $1 \leq i, j \leq m$ and $s+1 \leq k, \ell \leq n$. If (x_i, y_k) and (x_j, y_ℓ) are adjacent, then $i \neq j$ and $k \neq \ell$. Since S_i and S_j are two disjoint sets of colors, we know that $c_i \neq c_j$. Moreover, we know that $t_k \neq t_\ell$ since $k \neq \ell$. Thus, $|f_2((x_i, y_k)) \cap f_2((x_j, y_\ell))| = 0$. If $d((x_i, y_k), (x_j, y_\ell)) = 2$, then either $i \neq j$ or $k \neq \ell$. It follows that $|f_2((x_i, y_k)) \cap f_2((x_j, y_\ell))| \leq 1$. Therefore, f_2 is a proper 2-tone coloring of $(K_m \times K_n) \backslash H$.

Now define $g : V(K_m \times K_n) \to \mathcal{P}_2([m\lfloor t \rfloor + n - s])$ such that

$$g(u) = \begin{cases} f_1(u) & \text{if } u \in V(H), \\ f_2(u) & \text{otherwise.} \end{cases}$$

Figure 7 illustrates a labeling of $V(K_2 \times K_{11})$ assigned by g. To see that g is a proper 2-tone coloring, we only need to consider when $u \in V(H)$ and $v \notin V(H)$. Write $u = (x_i, y_k)$ and $v = (x_j, y_\ell)$ for some $i, j \in [m]$, $k \in [s]$, and $\ell \in \{s+1, \dots, n\}$. By definition $g(v) = \{c_j, t_\ell\}$, and we know $t_\ell \notin g(u)$ since $u \in V(H)$. So if u and v are located in the same column, then $|g(u) \cap g(v)| \leq 1$. If u and v are not located in the same column, then $i \neq j$ and $c_j \notin g(u)$ since $c_j \notin S_i$. It follows that $|g(u) \cap g(v)| = 0$. Therefore, g is a proper 2-tone coloring of $K_m \times K_n$ using $m\lfloor t \rfloor + n - \binom{\lfloor t \rfloor}{2}$ colors. ■

Using similar ideas found in Theorem 4, we can bound the value of $\tau_2(G \times H)$ given any graphs G and H. We make use of the following general lower bound given in [4] (p. 8).

Theorem 5 [4]**.** *Let G be a graph and let $\Delta(G) = d$. Then*

$$\tau_2(G) \geq \left\lceil \frac{\sqrt{8d+1}+5}{2} \right\rceil.$$

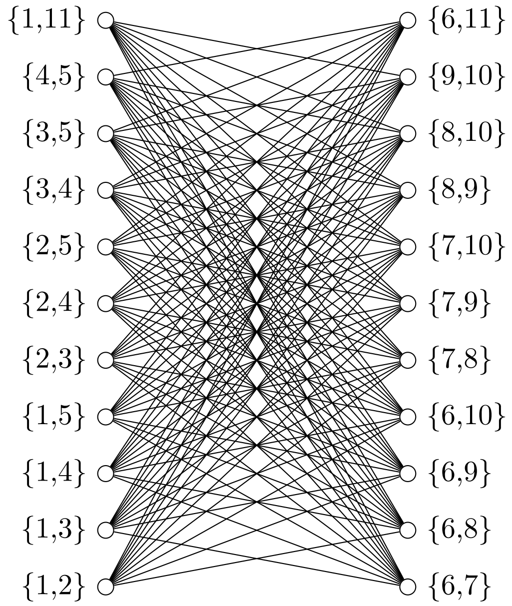

Figure 7. $K_2 \times K_{11}$.

Theorem 6. *Given two graphs G and H,*

$$\max\left\{\left\lceil\frac{5+\sqrt{1+8\Delta(G)\Delta(H)}}{2}\right\rceil, \tau_2(K_{\omega(G)}\times K_{\omega(H)})\right\} \le \tau_2(G\times H)$$
$$\le \chi(G^2)+\chi(H^2).$$

Proof. We first show that for any graphs G and H, we have $\tau_2(G\times H) \le \chi(G^2)+\chi(H^2)$. Assume $\chi(G^2)=k_1$ and $\chi(H^2)=k_2$. Let $f_1 : V(G)\to[k_1]$ be a distance $(2,k_1)$-coloring of G, and let $f_2 : V(H)\to\{k_1+1,\dots,k_1+k_2\}$ be a distance $(2,k_2)$-coloring of H. Define

$$g : V(G\times H)\to\mathcal{P}_2([k_1+k_2])$$

such that

$$(x,y)\mapsto\{f_1(x),f_2(y)\} \qquad \text{for all } x\in V(G) \text{ and } y\in V(H).$$

We claim that g is a proper 2-tone coloring of $G\times H$. Clearly, $|g((x,y))|=2$ for all $(x,y)\in V(G\times H)$. Let (u,v) and (w,z) be two distinct vertices of $V(G\times H)$. If (u,v) and (w,z) are adjacent, then $uw\in E(G)$ and $vz\in E(H)$. It follows that $f_1(u)\ne f_1(w)$ and $f_2(v)\ne f_2(z)$. Since f_1 is a mapping into the set $[k_1]$ and f_2 is a mapping into the set $\{k_1+1,\dots,k_1+k_2\}$, we have $|g((u,v))\cap g((w,z))|=0$.

Suppose that $d_{G\times H}((u,v),(w,z))=2$. If $u=w$, then $v\neq z$ and since $d_H(v,z)\leq 2$, it follows that $f_2(v)\neq f_2(z)$. Thus, $|g((u,v))\cap g((w,z))|\leq 1$. Similarly, if $v=z$, then $|g((u,v))\cap g((w,z))|\leq 1$. So we may assume that $u\neq w$ and $v\neq z$. There exists a vertex $(x,y)\in V(G\times H)$ such that uxw is a path in G and vyz is a path in H. Since $d_G(u,w)\leq 2$ and $d_H(v,z)\leq 2$, we know that $f_1(u)\neq f_1(w)$ and $f_2(v)\neq f_2(z)$. Thus, $|g((u,v))\cap g((w,z))|=0$, and we may conclude that g is a proper 2-tone coloring of $G\times H$, and $\tau_2(G\times H)\leq \chi(G^2)+\chi(H^2)$.

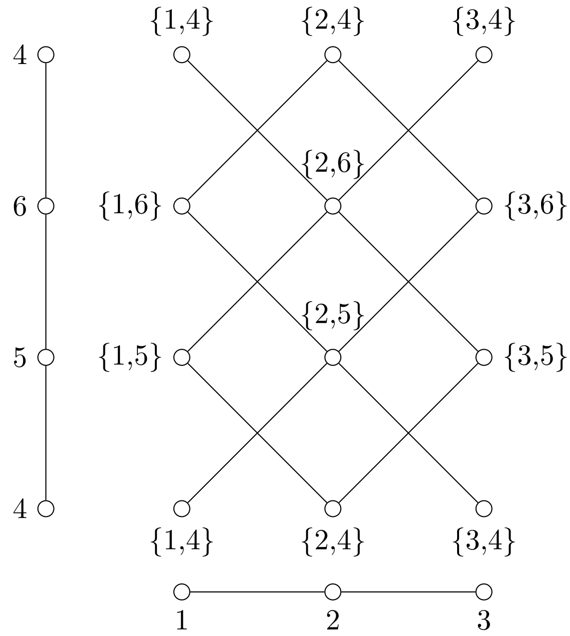

Figure 8. A 2-tone coloring of $P_3\times P_4$.

In terms of a lower bound, note that by definition of the direct product, $K_{\omega(G)}\times K_{\omega(H)}$ is a subgraph of $G\times H$. Thus, $\tau_2(K_{\omega(G)}\times K_{\omega(H)})\leq \tau_2(G\times H)$. On the other hand, we know $\Delta(G\times H)=\Delta(G)\Delta(H)$. So by Theorem 5, we know that $\left\lceil \frac{5+\sqrt{1+8\Delta(G)\Delta(H)}}{2}\right\rceil \leq \tau_2(G\times H)$. Therefore,

$$\max\left\{\left\lceil \frac{5+\sqrt{1+8\Delta(G)\Delta(H)}}{2}\right\rceil, \tau_2(K_{\omega(G)}\times K_{\omega(H)})\right\}\leq \tau_2(G\times H).$$

■

It should be noted that there exist graphs G and H such that the upper bound in Theorem 6 is better than applying Theorem 2. For example, consider the graph $P_3\times P_4$ in Figure 8. One can easily verify that the labeling shown in Figure 8 is in fact a 2-tone coloring. Thus, $\tau_2(P_3 \times P_4) \leq 6$, which is an improvement from the bound given in Theorem 2 of

$$\begin{aligned}\tau_2(P_3 \times P_4) &\leq \left\lceil (2+\sqrt{(2)})\Delta(P_3 \times P_4) \right\rceil \\ &= \left\lceil (2+\sqrt{2})4 \right\rceil = 14.\end{aligned}$$

3. Cartesian Product

We now focus on the Cartesian product of two graphs. Recall that the Cartesian product $G \square H$ has vertex set $V(G \square H) = V(G) \times V(H)$, whereby two vertices (u_1, u_2) and (v_1, v_2) are adjacent if $u_1v_1 \in E(G)$ and $u_2 = v_2$, or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

In this particular product, we have an obvious lower bound for the 2-tone chromatic number.

Proposition 7. *Given two graphs G and H,*

$$\max\{\tau_2(G), \tau_2(H)\} \leq \tau_2(G \square H).$$

Proof. This follows from the fact that G and H are both subgraphs of $G \square H$. ■

In terms of an upper bound, it is stated in [5] that $\tau_2(G \square H) \leq \tau_2(G)\tau_2(H)$, but that this bound could be improved. We give an upper bound for $\tau_2(G \square H)$ in terms of $\max\{\chi(G^2), \chi(H^2)\}$ depending on the parity of this value.

Theorem 8. *Given two graphs G and H where* $\max\{\chi(G^2), \chi(H^2)\} = \chi(G^2)$,

$$\tau_2(G \ \square \ H) \leq \begin{cases} 2\chi(G^2) & \text{if } \chi(G^2) \text{ is odd,} \\ 2(\chi(G^2)+1) & \text{otherwise.} \end{cases}$$

Proof. If $\chi(G^2)$ is an even integer, then we let $k = \chi(G^2) + 1$. Otherwise, we will let $k = \chi(G^2)$. Let $f_1 : V(G) \mapsto [k]$ be a proper distance $(2, k)$-coloring of G, and let $f_2 : V(H) \mapsto [k]$ be a proper distance $(2, k)$-coloring of H.

Define $g : V(G \square H) \mapsto \mathcal{P}_2([2k])$ such that

$$(x, y) \mapsto \big\{f_1(x) + f_2(y) \pmod k, (f_2(y) - f_1(x) \pmod k)k) + k\big\}.$$

Figure 9 depicts a labeling of $V(P_3 \square P_3)$ assigned by g. We will first show that g assigns two distinct colors to each vertex of $G \square H$. Let $(x, y) \in V(G \square H)$ and write $g((x, y)) = \{a, b\}$. Since $a = f_1(x) + f_2(y) \pmod k$, it follows that $a \in [k]$.

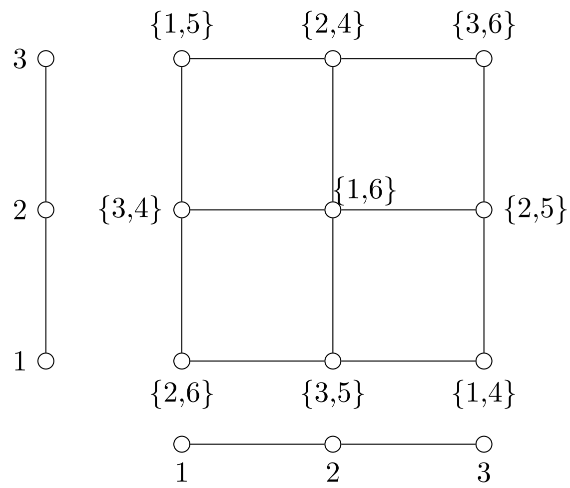

Figure 9. A 2-tone coloring of $P_3 \square P_3$.

On the other hand, $b = (f_2(y) - f_1(x) \pmod{k}) + k$. So $b \in \{k+1, \ldots, 2k\}$, which implies $|g((x, y))| = 2$.

Next, we show that g satisfies the distance criteria for 2-tone colorings. Let (u, v) and (w, z) be two distinct vertices of $V(G \square H)$.

Case 1. Suppose that $d_{G\square H}((u, v), (w, z)) = 1$. Then either $u = w$ and $d_H(v, z) = 1$ or $v = z$ and $d_G(u, w) = 1$. If $u = w$ and $d_H(v, z) = 1$, then we know that $f_1(u) = f_1(w)$ and $f_2(v) \neq f_2(z)$. This implies that

$$f_1(u) + f_2(v) \not\equiv f_1(w) + f_2(z) \pmod{k}.$$

Moreover,

$$f_2(v) - f_1(u) \not\equiv f_2(z) - f_1(w) \pmod{k},$$

which implies

$$(f_2(v) - f_1(u) \pmod{k}) + k \neq (f_2(z) - f_1(w) \pmod{k}) + k.$$

So $|g((u, v)) \cap g((w, z))| = 0$. A similar argument shows that $|g((u, v)) \cap g((w, z))| = 0$ if $v = z$ and $d_G(u, w) = 1$.

Case 2. Suppose that $d_{G\square H}((u, v), (w, z)) = 2$. Then exactly one of the following will be true:

(a) $u = w$ and $d_H(v, z) = 2$,

(b) $v = z$ and $d_G(u, w) = 2$,

(c) $d_G(u, w) = 1$ and $d_H(v, z) = 1$.

In the case of either (a) or (b), a similar argument as in Case 1 shows $|g((u,v)) \cap g((w,z))| = 0$. So assume $d_G(u,w) = 1$ and $d_H(v,z) = 1$. It follows that $f_1(u) \neq f_1(w)$ and $f_2(v) \neq f_2(z)$. If $|g((u,v)) \cap g((w,z))| \leq 1$, we are done. So suppose that $g((u,v)) = g((w,z))$. Thus,

$$(f_2(v) - f_1(u) \pmod{k}) + k = (f_2(z) - f_1(w) \pmod{k}) + k,$$

or equivalently $f_2(v) - f_1(u) \equiv f_2(z) - f_1(w) \pmod{k}$. Rearranging terms gives

$$f_2(v) - f_2(z) \equiv f_1(u) - f_1(w) \pmod{k}. \tag{1}$$

On the other hand, we have

$$f_1(u) + f_2(v) \equiv f_1(w) + f_2(z) \pmod{k},$$

which implies

$$f_1(u) - f_1(w) \equiv f_2(z) - f_2(v) \pmod{k}. \tag{2}$$

Combining (1) and (2), we have

$$f_2(v) - f_1(z) \equiv f_2(z) - f_2(v) \pmod{k},$$

which implies $2f_2(v) \equiv 2f_2(z) \pmod{k}$. However, this cannot happen since $f_2(v) \not\equiv f_2(z) \pmod{k}$ and $\gcd(2,k) = 1$. Thus, $|g((u,v)) \cap g((w,z))| \leq 1$. ■

Although the upper bound in Theorem 8 does not involve $\tau_2(G)$ or $\tau_2(H)$, it should be noted that there are graphs for which the upper bound is best possible. For example, consider the graph $P_3 \square P_3$. By Theorem 8, we know that $\tau_2(P_3 \square P_3) \leq 2\chi(P_3^2) = 6$. On the other hand, $P_3 \square P_3$ contains a cycle of length 4. Since $\tau_2(C_4) = 6$, it follows that $\tau_2(P_3 \square P_3) = 6$.

4. STRONG PRODUCT

The last graph product that we consider is the strong product $G \boxtimes H$. Recall that the strong product $G \boxtimes H$ has vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and edge set $E(G \boxtimes H) = E(G \square H) \cup E(G \times H)$.

Using similar ideas to those found in Sections 2 and 3, we have the following upper and lower bounds for $\tau_2(G \boxtimes H)$.

Theorem 9. *Given two graphs G and H,*

$$\max\{\tau_2(G \times H), \tau_2(G \square H)\} \leq \tau_2(G \boxtimes H) \leq \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\}.$$

Proof. Note that $G\square H$ and $G \times H$ are both subgraphs of $G \boxtimes H$. Thus, $\max\{\tau_2(G\square H), \tau_2(G \times H)\} \leq \tau_2(G \boxtimes H)$.

Next, we will prove that $\tau_2(G \boxtimes H) \leq \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\}$. Without loss of generality, we may assume $\tau_2(G)\chi(H^2) \leq \chi(G^2)\tau_2(H)$. Let f_1 be a proper 2-tone coloring of G using the colors $\{1, 2, \ldots, \tau_2(G)\}$. Let f_2 be a proper distance $(2, k)$-coloring of H using the colors $\{1, \tau_2(G) + 1, 2\tau_2(G) + 1, \ldots, (k-1)\tau_2(G) + 1\}$ where $k = \chi(H^2)$.

Define $g : V(G \boxtimes H) \to \mathcal{P}_2([k\tau_2(G)])$ such that for each $(x, y) \in V(G \boxtimes H)$ and for each $c \in f_1(x)$, we have $c + f_2(y) \in g((x, y))$. We show that g is a proper 2-tone coloring of $G \boxtimes H$. Let (u, v) and (w, z) be vertices of $V(G \boxtimes H)$.

Case 1. Assume that $d_{G\boxtimes H}((u, v), (w, z)) = 1$. By definition of the strong product, exactly one of the following will be true:

(a) $d_G(u, w) = 1$ and $v = z$,

(b) $u = w$ and $d_H(v, z) = 1$,

(c) $d_G(u, w) = 1$ and $d_H(v, z) = 1$.

We show that $|g((u, v)) \cap g((w, z))| = 0$ in each of the above cases.

(a) Assume $d_G(u, w) = 1$ and $v = z$. Since f_1 is a proper 2-tone coloring of G, $f_1(u) \cap f_1(w) = \emptyset$. Thus, we can write $f_1(u) = \{c_1, c_2\}$ and $f_1(w) = \{c_3, c_4\}$ where $c_i \neq c_j$ for $1 \leq i < j \leq 4$. Since $f_2(v) = f_2(z)$, we know for $i \in \{1, 2\}$ and $j \in \{3, 4\}$ that $c_i + f_2(v) \neq c_j + f_2(z)$. Therefore, $|g((u, v)) \cap g((w, z))| = 0$.

(b) Assume $u = w$ and $d_H(v, z) = 1$. Since f_2 is a proper distance $(2, k)$-coloring of H, $f_2(v) \neq f_2(z)$ and we may write $f_2(v) = i\tau_2(G) + 1$ and $f_2(z) = j\tau_2(G) + 1$ for some $0 \leq i < j \leq k - 1$. Let $f_1(u) = \{c_1, c_2\}$ where $c_1 \neq c_2$. Thus,

$$g((u, v)) = \{c_1 + i\tau_2(G) + 1, c_2 + i\tau_2(G) + 1\}$$

and

$$g((w, z)) = \{c_1 + j\tau_2(G) + 1, c_2 + j\tau_2(G) + 1\}.$$

It is clear that $c_1 + i\tau_2(G) + 1 \neq c_1 + j\tau_2(G) + 1$ since $i < j$. Similarly, $c_2 + i\tau_2(G) + 1 \neq c_2 + j\tau_2(G) + 1$. Note that if $c_1 + i\tau_2(G) + 1 = c_2 + j\tau_2(G) + 1$, then

$$c_1 - c_2 = (j - i)\tau_2(G).$$

We know that $c_1 - c_2 \neq 0$ since $i < j$. On the other hand, $c_1 - c_2$ cannot be a multiple of $\tau_2(G)$ since $1 \leq c_1, c_2 \leq \tau_2(G)$. Therefore,

$$c_1 + i\tau_2(G) + 1 \neq c_2 + j\tau_2(G) + 1,$$

and a similar argument shows that

$$c_2 + i\tau_2(G) + 1 \neq c_1 + j\tau_2(G) + 1.$$

Thus, $|g((u,v)) \cap g((w,z))| = 0$.

(c) Assume $d_G(u,w) = 1$ and $d_H(v,z) = 1$. It follows that $f_1(u) \cap f_1(w) = \emptyset$ and $f_2(v) \neq f_2(z)$. As before, let $f_1(u) = \{c_1, c_2\}$ and $f_1(w) = \{c_3, c_4\}$ where $c_a \neq c_b$ when $1 \leq a < b \leq 4$. Also, write $f_2(v) = i\tau_2(G)+1$ and $f_2(z) = j\tau_2(G)+1$ for some $0 \leq i < j \leq k-1$. Thus,

$$g((u,v)) = \{c_1 + i\tau_2(G) + 1, c_2 + i\tau_2(G) + 1\}$$

and

$$g((w,z)) = \{c_3 + j\tau_2(G) + 1, c_4 + j\tau_2(G) + 1\}.$$

Again, we see that $c_1 + i\tau_2(G) + 1 \neq c_2 + i\tau_2(G) + 1$ since $c_1 \neq c_2$. Similarly, $c_3 + j\tau_2(G) + 1 \neq c_4 + j\tau_2(G) + 1$ since $c_3 \neq c_4$. Furthermore, for any $a \in \{1,2\}$ and $b \in \{3,4\}$, we know

$$c_a + i\tau_2(G) + 1 \neq c_b + j\tau_2(G) + 1$$

since $c_a - c_b$ cannot be a multiple of $\tau_2(G)$. Therefore, $|g((u,v)) \cap g((w,z))| = 0$.

Case 2. Assume that $d_{G \boxtimes H}((u,v),(w,z)) = 2$. Necessarily, $d_G(u,w) \leq 2$ and $d_H(v,z) \leq 2$. Thus, $|f_1(u) \cap f_1(w)| \leq 1$ so there exist $a \in f_1(u)$ and $b \in f_1(w)$ such that $a \neq b$. Furthermore, since $d_H(v,z) \leq 2$, we may assume there exist $0 \leq i < j \leq k-1$ such that $f_2(v) = i\tau_2(G)+1$ and $f_2(z) = j\tau_2(G)+1$. We have already seen that this implies $a + i\tau_2(G) + 1 \neq b + j\tau_2(G) + 1$ since $i < j$ and $1 \leq a, b \leq \tau_2(G)$. Therefore, $|g((u,v)) \cap g((w,z))| \leq 1$. ■

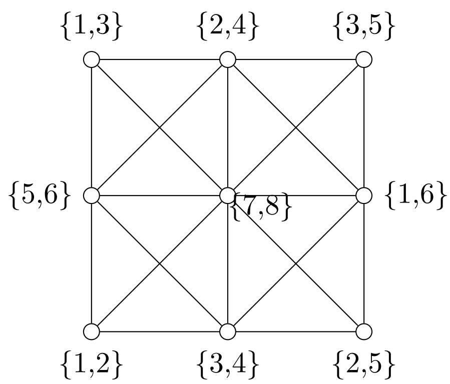

Figure 10. $P_3 \boxtimes P_3$

Note that for $P_3 \boxtimes P_3$, we can find a 2-tone 8-coloring as shown in Figure 10. This coloring is best possible since $P_3 \boxtimes P_3$ contains K_4 and $\tau_2(K_4) = 8$. However, in this case Theorem 9 gives bounds of

$$\begin{aligned}5 = \max\{\tau_2(P_3 \square P_3), \tau_2(P_3 \times P_3)\} &\leq \tau_2(P_3 \boxtimes P_3) \\ &\leq \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\} = 15.\end{aligned}$$

This alone shows that perhaps an upper bound in terms of other graph parameters would be more useful. On the other hand, since $K_3 \boxtimes K_3 \cong K_9$, it follows that $\tau_2(K_3 \boxtimes K_3) = 18$. In this particular case, we have

$$\begin{aligned}6 = \min\{\tau_2(K_3 \square K_3), \tau_2(K_3 \times K_3)\} &\leq \tau_2(K_3 \boxtimes K_3) \\ &\leq \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\} = 18,\end{aligned}$$

which shows the upper bound in Theorem 9 is sharp.

Acknowledgements

The authors would like to thank the NSF for making this project possible in the Clemson REU through the grant NSF DMS-1156734. We would also like to thank Jim Brown for his contributions in proving Theorem 4, and general suggestions throughout this project. We also wish to thank Neil Calkin for several useful discussions and the anonymous referees for their suggestions and comments.

References

[1] A. Bickle and B. Phillips, *t-tone colorings of graphs*, submitted (2011).

[2] D. Cranston, J. Kim and W. Kinnersley, *New results in t-tone colorings of graphs*, Electron. J. Comb. **20(2)** (2013) #17.

[3] D. Bal, P. Bennett, A. Dudek and A. Frieze, *The t-tone chromatic number of random graphs*, Graphs Combin. **30** (2013) 1073–1086. doi:10.1007/s00373-013-1341-9

[4] N. Fonger, J. Goss, B. Phillips and C. Segroves, *Math 6450: Final Report*, (2011). http://homepages.wmich.edu/~zhang/finalReport2.pdf

[5] D. West, REGS in Combinatorics. *t*-tone colorings, (2011). http://www.math.uiuc.edu/~west/regs/ttone.html

[6] R. Hammack, W. Imrich and S. Klavžar, Handbook of Product Graphs, Second Edition (CRC Press, Boca Raton, 2011).

[7] S. Krumke, M. Marathe and S. Ravi, *Approximation algorithms for channel assignment in radio networks*, Wireless Networks **7** (2001) 575–584. doi:10.1023/A:1012311216333

[8] V. Vazirani, Approximation Algorithms (Springer, 2001).

Received 13 September 2013
Revised 27 January 2014
Accepted 31 January 2014