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# 2-TONE COLORINGS IN GRAPH PRODUCTS

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**Abstract**

A variation of graph coloring known as a $t$-tone $k$-coloring assigns a set of $t$ colors to each vertex of a graph from the set $\{1, \ldots, k\}$, where the sets of colors assigned to any two vertices distance $d$ apart share fewer than $d$ colors in common. The minimum integer $k$ such that a graph $G$ has a $t$-tone $k$-coloring is known as the $t$-tone chromatic number. We study the 2-tone chromatic number in three different graph products. In particular, given graphs $G$ and $H$, we bound the 2-tone chromatic number for the direct product $G \times H$, the Cartesian product $G \square H$, and the strong product $G \boxtimes H$.

**Keywords:** $t$-tone coloring, Cartesian product, direct product, strong product.

**2010 Mathematics Subject Classification:** 05C15, 05C76.

## 1. Introduction

Many variations of classic graph $k$-colorings abound, whereby we take a $k$-coloring of a graph to mean an assignment of an element from $\{1, \ldots, k\}$, called a color, to each of the vertices of the graph. Chartrand was the first to introduce a $t$-tone $k$-coloring [4], which is an assignment of $t$ elements from the set $\{1, \ldots, k\}$ to each vertex such that the sets of colors assigned to any two distinct vertices within distance $d$ share fewer than $d$ colors. This $t$-tone $k$-coloring variation can be viewed as a generalization of classic graph coloring since a 1-tone $k$-coloring of a graph is simply a $k$-coloring.

For the purpose of this paper, we consider only simple, undirected graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. For each vertex $v \in V(G)$, $\deg_G(v)$ denotes the number of vertices adjacent to $v$ and the maximum degree of $G$ is defined to be $\Delta(G) = \max_{v \in V(G)} \deg_G(v)$. The distance between two vertices $u$ and $v$ of $V(G)$ is the size of the shortest length path between $u$ and $v$, and is denoted by $d_G(u, v)$. When the context is clear, we use the shorthand notation $d(u, v)$.

As stated above, a proper $k$-coloring of a graph $G$ is an assignment of an element from $\{1, \ldots, k\}$, called a *color*, to each vertex in $V(G)$ such that no two adjacent vertices are assigned the same color. The chromatic number of $G$, denoted $\chi(G)$, is the minimum number $k$ such that $G$ has a proper $k$-coloring. We use $K_n$ to denote the complete graph on $n$ vertices. Given a graph $G$, a clique is any complete subgraph of $G$, and the clique number of $G$, denoted $\omega(G)$, is the cardinality of the maximum clique of $G$. For positive integers $t$ and $k$ where $t \leq k$, we let $[k]$ represent the set $\{1, \ldots, k\}$ and denote the family of $t$-element subsets of $[k]$ by $\mathcal{P}_t([k])$. The following is a formal definition of the $t$-tone chromatic number of a graph.

**Definition.** Let $G$ be a graph, and let $t$ and $k$ be positive integers such that $t \leq k$. A *$t$-tone $k$-coloring of $G$* is a function $f : V(G) \to \mathcal{P}_t([k])$ such that $|f(u) \cap f(v)| < d_G(u, v)$ for all distinct vertices $u$ and $v$. A graph that has a $t$-tone $k$-coloring is said to be *$t$-tone $k$-colorable*. The *$t$-tone chromatic number of $G$*, denoted $\tau_t(G)$, is the minimum integer $k$ such that $G$ is $t$-tone $k$-colorable.

Figure 1 depicts a 2-tone 5-coloring of $P_5$ which is, indeed, minimum. Given a $t$-tone $k$-coloring $f$ of $G$, we call $f(v)$ the *label* of $v$ and the elements of $[k]$ *colors*.

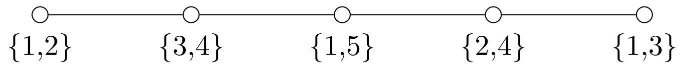


Figure 1. A 2-tone coloring of $P_5$.

Given a graph $G$, a proper distance $(d,k)$-coloring of $G$ is a map $f : V(G) \to [k]$ such that for any two distinct vertices $u$ and $v$ of $V(G)$ with $d_G(u,v) \leq d$, we have $f(u) \neq f(v)$. A $t$-tone $k$-coloring can also be viewed as a generalization of a $(d,k)$-coloring in that both incorporate similar conditions based on the distance between vertices. Applications of $(2,k)$-colorings include channel assignment, or broadcast scheduling, for packet radio networks [7] and facility location problems [8].

Recall that the square of $G$, denoted $G^2$, is the graph with $V(G^2) = V(G)$ and edge set $E(G^2) = \{\ uv :\ d_G(u,v) \leq 2\ \}$. We call the reader's attention to the fact that any proper $k$-coloring of $G^2$ is a proper distance $(2,k)$-coloring of $G$, and vice versa. Therefore, we use $\chi(G^2)$ to denote the smallest integer $k$ such that $G$ has a proper distance $(2,k)$-coloring. Fonger *et al.* [4] (p. 11) noted that the relationship between $\chi(G^2)$ and $\tau_2(G)$ can at times seem counterintuitive. For instance, one can show that $\chi(P_5^2) = 3 < 5 = \tau_2(P_5)$, but that $\tau_2(G) < \chi(G^2)$ when $G$ is the Petersen graph. However, the following was shown to be true for any graph $G$.

**Theorem 1** [4]**.** *Given any graph $G$, $\tau_2(G) \leq \chi(G) + \chi(G^2)$.*

Although a better general upper bound for $\tau_2(G)$ exists, the relationship between $\chi(G^2)$ and $\tau_2(G)$ found in Theorem 1 will be useful for our results. In 2011, Bickle and Phillips [1] gave general bounds for the $t$-tone chromatic number of a graph $G$ in terms of $\Delta(G)$. Shortly thereafter, Cranston, Kim, and Kinnersley [2] (p. 3) gave the following upper bound, which we will refer to in subsequent sections.

**Theorem 2** [2]**.** *For any graph $G$, $\tau_2(G) \leq \lceil (2+\sqrt{2})\Delta(G) \rceil$.*

In addition to the above, Bal *et al.* recently studied the $t$-tone chromatic number of random graphs [3]. We now shift our focus to $t$-tone colorings in graph products. Recall the definition of the direct product of two graphs.

**Definition.** Given two graphs $G$ and $H$, the *direct product* of $G$ and $H$, denoted $G \times H$, is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$, and whose edge set is

$$E(G \times H) = \{(x_1,y_1)(x_2,y_2) : x_1x_2 \in E(G) \text{ and } y_1y_2 \in E(H)\}.$$

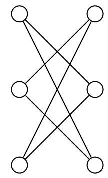

Figure 2. $K_2 \times K_3$.

Figure 2 depicts the direct product of $K_2$ and $K_3$. In Section 2, we use similar proof techniques to those used in [4](p. 6) to determine the exact value of $\tau_2(K_m \times K_n)$, and we give general upper and lower bounds for $\tau_2(G \times H)$.

Next, recall the definition of the Cartesian product.

**Definition.** The *Cartesian product* of graphs $G$ and $H$, denoted $G \Box H$, is the graph whose vertex set is $V(G) \times V(H)$, whereby two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if $u_1 v_1 \in E(G)$ and $u_2 = v_2$, or $u_1 = v_1$ and $u_2 v_2 \in E(H)$.

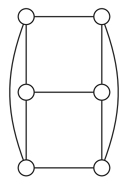

Figure 3. $K_2 \Box K_3$.

Figure 3 depicts the Cartesian product of $K_2$ and $K_3$. As mentioned in [5], a straight-forward argument shows that $\tau_t(G \Box H) \leq \tau_t(G)\tau_t(H)$, but that this bound can be improved. Focusing only on the case when $t = 2$, in Section 3 we give an upper bound based on the value of $\max\{\chi(G^2), \chi(H^2)\}$.

Finally, recall the definition of the strong product of graphs.

**Definition.** The *strong product* of graphs $G$ and $H$, denoted $G \boxtimes H$, is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edge set is given by $E(G \boxtimes H) = E(G \Box H) \cup E(G \times H)$.

Figure 4 depicts the strong product of $K_2$ and $K_3$. In Section 4, we show that $\tau_2(G \boxtimes H) \leq \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\}$ using similar techniques and results for the direct product and the Cartesian product.

## 2. Direct Product

In this section, we focus on the direct product of two graphs, whose definition we restate for ease of reference. The direct product of two graphs $G$ and $H$

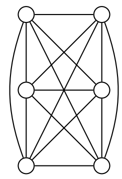

Figure 4. $K_2 \boxtimes K_3$.

is denoted $G \times H$ with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = \{(x_1, y_1)(x_2, y_2) :\ x_1x_2 \in E(G) \text{ and } y_1y_2 \in E(H)\}$.

Throughout this section, when we consider the direct product $G \times H$, where $|V(G)| = m$ and $|V(H)| = n$, we will represent the vertices of $G$ as $x_1, \ldots, x_m$ and the vertices of $H$ as $y_1, \ldots, y_n$. Using this notation, for each $i \in [m]$ we define the *column* $C_i$ as the set of all vertices with first coordinate $x_i$. In particular, for $i \in [m]$, the $i^{th}$ column is given by $C_i = \{(x_i, y_j) : j \in [n]\}$. Similarly, for $j \in [n]$, the $j^{th}$ *row* is the set $R_j = \{(x_i, y_j) : i \in [m]\}$.

In order to find an upper and lower bound of the 2-tone chromatic number of the direct product of any two graphs $G$ and $H$, we first consider the direct product of two complete graphs. By definition of the direct product, we know for $m, n \in \mathbb{N}$ such that $2 \leq m \leq n$,

$$V(K_m \times K_n) = \{(x_i, y_k) : i \in [m] \text{ and } k \in [n]\},$$

and

$$E(K_m \times K_n) = \{(x_i, y_k)(x_j, y_\ell) :\ i \neq j \text{ and } k \neq \ell\}.$$

The following is a direct consequence of the distance formula for the direct product found in [6](p. 54)

**Proposition 3.** *Let $m, n \in \mathbb{N}$ such that $m \geq 2$ and $n \geq 3$. If $u$ and $v$ are any two distinct vertices of $V(K_m \times K_n)$ that are contained within the same column, then $d(u, v) = 2$.*

Recall from Section 1 that given a graph $G$ and a $t$-tone $k$-coloring $f$ of $G$, we call $f(v)$ the label of $v$ and the elements of $[k]$ colors. Additionally, for any set of vertices $A \subseteq V(G)$, we define the *set of colors contained in the labels associated with* $A$ to be

$$c(A) = \{c \in [k] :\ c \in f(v) \text{ for some } v \in A\}.$$

**Theorem 4.** *If $m, n \in \mathbb{N}$, where $2 \leq m \leq n$ and $t = \frac{1+\sqrt{1+8n}}{2}$, then*

$$\tau_2(K_m \times K_n) = \min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor - 1)}{2}\right\}.$$

***Proof.*** First consider the case where $m = n = 2$. Since $K_2 \times K_2 \cong 2K_2$,

$$\tau_2(K_2 \times K_2) = \tau_2(K_2) = 4.$$

One can easily verify that this is the minimum of the three functions.

Now consider all other cases where $m \geq 2, n \geq m$ and $n \neq 2$. Let $f$ be a minimum 2-tone $k$-coloring of $K_m \times K_n$. For each $1 \leq i \leq m$, define $A_i$ as the set of all vertices $v \in C_i$ such that for each $a \in f(v)$, there exists a vertex $w \in C_i$ with $w \neq v$ and $a \in f(w)$. Note the following property of this subset $A_i \subseteq C_i$. Fix $i \in [m]$ and let $v \in A_i$ as described above; that is, for $a \in f(v)$ there exists $w \in C_i$ with $a \in f(w)$. This implies that for all $1 \leq j \leq m$ such that $j \neq i$ and for any $u \in C_j$, $a \notin f(u)$ since $u$ is adjacent to at least one of $v$ or $w$. Therefore, the set of colors contained in the labels associated with $A_i$ is disjoint from the set of colors contained in the labels associated with $C_j$; that is, $c(A_i) \cap c(C_j) = \emptyset$.

For each $1 \leq i \leq m$, let $s_i = |c(A_i)|$. Let $s_\ell = \min_{1 \leq i \leq m} s_i$ for some $\ell \in [m]$. Thus, the number of distinct colors contained in the labels associated with $\cup_{i=1}^m A_i$ is at least $ms_\ell$. By definition, for each $(x_\ell, y_j) \in C_\ell \backslash A_\ell$, there exists a color $a \in f((x_\ell, y_j))$ such that $a$ is not contained in any other label associated with $C_\ell$. Furthermore, if for some $1 \leq i \leq m$ where $i \neq \ell$ we have $a \in c(A_i)$, then there would exist $v \in A_i$ and $w \in C_i$ such that $a \in f(v) \cap f(w)$. However, this would contradict the fact that $f$ is a proper 2-tone coloring since one of $v$ or $w$ is adjacent to $(x_\ell, y_j)$. Thus, for each $1 \leq i \leq m$, $a \notin c(A_i)$. It follows that $k \geq ms_\ell + |C_\ell \backslash A_\ell|$. We now determine the minimum $k$ based on the value of $|C_\ell \backslash A_\ell|$. We do this by considering the following three cases.

*Case* 1. Assume that $|C_\ell \backslash A_\ell| = n$. Thus, $A_\ell = \emptyset$ and $s_\ell = 0$. Let $Q = \{C_i : \ i \in [m] \text{ and } s_i = 0\}$ and $T = \{C_i : i \in [m] \text{ and } s_i > 0\}$, where $|Q| = q$ and $|T| = t$. Note that $q + t = m$. Since $s_\ell = 0$, we know $t < m$ or equivalently $t + 1 \leq m$. For indexing purposes, we shall write $Q = \{C_{\alpha(1)}, \ldots, C_{\alpha(q)}\}$, where $\alpha(i) \in [m]$ for $1 \leq i \leq q$. Since $q = m - t \leq m \leq n$, there exists a set $W = \{v_{\alpha(1)}, \ldots, v_{\alpha(q)}\}$ such that $v_{\alpha(i)} \in C_{\alpha(i)}$ for each $\alpha(i)$, and if $\alpha(i) \neq \alpha(j)$, then $v_{\alpha(i)}$ and $v_{\alpha(j)}$ are in different rows. Notice that the induced subgraph of $W$ is a clique so that $|f(v_{\alpha(i)}) \cap f(v_{\alpha(j)})| = 0$ when $\alpha(i) \neq \alpha(j)$. Define, $B = \{(x_i, y_j) \in C_i : \ C_i \in Q \text{ and } R_j \cap W = \emptyset\}$.

Note that there exist at least $n - q$ colors that are contained in $c(B)$ which are not contained in $c(W)$. Thus, $|c(B) \cup c(W)| \geq 2q + n - q = n + q$. Finally, for each column $C_i \in T$, we know that $s_i \geq 2$. Thus,

$$\begin{aligned} k &\geq n + q + 2t \\ &= m + n + t \\ &\geq m + n. \end{aligned}$$

*Case* 2. Assume that $|C_\ell \backslash A_\ell| = 0$. It follows that $A_\ell = C_\ell$ and $|A_\ell| = n$. Furthermore, since $s_\ell$ represents the number of distinct colors contained in the labels associated with $A_\ell$, we know that $s_\ell \geq 2$. Since any two distinct vertices $u, v \in A_\ell$ satisfy $d(u,v) = 2$, we know that $\binom{s_\ell}{2} \geq n$. Using the quadratic formula, this implies that $s_\ell \geq \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$. Consequently, $k \geq m \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$.

*Case* 3. Assume that $n > |C_\ell \backslash A_\ell| > 0$. If $\binom{s_\ell}{2} > n$, then clearly $k \geq m \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$. So assume $\binom{s_\ell}{2} \leq n$, or equivalently $2 \leq s_\ell \leq \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor$. We have $n = |A_\ell| + |C_\ell \backslash A_\ell|$, which implies $|C_\ell \backslash A_\ell| = n - |A_\ell|$. As in Case 2, we know $\binom{s_\ell}{2} \geq |A_\ell|$. Thus, $n - \binom{s_\ell}{2} \leq n - |A_\ell|$, which implies $n - \binom{s_\ell}{2} \leq |C_\ell \backslash A_\ell|$.

Therefore,

$$ms_\ell + |C_\ell \backslash A_\ell| \geq ms_\ell + n - \binom{s_\ell}{2} = ms_\ell + n - \frac{s_\ell(s_\ell - 1)}{2}.$$

So we consider the function $g(s) = ms + n - \frac{s(s-1)}{2}$ over the interval $2 \leq s \leq \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor$. One can easily verify that $g'(s) = m - s + \frac{1}{2}$ and $g''(s) = -1$. Thus, $g$ is concave down for all values of $2 \leq s \leq \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor$, and over this interval $g$ has a local maximum when $s = m + \frac{1}{2}$. Therefore, the local minimums for $g$ occur when $s = 2$ and $s = \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor$. Letting $t = \frac{1+\sqrt{1+8n}}{2}$, it follows that

$$\begin{aligned}
k &\geq ms_\ell + |C_\ell \backslash A_\ell| \\
&\geq \min_s ms + n - \frac{s(s-1)}{2} \\
&\quad \text{such that } 2 \leq s \leq \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor \\
&\geq \min \left\{ 2m + n - 1, m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2} \right\}.
\end{aligned}$$

Since $2m + n - 1 > m + n$, we may conclude that

$$k \geq \min \left\{ m + n, m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2} \right\}.$$

Note that these cases sometimes overlap. For example, $\binom{s}{2} = n$ implies that $\lfloor t \rfloor = \lceil t \rceil = t$ and $n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2} = 0$, resulting in $m\lceil t \rceil = m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2}$. In any case, we have

$$\tau_2(K_m \times K_n) \geq \min \left\{ m\lceil t \rceil, m + n, m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2} \right\}.$$

It remains to be shown that

$$\tau_2(K_m \times K_n) \leq \min \left\{ m\lceil t \rceil, m + n, m\lfloor t \rfloor + n - \frac{\lfloor t \rfloor(\lfloor t \rfloor - 1)}{2} \right\}.$$

Given $m, n \in \mathbb{N}$ where $2 \leq m \leq n$ and $t = \frac{1+\sqrt{1+8n}}{2}$, we construct different 2-tone colorings, which depend on the value of $\min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}\right\}$.

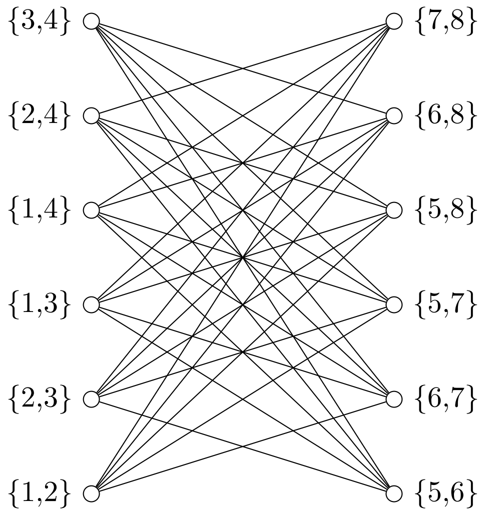


Figure 5. $K_2 \times K_6$.

*Case* 1. First, assume that

$$\min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}\right\} = m\lceil t\rceil.$$

Choose $m$ pairwise disjoint sets each containing $\lceil t\rceil$ distinct colors, and denote each set of colors $S_i$ for $1 \leq i \leq m$. Since $\binom{\lceil t\rceil}{2} \geq n$, for each $1 \leq i \leq m$ there exist $n$ distinct combinations containing two colors from the set $S_i$. Thus, we may define $f : V(K_m \times K_n) \to \mathcal{P}_2\left(\left[\lceil t\rceil\right]\right)$ to be any mapping such that for each $1 \leq i \leq m$ the restriction of $f$ to the set of vertices in $C_i$ is an injective mapping to the set of combinations containing two colors from the set $S_i$. Figure 5 illustrates a labeling of $V(K_2 \times K_6)$ assigned by $f$. To see that $f$ is a proper 2-tone coloring of $K_m \times K_n$, let $u$ and $v$ be distinct vertices of $V(K_m \times K_n)$. If $u$ and $v$ are not contained in the same column, then $f(u) \cap f(v) = \emptyset$. So assume $u, v \in C_i$ for some $i \in [m]$. We know by Proposition 3 that $d(u, v) = 2$. So we must show that $|f(u) \cap f(v)| \leq 1$. However, this follows from the fact that $f$ does not assign any label to more than one vertex of $C_i$. Therefore, $f$ is a proper 2-tone coloring.

*Case* 2. Next, assume that $\min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}\right\} = m+n$. Let $f_1$ be a proper coloring of $K_m$ and $f_2$ be a proper coloring of $K_n$ defined as follows:

$$\begin{aligned} f_1 : V(K_m) &\to \{1, \ldots, m\} \\ x_i &\mapsto i \\ f_2 : V(K_n) &\to \{m+1, \ldots, m+n\} \\ y_j &\mapsto m + j. \end{aligned}$$

Define the following function on $V(K_m \times K_n)$:

$$\begin{aligned} g : V(K_m \times K_n) &\to \mathcal{P}_2([m+n]) \\ (x_i, y_j) &\mapsto \{f_1(x_i), f_2(y_j)\}. \end{aligned}$$

Figure 6 illustrates a labeling of $V(K_2 \times K_3)$ assigned by $g$.

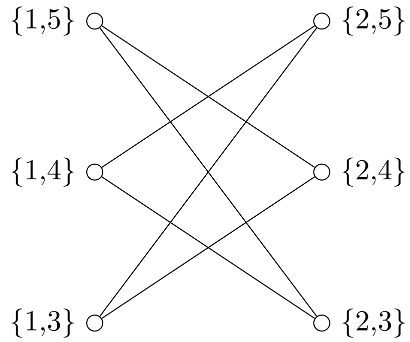


Figure 6. $K_2 \times K_3$.

We claim $g$ is a proper 2-tone coloring of $K_m \times K_n$. Clearly, $|g((x, y))| = 2$ for all $(x, y) \in V(K_m \times K_n)$. Let $(x_i, y_k)$ and $(x_j, y_\ell)$ be two distinct vertices of $V(K_m \times K_n)$, where $1 \le i, j \le m$ and $1 \le k, \ell \le n$. Then $g((x_i, y_k)) = \{i, k+m\}$ and $g((x_j, y_\ell)) = \{j, \ell + m\}$. If $(x_i, y_k)$ and $(x_j, y_\ell)$ are adjacent, then $i \ne j$ and $k \ne \ell$. Thus, $|g((x_i, y_k)) \cap g((x_j, y_\ell))| = 0$. If $d((x_i, y_k), (x_j, y_\ell)) = 2$, then either $i \ne j$ or $k \ne \ell$. In any case, $|g((x_i, y_k)) \cap g((x_j, y_\ell))| \le 1$. Therefore, $g$ is a proper 2-tone coloring of $K_m \times K_n$, and we may conclude that $\tau_2(K_m \times K_n) \le m + n$.

*Case* 3. Assume that $\min\left\{m\lceil t\rceil, m+n, m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor - 1)}{2}\right\} = m\lfloor t\rfloor + n - \frac{\lfloor t\rfloor(\lfloor t\rfloor-1)}{2}$. Note that $t = \frac{1+\sqrt{1+8n}}{2}$ is the only positive solution to $\binom{t}{2} = n$. Therefore, $\lfloor t\rfloor$ satisfies $\binom{\lfloor t\rfloor}{2} \le n$. Let $s = \binom{\lfloor t\rfloor}{2}$ and consider the subgraph $H$ of $K_m \times K_n$ induced by the set $\{(x_i, y_j) :\ i \in [m],\ j \in [s]\}$. Thus, $H \cong K_m \times K_s$. As in Case 2, choose $m$ pairwise disjoint sets of $\lfloor t\rfloor$ distinct colors and denote each set $S_i$ for each $i \in [m]$. Define $f_1 : V(H) \to \mathcal{P}_2\left(\left[\lfloor t\rfloor\right]\right)$ to be any mapping such that for each $i \in [m]$, the restriction of $f_1$ to the set of vertices of $C_i$ is an injective mapping to the set of combinations containing two colors from the set $S_i$. A similar argument as in Case 2 can be used to show that $f_1$ is a proper 2-tone coloring of $H$.

Next, choose $n-s$ distinct colors each of which are not contained in the set $\cup_{i=1}^m S_i$, and label these colors $\{t_{s+1},\ldots,t_n\}$. Additionally, for each $i \in [m]$, choose one color from the set $S_i$ and call it $c_i$. Notice that $V(K_m \times K_n)\backslash V(H) = \{(x_i,y_j) :\ i \in [m], s+1 \le j \le n\}$. Define

$$\begin{aligned} f_2 : V(K_m \times K_n)\backslash V(H) &\to \mathcal{P}_2([m+n-s]) \\ (x_i,y_j) &\mapsto \{c_i,t_j\}. \end{aligned}$$

We claim that $f_2$ is a proper 2-tone coloring of $(K_m \times K_n)\backslash H$. To see this, let $(x_i,y_k)$ and $(x_j,y_\ell)$ be two distinct vertices of $V(K_m \times K_n)\backslash V(H)$ for some $1 \le i,j \le m$ and $s+1 \le k,\ell \le n$. If $(x_i,y_k)$ and $(x_j,y_\ell)$ are adjacent, then $i \ne j$ and $k \ne \ell$. Since $S_i$ and $S_j$ are two disjoint sets of colors, we know that $c_i \ne c_j$. Moreover, we know that $t_k \ne t_\ell$ since $k \ne \ell$. Thus, $|f_2((x_i,y_k)) \cap f_2((x_j,y_\ell))| = 0$. If $d((x_i,y_k),(x_j,y_\ell)) = 2$, then either $i \ne j$ or $k \ne \ell$. It follows that $|f_2((x_i,y_k)) \cap f_2((x_j,y_\ell))| \le 1$. Therefore, $f_2$ is a proper 2-tone coloring of $(K_m \times K_n)\backslash H$.

Now define $g : V(K_m \times K_n) \to \mathcal{P}_2([m\lfloor t\rfloor + n - s])$ such that

$$g(u) = \begin{cases} f_1(u) & \text{if } u \in V(H), \\ f_2(u) & \text{otherwise.} \end{cases}$$

Figure 7 illustrates a labeling of $V(K_2 \times K_{11})$ assigned by $g$. To see that $g$ is a proper 2-tone coloring, we only need to consider when $u \in V(H)$ and $v \notin V(H)$. Write $u = (x_i,y_k)$ and $v = (x_j,y_\ell)$ for some $i,j \in [m]$, $k \in [s]$, and $\ell \in \{s+1,\ldots,n\}$. By definition $g(v) = \{c_j,t_\ell\}$, and we know $t_\ell \notin g(u)$ since $u \in V(H)$. So if $u$ and $v$ are located in the same column, then $|g(u) \cap g(v)| \le 1$. If $u$ and $v$ are not located in the same column, then $i \ne j$ and $c_j \notin g(u)$ since $c_j \notin S_i$. It follows that $|g(u) \cap g(v)| = 0$. Therefore, $g$ is a proper 2-tone coloring of $K_m \times K_n$ using $m\lfloor t\rfloor + n - \binom{\lfloor t\rfloor}{2}$ colors. ■

Using similar ideas found in Theorem 4, we can bound the value of $\tau_2(G \times H)$ given any graphs $G$ and $H$. We make use of the following general lower bound given in [4] (p. 8).

**Theorem 5** [4]**.** *Let $G$ be a graph and let $\Delta(G) = d$. Then*

$$\tau_2(G) \ge \left\lceil \frac{\sqrt{8d+1}+5}{2} \right\rceil.$$

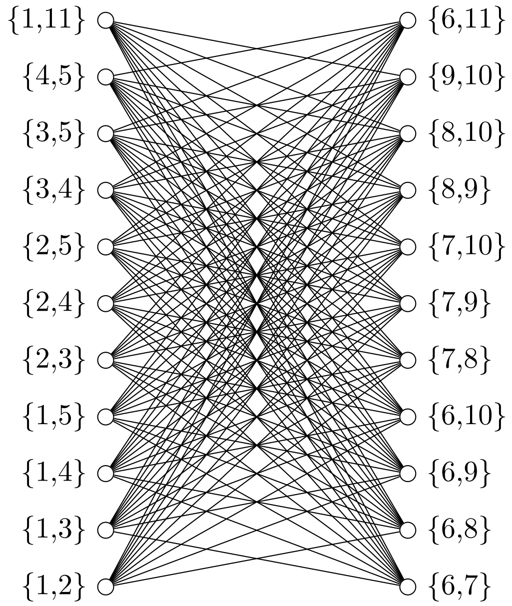


Figure 7. $K_2 \times K_{11}$.

**Theorem 6.** *Given two graphs $G$ and $H$,*

$$\max\left\{\left\lceil\frac{5+\sqrt{1+8\Delta(G)\Delta(H)}}{2}\right\rceil, \tau_2(K_{\omega(G)} \times K_{\omega(H)})\right\} \le \tau_2(G \times H)$$
$$\le \chi(G^2) + \chi(H^2).$$

***Proof.*** We first show that for any graphs $G$ and $H$, we have $\tau_2(G \times H) \le \chi(G^2) + \chi(H^2)$. Assume $\chi(G^2) = k_1$ and $\chi(H^2) = k_2$. Let $f_1 : V(G) \to [k_1]$ be a distance $(2, k_1)$-coloring of $G$, and let $f_2 : V(H) \to \{k_1 + 1, \ldots, k_1 + k_2\}$ be a distance $(2, k_2)$-coloring of $H$. Define

$$g : V(G \times H) \to \mathcal{P}_2([k_1 + k_2])$$

such that

$$(x, y) \mapsto \{f_1(x), f_2(y)\} \qquad \text{for all } x \in V(G) \text{ and } y \in V(H).$$

We claim that $g$ is a proper 2-tone coloring of $G \times H$. Clearly, $|g((x, y))| = 2$ for all $(x, y) \in V(G \times H)$. Let $(u, v)$ and $(w, z)$ be two distinct vertices of $V(G \times H)$. If $(u, v)$ and $(w, z)$ are adjacent, then $uw \in E(G)$ and $vz \in E(H)$. It follows that $f_1(u) \ne f_1(w)$ and $f_2(v) \ne f_2(z)$. Since $f_1$ is a mapping into the set $[k_1]$ and $f_2$ is a mapping into the set $\{k_1 + 1, \ldots, k_1 + k_2\}$, we have $|g((u, v)) \cap g((w, z))| = 0$.

Suppose that $d_{G\times H}((u,v),(w,z))=2$. If $u=w$, then $v \neq z$ and since $d_H(v,z) \leq 2$, it follows that $f_2(v) \neq f_2(z)$. Thus, $|g((u,v)) \cap g((w,z))| \leq 1$. Similarly, if $v = z$, then $|g((u,v)) \cap g((w,z))| \leq 1$. So we may assume that $u \neq w$ and $v \neq z$. There exists a vertex $(x,y) \in V(G\times H)$ such that $uxw$ is a path in $G$ and $vyz$ is a path in $H$. Since $d_G(u,w) \leq 2$ and $d_H(v,z) \leq 2$, we know that $f_1(u) \neq f_1(w)$ and $f_2(v) \neq f_2(z)$. Thus, $|g((u,v)) \cap g((w,z))| = 0$, and we may conclude that $g$ is a proper 2-tone coloring of $G \times H$, and $\tau_2(G \times H) \leq \chi(G^2) + \chi(H^2)$.

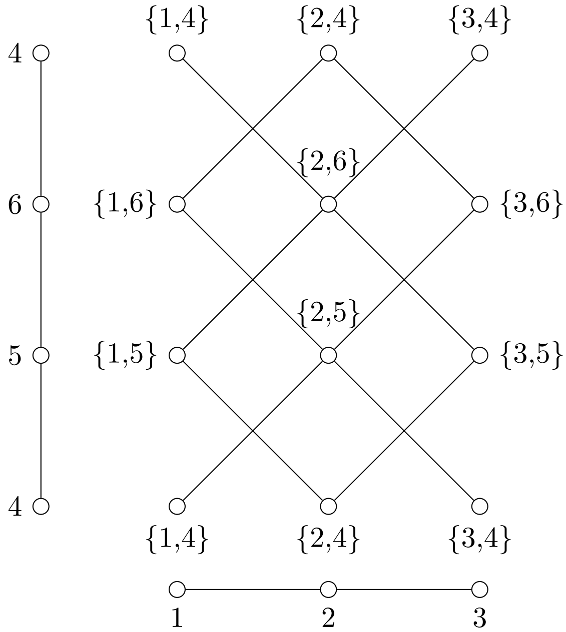


Figure 8. A 2-tone coloring of $P_3 \times P_4$.

In terms of a lower bound, note that by definition of the direct product, $K_{\omega(G)} \times K_{\omega(H)}$ is a subgraph of $G \times H$. Thus, $\tau_2(K_{\omega(G)} \times K_{\omega(H)}) \leq \tau_2(G \times H)$. On the other hand, we know $\Delta(G \times H) = \Delta(G)\Delta(H)$. So by Theorem 5, we know that $\left\lceil \frac{5+\sqrt{1+8\Delta(G)\Delta(H)}}{2} \right\rceil \leq \tau_2(G \times H)$. Therefore,

$$\max\left\{ \left\lceil \frac{5+\sqrt{1+8\Delta(G)\Delta(H)}}{2} \right\rceil, \tau_2(K_{\omega(G)} \times K_{\omega(H)}) \right\} \leq \tau_2(G \times H).$$

■

It should be noted that there exist graphs $G$ and $H$ such that the upper bound in Theorem 6 is better than applying Theorem 2. For example, consider the graph $P_3 \times P_4$ in Figure 8. One can easily verify that the labeling shown in Figure 8 is in fact a 2-tone coloring. Thus, $\tau_2(P_3 \times P_4) \leq 6$, which is an improvement from the bound given in Theorem 2 of

$$\begin{aligned}\tau_2(P_3 \times P_4) &\leq \left\lceil (2+\sqrt{(2)})\Delta(P_3 \times P_4) \right\rceil \\ &= \left\lceil (2+\sqrt{2})4 \right\rceil = 14.\end{aligned}$$

## 3. Cartesian Product

We now focus on the Cartesian product of two graphs. Recall that the Cartesian product $G \square H$ has vertex set $V(G \square H) = V(G) \times V(H)$, whereby two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if $u_1 v_1 \in E(G)$ and $u_2 = v_2$, or $u_1 = v_1$ and $u_2 v_2 \in E(H)$.

In this particular product, we have an obvious lower bound for the 2-tone chromatic number.

**Proposition 7.** *Given two graphs $G$ and $H$,*

$$\max\{\tau_2(G), \tau_2(H)\} \leq \tau_2(G \square H).$$

***Proof.*** This follows from the fact that $G$ and $H$ are both subgraphs of $G \square H$. ■

In terms of an upper bound, it is stated in [5] that $\tau_2(G \square H) \leq \tau_2(G)\tau_2(H)$, but that this bound could be improved. We give an upper bound for $\tau_2(G \square H)$ in terms of $\max\{\chi(G^2), \chi(H^2)\}$ depending on the parity of this value.

**Theorem 8.** *Given two graphs $G$ and $H$ where* $\max\{\chi(G^2), \chi(H^2)\} = \chi(G^2)$,

$$\tau_2(G \ \square \ H) \leq \begin{cases} 2\chi(G^2) & \text{if } \chi(G^2) \text{ is odd,} \\ 2(\chi(G^2)+1) & \text{otherwise.} \end{cases}$$

***Proof.*** If $\chi(G^2)$ is an even integer, then we let $k = \chi(G^2) + 1$. Otherwise, we will let $k = \chi(G^2)$. Let $f_1 : V(G) \mapsto [k]$ be a proper distance $(2, k)$-coloring of $G$, and let $f_2 : V(H) \mapsto [k]$ be a proper distance $(2, k)$-coloring of $H$.

Define $g : V(G \square H) \mapsto \mathcal{P}_2([2k])$ such that

$$(x, y) \mapsto \big\{f_1(x) + f_2(y) \pmod k, (f_2(y) - f_1(x) \pmod k)k) + k\big\}.$$

Figure 9 depicts a labeling of $V(P_3 \square P_3)$ assigned by $g$. We will first show that $g$ assigns two distinct colors to each vertex of $G \square H$. Let $(x, y) \in V(G \square H)$ and write $g((x, y)) = \{a, b\}$. Since $a = f_1(x) + f_2(y) \pmod k$, it follows that $a \in [k]$.

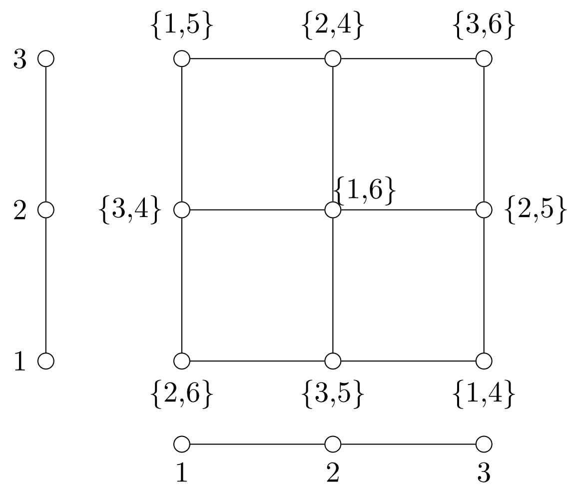


Figure 9. A 2-tone coloring of $P_3 \square P_3$.

On the other hand, $b = (f_2(y) - f_1(x) \pmod{k}) + k$. So $b \in \{k+1, \dots, 2k\}$, which implies $|g((x,y))| = 2$.

Next, we show that $g$ satisfies the distance criteria for 2-tone colorings. Let $(u,v)$ and $(w,z)$ be two distinct vertices of $V(G \square H)$.

*Case* 1. Suppose that $d_{G\square H}((u,v),(w,z)) = 1$. Then either $u = w$ and $d_H(v,z) = 1$ or $v = z$ and $d_G(u,w) = 1$. If $u = w$ and $d_H(v,z) = 1$, then we know that $f_1(u) = f_1(w)$ and $f_2(v) \neq f_2(z)$. This implies that

$$f_1(u) + f_2(v) \not\equiv f_1(w) + f_2(z) \pmod{k}.$$

Moreover,

$$f_2(v) - f_1(u) \not\equiv f_2(z) - f_1(w) \pmod{k},$$

which implies

$$(f_2(v) - f_1(u) \pmod{k}) + k \neq (f_2(z) - f_1(w) \pmod{k}) + k.$$

So $|g((u,v)) \cap g((w,z))| = 0$. A similar argument shows that $|g((u,v)) \cap g((w,z))| = 0$ if $v = z$ and $d_G(u,w) = 1$.

*Case* 2. Suppose that $d_{G\square H}((u,v),(w,z)) = 2$. Then exactly one of the following will be true:

(a) $u = w$ and $d_H(v,z) = 2$,

(b) $v = z$ and $d_G(u,w) = 2$,

(c) $d_G(u,w) = 1$ and $d_H(v,z) = 1$.

In the case of either (a) or (b), a similar argument as in Case 1 shows $|g((u,v))\cap g((w,z))| = 0$. So assume $d_G(u,w) = 1$ and $d_H(v,z) = 1$. It follows that $f_1(u) \neq f_1(w)$ and $f_2(v) \neq f_2(z)$. If $|g((u,v))\cap g((w,z))| \leq 1$, we are done. So suppose that $g((u,v)) = g((w,z))$. Thus,

$$(f_2(v) - f_1(u) \pmod k) + k = (f_2(z) - f_1(w) \pmod k) + k,$$

or equivalently $f_2(v) - f_1(u) \equiv f_2(z) - f_1(w) \pmod k$. Rearranging terms gives

$$f_2(v) - f_2(z) \equiv f_1(u) - f_1(w) \pmod k. \tag{1}$$

On the other hand, we have

$$f_1(u) + f_2(v) \equiv f_1(w) + f_2(z) \pmod k,$$

which implies

$$f_1(u) - f_1(w) \equiv f_2(z) - f_2(v) \pmod k. \tag{2}$$

Combining (1) and (2), we have

$$f_2(v) - f_1(z) \equiv f_2(z) - f_2(v) \pmod k,$$

which implies $2f_2(v) \equiv 2f_2(z) \pmod k$. However, this cannot happen since $f_2(v) \not\equiv f_2(z) \pmod k$ and $\gcd(2,k)=1$. Thus, $|g((u,v))\cap g((w,z))| \leq 1$. ■

Although the upper bound in Theorem 8 does not involve $\tau_2(G)$ or $\tau_2(H)$, it should be noted that there are graphs for which the upper bound is best possible. For example, consider the graph $P_3 \square P_3$. By Theorem 8, we know that $\tau_2(P_3 \square P_3) \leq 2\chi(P_3^2) = 6$. On the other hand, $P_3 \square P_3$ contains a cycle of length 4. Since $\tau_2(C_4) = 6$, it follows that $\tau_2(P_3 \square P_3) = 6$.

## 4. Strong Product

The last graph product that we consider is the strong product $G \boxtimes H$. Recall that the strong product $G \boxtimes H$ has vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and edge set $E(G \boxtimes H) = E(G \square H) \cup E(G \times H)$.

Using similar ideas to those found in Sections 2 and 3, we have the following upper and lower bounds for $\tau_2(G \boxtimes H)$.

**Theorem 9.** *Given two graphs $G$ and $H$,*

$$\max\{\tau_2(G \times H), \tau_2(G \square H)\} \leq \tau_2(G \boxtimes H) \leq \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\}.$$

***Proof.*** Note that $G \Box H$ and $G \times H$ are both subgraphs of $G \boxtimes H$. Thus, $\max\{\tau_2(G \Box H), \tau_2(G \times H)\} \leq \tau_2(G \boxtimes H)$.

Next, we will prove that $\tau_2(G \boxtimes H) \leq \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\}$. Without loss of generality, we may assume $\tau_2(G)\chi(H^2) \leq \chi(G^2)\tau_2(H)$. Let $f_1$ be a proper 2-tone coloring of $G$ using the colors $\{1, 2, \dots, \tau_2(G)\}$. Let $f_2$ be a proper distance $(2, k)$-coloring of $H$ using the colors $\{1, \tau_2(G)+1, 2\tau_2(G)+1, \dots, (k-1)\tau_2(G)+1\}$ where $k = \chi(H^2)$.

Define $g : V(G \boxtimes H) \to \mathcal{P}_2([k\tau_2(G)])$ such that for each $(x, y) \in V(G \boxtimes H)$ and for each $c \in f_1(x)$, we have $c + f_2(y) \in g((x, y))$. We show that g is a proper 2-tone coloring of $G \boxtimes H$. Let $(u, v)$ and $(w, z)$ be vertices of $V(G \boxtimes H)$.

*Case* 1. Assume that $d_{G \boxtimes H}((u, v), (w, z)) = 1$. By definition of the strong product, exactly one of the following will be true:

(a) $d_G(u, w) = 1$ and $v = z$,

(b) $u = w$ and $d_H(v, z) = 1$,

(c) $d_G(u, w) = 1$ and $d_H(v, z) = 1$.

We show that $|g((u, v)) \cap g((w, z))| = 0$ in each of the above cases.

(a) Assume $d_G(u, w) = 1$ and $v = z$. Since $f_1$ is a proper 2-tone coloring of $G$, $f_1(u) \cap f_1(w) = \emptyset$. Thus, we can write $f_1(u) = \{c_1, c_2\}$ and $f_1(w) = \{c_3, c_4\}$ where $c_i \neq c_j$ for $1 \leq i < j \leq 4$. Since $f_2(v) = f_2(z)$, we know for $i \in \{1, 2\}$ and $j \in \{3, 4\}$ that $c_i + f_2(v) \neq c_j + f_2(z)$. Therefore, $|g((u, v)) \cap g((w, z))| = 0$.

(b) Assume $u = w$ and $d_H(v, z) = 1$. Since $f_2$ is a proper distance $(2, k)$-coloring of $H$, $f_2(v) \neq f_2(z)$ and we may write $f_2(v) = i\tau_2(G) + 1$ and $f_2(z) = j\tau_2(G) + 1$ for some $0 \leq i < j \leq k - 1$. Let $f_1(u) = \{c_1, c_2\}$ where $c_1 \neq c_2$. Thus,

$$g((u, v)) = \{c_1 + i\tau_2(G) + 1, c_2 + i\tau_2(G) + 1\}$$

and

$$g((w, z)) = \{c_1 + j\tau_2(G) + 1, c_2 + j\tau_2(G) + 1\}.$$

It is clear that $c_1 + i\tau_2(G) + 1 \neq c_1 + j\tau_2(G) + 1$ since $i < j$. Similarly, $c_2 + i\tau_2(G) + 1 \neq c_2 + j\tau_2(G) + 1$. Note that if $c_1 + i\tau_2(G) + 1 = c_2 + j\tau_2(G) + 1$, then

$$c_1 - c_2 = (j - i)\tau_2(G).$$

We know that $c_1 - c_2 \neq 0$ since $i < j$. On the other hand, $c_1 - c_2$ cannot be a multiple of $\tau_2(G)$ since $1 \leq c_1, c_2 \leq \tau_2(G)$. Therefore,

$$c_1 + i\tau_2(G) + 1 \neq c_2 + j\tau_2(G) + 1,$$

and a similar argument shows that

$$c_2 + i\tau_2(G) + 1 \neq c_1 + j\tau_2(G) + 1.$$

Thus, $|g((u,v)) \cap g((w,z))| = 0$.

(c) Assume $d_G(u,w) = 1$ and $d_H(v,z) = 1$. It follows that $f_1(u) \cap f_1(w) = \emptyset$ and $f_2(v) \neq f_2(z)$. As before, let $f_1(u) = \{c_1, c_2\}$ and $f_1(w) = \{c_3, c_4\}$ where $c_a \neq c_b$ when $1 \leq a < b \leq 4$. Also, write $f_2(v) = i\tau_2(G)+1$ and $f_2(z) = j\tau_2(G)+1$ for some $0 \leq i < j \leq k-1$. Thus,

$$g((u,v)) = \{c_1 + i\tau_2(G) + 1, c_2 + i\tau_2(G) + 1\}$$

and

$$g((w,z)) = \{c_3 + j\tau_2(G) + 1, c_4 + j\tau_2(G) + 1\}.$$

Again, we see that $c_1 + i\tau_2(G) + 1 \neq c_2 + i\tau_2(G) + 1$ since $c_1 \neq c_2$. Similarly, $c_3 + j\tau_2(G) + 1 \neq c_4 + j\tau_2(G) + 1$ since $c_3 \neq c_4$. Furthermore, for any $a \in \{1,2\}$ and $b \in \{3,4\}$, we know

$$c_a + i\tau_2(G) + 1 \neq c_b + j\tau_2(G) + 1$$

since $c_a - c_b$ cannot be a multiple of $\tau_2(G)$. Therefore, $|g((u,v)) \cap g((w,z))| = 0$.

*Case* 2. Assume that $d_{G\boxtimes H}((u,v),(w,z)) = 2$. Necessarily, $d_G(u,w) \leq 2$ and $d_H(v,z) \leq 2$. Thus, $|f_1(u) \cap f_1(w)| \leq 1$ so there exist $a \in f_1(u)$ and $b \in f_1(w)$ such that $a \neq b$. Furthermore, since $d_H(v,z) \leq 2$, we may assume there exist $0 \leq i < j \leq k-1$ such that $f_2(v) = i\tau_2(G)+1$ and $f_2(z) = j\tau_2(G)+1$. We have already seen that this implies $a + i\tau_2(G) + 1 \neq b + j\tau_2(G) + 1$ since $i < j$ and $1 \leq a, b \leq \tau_2(G)$. Therefore, $|g((u,v)) \cap g((w,z))| \leq 1$. ■

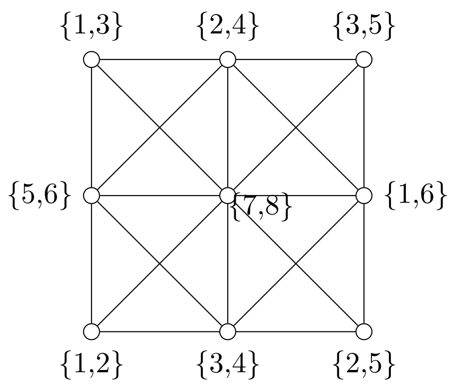


Figure 10. $P_3 \boxtimes P_3$

Note that for $P_3 \boxtimes P_3$, we can find a 2-tone 8-coloring as shown in Figure 10. This coloring is best possible since $P_3 \boxtimes P_3$ contains $K_4$ and $\tau_2(K_4) = 8$. However, in this case Theorem 9 gives bounds of

$$\begin{aligned}5 = \max\{\tau_2(P_3 \square P_3), \tau_2(P_3 \times P_3)\} &\le \tau_2(P_3 \boxtimes P_3)\\ &\le \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\} = 15.\end{aligned}$$

This alone shows that perhaps an upper bound in terms of other graph parameters would be more useful. On the other hand, since $K_3 \boxtimes K_3 \cong K_9$, it follows that $\tau_2(K_3 \boxtimes K_3) = 18$. In this particular case, we have

$$\begin{aligned}6 = \min\{\tau_2(K_3 \square K_3), \tau_2(K_3 \times K_3)\} &\le \tau_2(K_3 \boxtimes K_3)\\ &\le \min\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\} = 18,\end{aligned}$$

which shows the upper bound in Theorem 9 is sharp.

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## References

[1] A. Bickle and B. Phillips, *t-tone colorings of graphs*, submitted (2011).

[2] D. Cranston, J. Kim and W. Kinnersley, *New results in t-tone colorings of graphs*, Electron. J. Comb. **20(2)** (2013) #17.

[3] D. Bal, P. Bennett, A. Dudek and A. Frieze, *The t-tone chromatic number of random graphs*, Graphs Combin. **30** (2013) 1073–1086.
doi:10.1007/s00373-013-1341-9

[4] N. Fonger, J. Goss, B. Phillips and C. Segroves, *Math 6450: Final Report*, (2011).
http://homepages.wmich.edu/~zhang/finalReport2.pdf

[5] D. West, REGS in Combinatorics. *t*-tone colorings, (2011).
http://www.math.uiuc.edu/~west/regs/ttone.html

[6] R. Hammack, W. Imrich and S. Klavžar, Handbook of Product Graphs, Second Edition (CRC Press, Boca Raton, 2011).

[7] S. Krumke, M. Marathe and S. Ravi, *Approximation algorithms for channel assignment in radio networks*, Wireless Networks **7** (2001) 575–584.
doi:10.1023/A:1012311216333

[8] V. Vazirani, Approximation Algorithms (Springer, 2001).

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