THE WELL-COVERED DIMENSION OF PRODUCTS OF GRAPHS

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Abstract

We discuss how to find the well-covered dimension of a graph that is the Cartesian product of paths, cycles, complete graphs, and other simple graphs. Also, a bound for the well-covered dimension of $K_n \times G$ is found, provided that G has a largest greedy independent decomposition of length $c < n$.

Formulae to find the well-covered dimension of graphs obtained by vertex blowups on a known graph, and to the lexicographic product of two known graphs are also given.

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1. Introduction

In this paper, a graph is understood to be undirected and have no loops or multiple edges. While graphs with multiple edges could be taken under consideration, it is not necessary to do so as multiple edges do not add any difficulty or important properties.

A set of vertices in a graph G is said to be *independent* if no two vertices in the set are joined by an edge. An independent set M of G is called *maximal* if no independent set of G properly contains M. The largest (in terms of cardinality) maximal independent set (or sets) of G is called a *maximum independent set* of G, and a graph is said to be *well-covered* if every maximal independent set of G is also maximum. A well-covered graph could also be defined by the property of all maximal independent sets having the same cardinality. This notion was introduced by Plummer in [5]. In [1], Brown and Nowakowski defined a *well-covered weighting* of a graph G as a function $w : V(G) \to \mathbf{F}$ that assigns values to the vertices of G in such a way that $\sum_{x \in M} w(x)$ is a constant for all maximal independent sets M of G. It is immediate from the latter definition that one could re-define well-coveredness by saying that a well-covered graph is a graph that admits the constant function equal to 1 as a well-covered weighting of G. We will use Brown and Nowakowski's presentation (notation, nomenclature, etc), although this problem was originally introduced by Caro, Ellingham, Ramey, and Yuster in [2] and [3].

It is easy to show that, once a field $\mathbf{F}$ is fixed, the set of all well-covered weightings of a graph G is an $\mathbf{F}$-vector space, which is called *the well-covered space* of G. The dimension of this vector space over $\mathbf{F}$ is called the *well-covered dimension* of G and is denoted by $wcdim(G, \mathbf{F})$. If $wcdim(G, \mathbf{F})$ does not depend on the field used then the well-covered dimension of G is instead denoted as $wcdim(G)$. Note that $wcdim(G, \mathbf{F})$ may change depending on $char(\mathbf{F})$. In [1], and later in this article, examples of graphs with variable dimension are discussed. When the characteristic becomes something to consider we will be careful to remark on it.

Our graph theoretic notation, algebraic notation, and matrix theoretic notation are standard; the reader can look at [6] for any concepts we fail to define. The vertex set of a graph G is denoted by $V(G)$. The cardinality of a set of vertices V is denoted by $|V|$. A field with $q = p^h$ (p prime) elements is denoted by $\mathbf{F}_q$. The $n \times n$ identity matrix is denoted by I_n. The $n \times n$ matrix where each entry is a 1 is denoted by J_n. An $m \times 1$ column vector where each entry is a 1 is denoted by $\mathbf{1}_m$. An $m \times 1$ column vector where each entry is a 0 is denoted by $\mathbf{0}_m$.

It is relatively simple to calculate the well-covered dimension of a graph G, provided G is not too large. One first needs to find all the independent sets

of G, which can be done using a greedy algorithm. Suppose that the maximal independent sets of G are M_i for $i = 0, \ldots, k - 1$. Then a well-covered weighting w of G is determined by a solution of the linear system of equations formed by selecting a maximal independent set, in this particular instance M_0, and setting the system $M_i - M_0 = 0$ for $i = 1, \ldots, k - 1$. This system is homogeneous, and can therefore be written in the form $A\mathbf{x} = \mathbf{0}$. Note that A is an $m \times n$ matrix where $m = k - 1$ and $n = |V(G)|$. As this system is homogeneous, the nullity of A (note that $char(\mathbf{F})$ could be relevant here) is equivalent to $wcdim(G, \mathbf{F})$. So, $wcdim(G, \mathbf{F}) = n - rank(A)$. In the case when $n = rank(A)$, then $wcdim(G, \mathbf{F}) = 0$, which implies that in this case the only possible well-covered weighting is the 0 function.

For the remainder of this paper, we shall concern ourselves only with the determining of the well-covered dimensions for various individual graphs and graph families. We start by recalling a lemma from [1], as it will allow us to focus only on connected graphs.

Lemma 1 (Brown & Nowakowski) [1]**.** *Let G and H be graphs. Then*

$$wcdim(G \cup H, \boldsymbol{F}) = wcdim(G, \boldsymbol{F}) + wcdim(H, \boldsymbol{F}).$$

Although our main focus is to find the well-covered dimension of products of graphs, we will start with a few general results.

2. THE WELL-COVERED DIMENSION OF CERTAIN FAMILIES OF GRAPHS

The family of complete graphs has the easiest to find well-covered dimension among all (connected) graphs. In fact, by simply looking at the maximal independent sets of K_n we get that $wcdim(K_n) = 1$. Also, only using the technique mentioned above, it is easy to find the well-covered dimension of several families of graphs. In this section we discuss crown graphs, complete multipartite graphs, paths, cycles, and gear graphs.

Recall that, for any $n > 2$, the crown graph S_n^0 is formed by removing a perfect matching from $K_{n,n}$. Though not specifically stated as such, it was proven in [1] that $wcdim(S_n^0, \mathbf{F}) = n - 1$, if $char(\mathbf{F}) = 0$, and $wcdim(S_n^0, \mathbf{F}) = n$ if both $char(\mathbf{F})$ and n are even. We shall extend this result to allow us to calculate the well-covered dimensions of all crown graphs over all fields.

Theorem 2. *Let S_n^0 denote a crown graph, for all $n \in \mathbb{N}$. Then,*

$$wcdim(S_n^0, \boldsymbol{F}) = \begin{cases} n & \text{if } char(\boldsymbol{F}) = p \neq 0 \text{ and } p | (n - 2), \\ n - 1 & \text{otherwise.} \end{cases}$$

Proof. Let K_{V_1,V_2} be the complete bipartite graph with $V_1 = \{a_1, \ldots, a_n\}$ and $V_2 = \{b_1, \ldots, b_n\}$ and let $a_1 b_1, \ldots, a_n b_n$ be the perfect matching that is removed from K_{V_1,V_2} to form S_n^0. The maximal independent sets of S_n^0 are $\{a_i, b_i\}$ for $i = 1, 2, 3, \ldots, n$, and V_1 and V_2. Setting the sum of each of the weights on the maximal independent sets equal to that of the weights on the vertices of V_2, we find that the linear system corresponding to the well-covered weightings is $A\mathbf{x} = 0$, where

$$A = \left[\begin{array}{cc} I_n & I_n - J_n \\ \mathbf{1}_n^T & -\mathbf{1}_n^T \end{array} \right],$$

an $(n+1) \times 2n$ matrix. Subtracting the top n rows from the bottom yields

$$\left[\begin{array}{cc} I_n & I_n - J_n \\ \mathbf{0}_n^T & (n-2)\,\mathbf{1}_n^T \end{array} \right].$$

It follows that we have two possibilities depending on whether or not $char(F)$ divides $n - 2$. The theorem follows after finding the rank of this matrix in either case.   ∎

Theorem 3. *Let $G = K_{n_0, \ldots, n_{k-1}}$ be a complete k-partite graph. Then*

$$wcdim(K_{n_0, \ldots, n_{k-1}}) = \sum_{i=0}^{k-1} n_i - (k-1).$$

Proof. Let f be a well-covered weighting of G. We denote the maximal independent sets of G by N_i, where $|N_i| = n_1$, for all $i = 0, \ldots, k-1$. Setting the sum of each of the weights on the maximal independent sets equal to that of the weights on the vertices of N_{k-1}, we find that the linear system corresponding to the well-covered weightings is $A\mathbf{x} = 0$, where

$$A = \left[\begin{array}{cccccc} \mathbf{1}_{n_0}^T & \mathbf{0}_{n_1}^T & \mathbf{0}_{n_2}^T & \cdots & \mathbf{0}_{n_{k-2}}^T & -\mathbf{1}_{n_{k-1}}^T \\ \mathbf{0}_{n_0}^T & \mathbf{1}_{n_1}^T & \mathbf{0}_{n_2}^T & \cdots & \mathbf{0}_{n_{k-2}}^T & -\mathbf{1}_{n_{k-1}}^T \\ \mathbf{0}_{n_0}^T & \mathbf{0}_{n_1}^T & \mathbf{1}_{n_2}^T & \cdots & \mathbf{0}_{n_{k-2}}^T & -\mathbf{1}_{n_{k-1}}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{n_0}^T & \mathbf{0}_{n_1}^T & \mathbf{0}_{n_2}^T & \cdots & \mathbf{0}_{n_{k-2}}^T & -\mathbf{1}_{n_{k-1}}^T \end{array} \right],$$

which is a $(k-1) \times \left(\sum_{i=0}^{k-1} n_i \right)$ matrix. A has rank $k - 1$. Hence the nullity is $\sum_{i=0}^{k-1} n_i - (k-1)$, which is what we wanted to prove.   ∎

Corollary 4. *Let $T(n, r)$ be a Turán graph. Then*

$$wcdim\left(T(n,r)\right) = (n \bmod r)\lceil n/r \rceil + (r - (n \bmod r))\lfloor n/r \rfloor - (r - 1).$$

Moreover, if r divides n, then $wcdim\left(T(n,r)\right) = n - r + 1$.

The behavior, in terms of well-covered weightings, of paths and cycles is very similar. Hence, we will study these two families simultaneously.

Consider G to be an n-path or an $(n+2)$-cycle, for $n \geq 6$. Label six 'consecutive' vertices a, b, c, d, e and f as in Figure 1. Let w be a well-covered weighting of G, and let M_1 and M_2 be two maximal independent sets of G that contain the same vertices, except that M_1 contains $\{a, c, f\}$ and M_2 contains $\{a, d, f\}$ instead. Locally, these two independent sets are represented in the figure below.
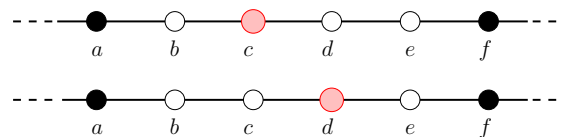

Figure 1. M_1 and M_2 on six consecutive vertices.

Since M_1 and M_2 just 'interchange' c and d, then $w(c) = w(d)$. It is now immediate that all vertices of C_n, for $n \geq 8$, have the same weight for all well-covered weightings of this graph. Hence, $wcdim(C_n) \leq 1$ for all $n \geq 8$.

Now consider two maximal independent sets N_1 and N_2 of C_n, with $n \geq 8$, that contain the same vertices outside of a string of seven consecutive vertices, where N_1 and N_2 contain four and three vertices respectively. These seven vertices, with the vertices contained in N_1 and N_2 are represented in Figure 2 below.
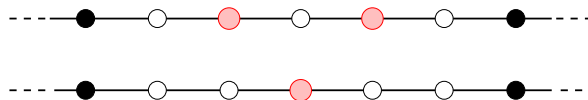

Figure 2. Two maximal independent sets with different cardinality.

It follows that this graph admits maximal independent sets with different cardinalities, and thus $wcdim(C_n) = 0$ for all $n \geq 8$.

Similarly, from the argument associated to Figure 1, if $n \geq 6$ and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ (edges connecting v_i with v_{i+1}) then vertices $v_3, \ldots, v_{n-2}$ must have the same weight for all well-covered weightings of P_n. Moreover, for small values of n it is easy to see that these weights must be zero. For larger values of n Figure 2 provides a way to construct maximal independent sets with different cardinality, which forces $w(v_3) = \cdots = w(v_{n-2})$.

Finally, we can construct two maximal independent sets of P_n that share all but one vertex, which is v_1 for one of them and v_2 for the other. This can be seen in the next figure.
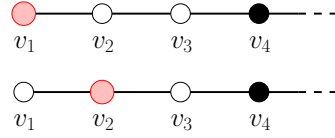
Figure 3. Two maximal independent sets on the first four vertices.

It follows that $w(v_1) = w(v_2)$, and symmetrically that $w(v_{n-1}) = w(v_n)$, for all well-covered weightings w of P_n. Lastly, we want to remark that $w(v_1)$ is independent of $w(v_n)$, and thus, adding simple computations to the arguments above we obtain the following result.

Theorem 5. *If w is a well-covered weighting of P_n and $n \geq 5$, then*

$$w(v_1) = w(v_2), \quad w(v_3) = \cdots = w(v_{n-2}) = 0, \quad w(v_{n-1}) = w(v_n).$$

Moreover,

$$wcdim(P_n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } n > 2, \end{cases} \quad and \quad wcdim(C_n) = \begin{cases} 3 & \text{if } n = 4, \\ 2 & \text{if } n = 6, \\ 1 & \text{if } n = 3, 5, 7, \\ 0 & \text{if } n \geq 8. \end{cases}$$

Remark 6. The well-covered dimensions of paths had already been computed in [3] using methods different from the one used in this paper.

Now we look at the family of gear graphs. A *gear graph* over $2n + 1$ vertices, denoted G_n, is the graph with vertex set $V(G_n) = \{v_0, \ldots, v_{2n-1}, v_c\}$ where
(a) v_i is adjacent to $v_{i-1 \bmod 2n}$ and $v_{i+1 \bmod 2n}$ for $0 \leq i \leq 2n - 1$,
(b) if $i \in 2\mathbb{Z}$, then v_i is adjacent to v_c.

We can compute the well-covered dimensions of the gear graphs using the same methods we used to compute the well-covered dimensions of the cycles.

Corollary 7. *Let G_n be the gear graph in $2n + 1$ vertices. Then*

$$wcdim(G_n) = \begin{cases} 3 & \text{if } n = 3, \\ 0 & \text{if } n > 3. \end{cases}$$

We have used that if we had maximal independent sets that share many vertices then some relations between the weights of the vertices may be found. We close this section with a generic result that has some relation with the technique just mentioned.

Lemma 8. *Let G and H be graphs such that G is a subgraph of H, $wcdim(G, \boldsymbol{F}) = 0$, and that there is a maximal independent set M of $H \setminus G$ such that $M \cup N$ is a maximal independent set of H, for all maximal independent sets N of G. Then, every well-covered weighting of H (over $\boldsymbol{F}$) is constant equal to zero on $V(G)$.*

Proof. We look at the system created by considering the maximal independent sets of H of the form $M \cup N$, where N is a maximal independent set of G. This system yields no restrictions on the vertices of $H \setminus G$ but, since $wcdim(G, \mathbf{F}) = 0$, we get that the weights for the vertices of G must all be equal to zero. Since the equations in this system are a subset of the equations in the system that we would need to analyze to get $wcdim(H, \mathbf{F})$ then the result follows.   ∎

3. BLOWUPS AND LEXICOGRAPHIC PRODUCTS

In this section we look at the well-covered dimension of graphs that can be constructed from known ones by using various techniques. We begin with a definition.

Definition. Let G be a graph and $t \in \mathbb{N}$. A *t-blowup of a vertex $v_i \in V(G)$* is an independent set $V_{v_i} = \{v_{i1}, v_{i2}, \ldots, v_{it}\}$ that 'takes the place' of v. More precisely, wherever there was an edge joining v to $w \in V(G)$ there is an edge joining v_i with w. The graph obtained by the t-blowup of v will be denoted $G(tv)$. Similarly, for $v, w \in V(G)$ and $s, t \in \mathbb{N}$ we denote a 'double blowup' $G(tv)(sw)$ as $G(tv, sw)$. For multiple blowups we extend in the natural way the notation set of double blowups.

Note that $G(v) = G$ for all $v \in V(G)$.

Lemma 9. *Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$, and $m = wcdim(G, \boldsymbol{F})$. Let $H = G(tv_1)$, where $t \in \mathbb{N}$. Then, $wcdim(H, \boldsymbol{F}) = m + t - 1$.*

Proof. We begin by noticing that a maximal independent set of G not containing v_1 is also a maximal independent set of H, and if $S = \{v_1, v_{i_2}, \ldots, v_{i_r}\}$ is a maximal independent set of G then

$$S' = \{v_{11}, \ldots, v_{1t}, v_{i_2}, \ldots, v_{i_r}\}$$

is a maximal independent set of H. Moreover, it is easy to see that every maximal independent set of H must be of one of these two types.

Let M and $M(t)$ be the matrices associated to the systems of equations arising from looking for well-covered weightings of G and H respectively. We notice that $M(t)$ has $t - 1$ more columns than M but that it has exactly the same number of rows, and in fact the same rank as M, which is $n - m$. The result follows.   ∎

By using this lemma repeatedly in a graph that is constructed from G by a sequence of blowups of vertices of G we get the following theorem.

Theorem 10. *Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and $m = wcdim(G, \boldsymbol{F})$. Let $H = G(t_1 v_1, t_2 v_2, \ldots, t_n v_n)$, where $t_i \in \mathbb{N}$ for all $i = 1, 2, \ldots, n$. Then,*

$$wcdim(H, \boldsymbol{F}) = (m - n) + \sum_{i=1}^{n} t_i.$$

Now we will look at the lexicographic product of graphs. We start with a definition.

Definition. The *lexicographic product* of G and H, denoted $G \bullet H$, is the graph with vertex set $V(G) \times V(H)$ and edges joining (g, h) and (g', h') if and only if $gg' \in E(G)$ or $g = g'$ and $hh' \in E(H)$.

Corollary 11. *Let G be a graph in n vertices with $wcdim(G, \boldsymbol{F}) = m$. Then,*

$$wcdim\left(G \bullet \overline{K_t}, \boldsymbol{F}\right) = m + n(t - 1)$$

where $t \in \mathbb{N}$.

Proof. Assume that $V(G) = \{v_1, \ldots, v_n\}$. The result follows from the previous theorem and the fact that $G(tv_1, tv_2, \ldots, tv_n) \cong G \bullet \overline{K_t}$.   ∎

The previous corollary is also a corollary of Theorem 14. In order to prove this theorem we need a couple of linear algebra results that we will not prove, but will mention in full detail.

Lemma 12. *Let M be an $n \times m$ matrix and let N be the $(n-1) \times m$ matrix obtained by subtracting the first row R_1 of M from all the other rows of M, and then deleting R_1. Assume $rank(N) = k$, then,*

$$rank(M) = \begin{cases} k & \text{if } R_1 \text{ is dependent of other rows of } M, \\ k+1 & \text{if } R_1 \text{ is independent from other rows of } M. \end{cases}$$

For the next couple of results, we denote the Kronecker (or tensor) product of two matrices, M and A, by $M \otimes A$.

Remark 13. Let N, B and C be matrices obtained by using the construction described in Lemma 12 from matrices M, A, and $M \otimes A$ respectively, we will rearrange rows in the matrices if necessary to get, when possible, the first row to be dependent of the others. Then, $rank(C) = rank(M \otimes A)$ whenever there is a row that is dependent of others in A or M, as in these cases we can always choose a row of $M \otimes A$ that depends on the other rows of this matrix. On the other hand, if both M and A have linearly independent rows, then $M \otimes A$ also has this property (using $rank(A \otimes M) = rank(A)rank(M)$), and thus $rank(C) = rank(M \otimes A) - 1$.

If we now use Lemma 12, and assume $rank(N) = k$ and $rank(B) = q$, then

$$
rank(C) = \begin{cases}
kq & \text{if both } M \text{ and } A \text{ have linearly dependent rows,} \\
k(q{+}1) & \text{if } M \text{ has linearly dependent rows and} \\
& \quad A \text{ does not,} \\
(k{+}1)q & \text{if } A \text{ has linearly dependent rows and} \\
& \quad M \text{ does not,} \\
(k{+}1)(q{+}1){-}1 & \text{if both } M \text{ and } A \text{ have linearly independent rows.}
\end{cases}
$$

Now we have all the tools needed to prove.

Theorem 14. *Let G and H be graphs with $|V(G)| = a$, $|V(H)| = b$, $wcdim(G, \boldsymbol{F}) = n$, and $wcdim(H, \boldsymbol{F}) = m$. Then,*

$$
wcdim(G \bullet H, \boldsymbol{F}) = nb + am - nm + \delta_{m-b+1,i}(n - a) + \delta_{n-a+1,j}(m - b)
$$

where δ_{xy} represents the Kronecker delta, and i, j are the number of maximal independent sets of H and G respectively.

Proof. We first notice that if $S = \{v_{1_1}, v_{i_2}, \ldots, v_{i_r}\}$ is a maximal independent set of G then $S' = \{w_{1i_1}, \ldots, w_{t_1 i_1}, w_{1i_2}, \ldots, w_{t_2 i_2}, \ldots, w_{1i_r}, \ldots, w_{t_r i_r}\}$ is a maximal independent set of $G \bullet H$, where $\{w_{1i_j}, \ldots, w_{t_j i_j}\}$ is a maximal independent set of H for all $j = 1, 2, \ldots, r$. Moreover, it is easy to see that every maximal independent set of $G \bullet H$ must be obtained this way.

Set the weight-sums of each of the independent sets of G equal to zero. Let M be the matrix representing that homogeneous system of equations. Note that the matrix N needed to find $wcdim(G, \mathbf{F})$ is obtained from M by using the construction described in Lemma 12. Similarly, by repeating this process with H we obtain B (needed for finding $wcdim(H, \mathbf{F})$) out of A (found by setting the weight-sums of the maximal independent sets of H equal to zero).

Now we notice that (because of the first paragraph in this proof) $A \otimes M$ is the matrix associated to the homogeneous system given by setting the weight-sums of all the independent sets of $G \bullet H$ equal to zero. It follows that we are interested in finding the rank of the matrix C obtained from $A \otimes M$ by using the construction described in Lemma 12.

Since $rank(N) = a - n$, $rank(B) = b - m$, and $|V(G \bullet H)| = ab$, then using that a matrix has linearly dependent rows if and only if its rank is not equal to its number of rows, and Remark 13, we get

$$
wcdim(G \bullet H, \mathbf{F}) = \begin{cases}
nb + am - nm & \text{if } i \neq b - m + 1,\ j \neq a - n + 1, \\
nb + am - nm + n - a & \text{if } i = b - m + 1,\ j \neq a - n + 1, \\
nb + am - nm + m - b & \text{if } i \neq b - m + 1,\ j = a - n + 1, \\
nb + am - nm + m - b + n - a & \text{if } i = b - m + 1,\ j = a - n + 1,
\end{cases}
$$

where i, j represent the number of maximal independent sets of H and G respectively (which are the number of rows of A and M respectively).

The result follows from the definition of the Kronecker delta. ∎

Corollary 15. *Let G and H be graphs with more maximal independent sets than vertices, and such that $|V(G)| = a$, $|V(H)| = b$, $wcdim(G, \mathbf{F}) = n$ and $wcdim(H, \mathbf{F}) = m$. Then,*

$$wcdim(G \bullet H, \mathbf{F}) = nb + am - nm.$$

Remark 16. As mentioned above, Corollary 11 is a corollary of Theorem 14. In order to see this we just need to notice that $\overline{K_t}$ has one maximal independent set and that $wcdim(\overline{K_t}) = V(\overline{K_t}) = t$.

4.   Cartesian Products: Paths and Cycles

The *Cartesian product* of $G \times H$ is the graph with vertices (u, v) where $u \in V(G)$ and $v \in V(H)$ and there exists an edge joining (u_1, v_1) with (u_2, v_2) iff there exists an edge in G joining u_1 and v_2 and $v_1 = v_2$ or there exists an edge in H joining v_1 and v_2 and $u_1 = u_2$.

We start by exploring products of paths and/or cycles. Just like in Section 2 we will study these two classes of graphs almost simultaneously. We will also borrow from that section the idea of comparing maximal independent sets that agree in all but a few of their vertices. Also, given that we will use many pictures in this section we need the following definitions.

Definition. If $G = P_n \times P_m$ or $G = P_n \times C_m$, where $n, m > 1$, we will say that a *vertex is on the interior* of G (or that it is an *internal vertex*) if its degree is equal to 4. A vertex of degree 2 or 3 will be said *to be on the boundary* of G (or that it is a *boundary vertex*). A vertex of degree 2 will be also called a *corner* of G.

Lemma 17. *Let w be any well-covered weighting of the graph G.*
 (1) *If $G = P_n \times P_m$, where $m \geq 2$ and $n \geq 4$, or*
 (2) *If $G = P_n \times C_3$, where $n \geq 2$, or*
 (3) *If $G = P_n \times C_m$, where $m \geq 5$ and $n \geq 2$,*
then any two adjacent boundary non-corner vertices, a and b, of G must satisfy $w(a) = w(b)$.

Proof. Let a and b be two adjacent boundary non-corner vertices of G. Consider the following two maximal independent sets of $P_4 \times P_2$.
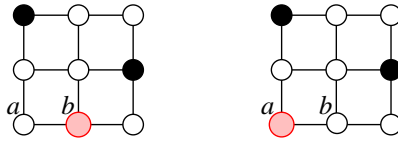
Here we can see that $w(a) = w(b)$, for all well-covered weightings of $P_4 \times P_2$. Using a greedy algorithm we can find maximal independent sets of G that are the same except at these 'local' pictures. This is immediate for cases (1) and (3). On the other hand, for case (2) we need (in each of these figures) to identify the vertices on their leftmost 'side' to those on their rightmost 'side' to create a local picture of $P_n \times C_3$. The result follows.   ∎

Now that we know how non-corner boundary vertices behave we need to take a look at corner vertices.

Lemma 18. *Let $G = P_n \times P_m$, where $n, m \geq 3$, and w be any well-covered weighting of G. Let b and c be two (boundary) vertices adjacent to a corner vertex a. Then, $w(a) = w(b) = w(c)$.*

Proof. Let a be a corner vertex and b one of its neighbors. We consider the two local pictures of this corner below.

Using the same ideas used in the proof of Lemma 17, we get that these pictures imply that $w(a) = w(b)$, for all well-covered weighting of G. Finally, by simply reflecting the picture we obtain $w(a) = w(c)$.   ∎

We summarize the results in Lemmas 17 and 18 in the following corollary.

Corollary 19. *Let w be any well-covered weighting of the graph G.*

(1) *If $G = P_n \times P_m$, where $n, m \geq 4$, then $w(a) = w(b)$, for any two boundary vertices, a and b, of G.*

(2) *If $G = P_n \times C_m$, where $n \geq 2$, and $m = 3$ or $m \geq 5$, then $w(a) = w(b)$, for any two boundary vertices, a and b, that are on the same 'boundary cycle' of G.*

Now that we understand boundary vertices we move on to study interior vertices.

Lemma 20. *Let w be any well-covered weighting of the graph G.*

(1) *If $G = P_n \times P_m$, where $m, n \geq 5$, or*

(2) *If $G = P_n \times C_m$, where $m \geq 6$ and $n \geq 5$, or*

(3) *If $G = C_n \times C_m$, where $m, n \geq 6$,*

then $w(a) = 0$, for all $a \in E(G)$.

Proof. In order to prove this we will embed a $P_5 \times P_5$ and a $P_4 \times P_4$ as 'local pictures', hence the bounds for m and n in the (three different) hypothesis.

We consider the following figures, which show maximal independent sets of $P_5 \times P_5$ and $P_4 \times P_4$ that share all but the grey vertices (red if you read this in color).
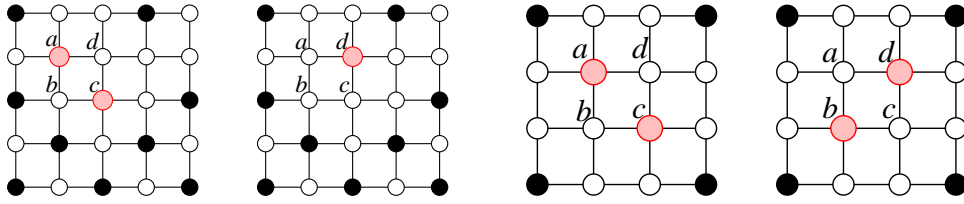
The picture on the left (where we consider $P_5 \times P_5$ as a subgraph of G) implies that $w(a) + w(c) = w(d)$, for all well-covered weightings of G. However, by considering the picture on the right (where we consider $P_4 \times P_4$ as a subgraph of G) we obtain $w(a) + w(c) = w(b) + w(d)$, for all well-covered weightings of G. It follows that $w(b) = 0$, for all well-covered weightings of G.

By embedding reflected and/or rotated versions of these figures into G we obtain the desired result. ∎

Since all vertices in $C_n \times C_m$ are interior, the following result holds trivially.

Lemma 21. *Let* $m, n \in \mathbb{N}$ *such that* $m, n \geq 6$. *Then* $wcdim(C_n \times C_m, \boldsymbol{F}) = 0$, *for all fields* $\boldsymbol{F}$.

We will now get results similar to Lemma 21, but for $P_n \times P_m$ and $P_n \times C_m$.

Lemma 22. *Let* $m, n \in \mathbb{N}$.

 (1) *If* $m, n \geq 5$, *then* $wcdim(P_n \times P_m, \boldsymbol{F}) = 0$, *for all fields* $\boldsymbol{F}$.
 (2) *If* $m \geq 6$ *and* $n \geq 5$, *then* $wcdim(P_n \times C_m, \boldsymbol{F}) = 0$, *for all fields* $\boldsymbol{F}$.

Proof. We consider the following figures, where b is a boundary vertex.
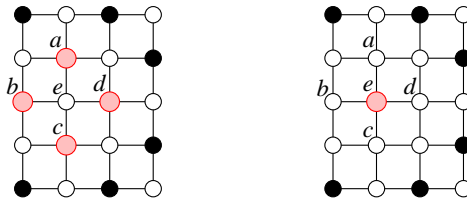
Given the bounds for m and n, we may embed these figures into G for either case. It follows that $w(e) = w(a) + w(b) + w(c) + w(d)$, for all well-covered weightings of G. However, Lemma 20 says that $w(a) = w(c) = w(d) = w(e) = 0$. Since b can be chosen to be on any 'boundary cycle' of G the result follows from Lemma 19. ∎

Before phrasing our main theorem for this section (Theorem 24) we add one more result, which shows that the lower bound for the size of a cycle in Lemma 22 is sharp.

Lemma 23. $wcdim(P_n \times C_5, \boldsymbol{F}) = 2$, *for all fields $\boldsymbol{F}$, and all $n \in \mathbb{N}$ such that $n \geq 6$.*

**Proof.** Let $G = P_n \times C_5$, and w be a well-covered weighting of G.

As we have done before, we will use pictures to find relations between the weights of vertices of G. Hence, we consider

Note that we are assuming that the left cycle of the figure on the right is a 'boundary cycle' of G.

It is easy to see that the figure on the left implies $w(a) = w(b)$, for any two vertices, a and b, of G that are on any given non-boundary cycle. We use this to realize that in the figure on the right we get $w(c) = w(d)$. Moreover, since $w(a) + w(c) = w(b) + w(d)$ then $w(a) = w(b)$. Hence, any two vertices on any given cycle have the same weight.

Next we consider yet two more pairs of figures.
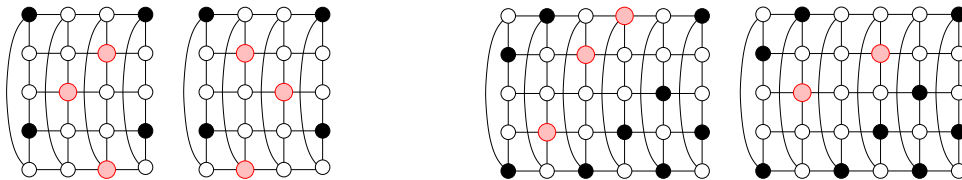
The figure on the left assures that every non-boundary vertex has the same weight. Now we use this to obtain, from the figure on the right, that all the non-boundary vertices of G have weight equal to zero.

So far, we know that $wcdim(P_n \times C_5, \mathbf{F}) \leq 2$. We notice that a boundary cycle of G must contain exactly two vertices of every maximal independent set of $P_n \times C_5$. Hence, the functions defined by

$$f_1(v) = \begin{cases} 1 & \text{if } v \text{ is a vertex of } \mathcal{C}_1, \\ 0 & \text{otherwise,} \end{cases} \qquad f_2(v) = \begin{cases} 1 & \text{if } v \text{ is a vertex of } \mathcal{C}_n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{C}_1$ and $\mathcal{C}_n$ are the two boundary cycles of $P_n \times C_5$, are linearly independent well-covered weightings of $P_n \times C_5$. ∎

We close this section with a theorem that collects all results in this section.

Theorem 24. *Let $m, n \in \mathbb{N}$.*
 (1) *If $m, n \geq 6$, then $wcdim(C_n \times C_m, \boldsymbol{F}) = 0$, for all fields $\boldsymbol{F}$.*
 (2) *If $m, n \geq 5$, then $wcdim(P_n \times P_m, \boldsymbol{F}) = 0$, for all fields $\boldsymbol{F}$.*
 (3) *If $m \geq 6$ and $n \geq 5$, then $wcdim(P_n \times C_m, \boldsymbol{F}) = 0$, for all fields $\boldsymbol{F}$.*
 (4) *If $n \geq 6$, then $wcdim(P_n \times C_5, \boldsymbol{F}) = 2$, for all fields $\boldsymbol{F}$.*

Remark 25. Theorem 24 may be improved for 'small' m and n. In these cases, the values of $wcdim(G)$ are found using similar techniques and the proofs are heavy in pictures and/or matrix computations. For this reason we will not include these results here, they may be found on `arxiv.org`, or the fifth author's web site. These results will not be published otherwise.

5.   Cartesian Products with Complete Graphs

The greedy independent decomposition of a graph provides a means to construct maximal independent sets of $G \times H$. The following definition and lemma may be found in [4].

Definition. Let I_1 be a maximal independent set of G. Choose $I_2, I_3, \ldots$ such that I_k is a maximal independent set in $G \backslash (I_1 \cup I_2 \cup \cdots \cup I_{k-1})$. Then, $\{I_j\}_{i=1}^{n}$ is a *greedy independent decomposition* of G.

The following result will be used later in this article.

Lemma 26 (Ovetsky) [4]**.** *Let $\{I_j\}_{i=1}^{n}$ and $\{J_j\}_{j=1}^{n}$ be greedy independent decompositions of graphs G and H respectively, and without loss of generality suppose $n \leq m$. Then $(I_1 \times J_1) \cup (I_2 \times J_2) \cup \cdots \cup (I_n \times J_n)$ is a maximal independent set in $G \times H$.*

Theorem 27. *Let H be a graph such that there exists a greedy independent decomposition of cardinality c. Then, $wcdim(K_n \times H, \boldsymbol{F}) \leq |V(H)|$ where $c < n$.*

Proof. Let w_j denote the vertices of H, where $j = 1, 2, \ldots, m$, and v_i denote the vertices of K_n, where $i = 1, 2, \ldots, n$. Let $\{S_i\}_{i=1}^{p}$ be a greedy independent decomposition of H such that $|S| = p < n$. Let $\{J_i\}_{i=1}^{n}$ be a greedy independent decomposition of K_n such that $J_i = \{v_i\}$. By the greedy independent theorem, $A = \bigcup_{i=1}^{p}(J_i \times S_i)$ is a maximal independent set of $K_n \times H$. Let $w_j \in S_1$ for some $j = 1, 2, \ldots, m$. Then, $(v_1, w_j) \in A$. Switch the vertex at (v_1, w_j) to (v_n, w_j) to create a new maximal independent set. Hence, $w(v_1, w_j) = w(v_n, w_j)$ for some well-covered weighting $w : V(K_n \times H) \to \mathbf{F}$. Then, switch each vertex of the

form (v_1, w_j) to (v_n, w_j) for every $w_j \in S_1$. Hence, $w(v_1, w_j) = w(v_n, w_j)$ for all w_j such that $w_j \in S_1$.

Set $a = 1$. Let $w_j \in S_{a+1}$ for some $j = 1, 2, \ldots, m$. Then, $(v_{a+1}, w_j) \in A$. Switch the vertex at (v_{a+1}, w_j) to (v_a, w_j) to create a new maximal independent set. Hence, $w(v_{a+1}, w_j) = w(v_a, w_j)$. Repeat for every $w_j \in S_{a+1}$. Hence, $w(v_{a+1}, w_j) = w(v_a, w_j)$ for every $w_j \in S_{a+1}$. Set $a = a + 1$ and repeat for every $a \leq p - 1$.

After the above process has been completed, A has been shifted vertically by exactly one vertex to create a new maximal independent set A'. Continue to shift the maximal independent set vertically in the same manner until $A' = A$. Then, you will have shown that $w(v_i, w_j) = w(v_k, w_j)$ for every $i, k = 1, 2, \ldots, n$. Hence, $wcdim(K_n \times H, \mathbf{F}) \leq |V(H)|$.   ∎

Lemma 28. *For every P_n, there is a greedy independent decomposition of cardinality exactly 2.*

Proof. Let $J = \{J_i\}_{i=1}^p$ be a greedy independent decomposition for P_n. Set $J_1 = \{v_i : i = 2n - 1 \text{ for some } n \in \mathbb{N}\}$. Then, $J_2 = \{v_i : i = 2n \text{ for some } n \in \mathbb{N}\}$. Hence, $|J| = 2$.   ∎

Corollary 29. $wcdim(K_n \times P_m, \mathbf{F})$ *for all $n, m \geq 3$.*

Proof. By Lemma 28 and Theorem 27, $wcdim(K_n \times P_m, \mathbf{F}) \leq |V(P_m)| = m$. Let $f_i : V(K_n \times P_m) \to \mathbf{F}$ where

$$f_i = \begin{cases} 1 & \text{for all vertices of column } i, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\sum_{v \in A} f_i(v) = 1$ for every i and every maximal independent set A. Hence, $F = \{f_i\}_{i=1}^m$ is a linearly independent set of well-covered weightings. Hence, $wcdim(K_n \times P_m, \mathbf{F}) \geq m$ because $|F| = m$. Therefore, $wcdim(K_n \times P_m, \mathbf{F}) = m$.   ∎

Lemma 30. *If m is even, there is a greedy independent decomposition of C_m of cardinality exactly 2. If m is odd, there is a greedy independent decomposition of C_m of cardinality exactly 3.*

Proof. Let $J = \{J_i\}_{i=1}^p$ be a greedy independent decomposition of a cycle.

Case 1: Let m be even. Set $J_1 = \{v_i : i = 2n - 1 \text{ for some } n \in \mathbb{N}\}$. Then, $J_2 = \{v_i : i = 2n \text{ for some } n \in \mathbb{N}\}$. Hence, $|J| = 2$.

Case 2: Let m be odd. Set $J_1 = \{v_i : i = 2n - 1 \text{ for some } n \in \mathbb{N}\} \backslash \{v_m\}$. Then, $J_2 = \{v_i : i = 2n \text{ for some } n \in \mathbb{N}\}$ and $J_3 = \{v_m\}$. Therefore, $|J| = 3$.   ∎

Corollary 31. $wcdim(K_n \times C_m, \mathbf{F})$ *for all $n, m \geq 4$.*

Proof. By Lemma 30 and Theorem 27, $wcdim(K_n \times P_m, \mathbf{F}) \leq |V(C_m)| = m$.
Let $f_i : V(K_n \times C_m) \to \mathbf{F}$ where

$$f_i = \begin{cases} 1 & \text{for all vertices of column } i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $F = \{f_i\}_{i=1}^m$ is a set of linearly independent well-covered weightings of
$K_n \times C_m$. Hence, $wcdim(K_n \times C_m) \geq m$. Therefore, $wcdim(K_n \times C_m, \mathbf{F}) = m$.
■

Corollary 32. $wcdim(K_n \times K_m, \mathbf{F}) = m$ for all $n > m \geq 3$.

Proof. Let $J = \{J_i\}_{i=1}^m$ be a greedy independent decomposition of K_m. Hence,
$|J| = m < n$. By Theorem 27, $wcdim(K_n \times K_m) \leq |V(K_m)| = m$. Let $f_i :
V(K_n \times K_m) \to \mathbf{F}$ where

$$f_i = \begin{cases} 1 & \text{for all vertices of column } i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $F = \{f_i\}_{i=1}^m$ is a set of linearly independent well-covered weightings of
$K_n \times K_m$. Hence, $wcdim(K_n \times K_m) \geq m$. Therefore, $wcdim(K_n \times K_m, \mathbf{F}) = m$.
■

Theorem 33. $wcdim(K_n \times K_n, \mathbf{F}) = 2n - 1$ for all $n \geq 3$.

Proof. Let A be a maximal independent set of $K_n \times K_n$ such that $(v_i, w_j), (v_k, w_l)$
$\in A$ where $i \neq k$ and $j \neq l$. We can create a new maximal independent set B
such that $(v_i, w_l), (v_k, w_j) \in B$ and $A \backslash \{(v_i, w_j), (v_k, w_l)\} = B \backslash \{(v_i, w_l), (v_k, w_j)\}$.
Hence, $w(v_i, w_j) + w(v_k, w_l) = w(v_i, w_l) + w(v_k, w_j)$ for any well-covered weighting
w. Therefore, $w(v_i, w_j) = w(v_i, w_l) + w(v_k, w_j) - w(v_k, w_l)$ for any i, j, k, l such
that $i \neq k$ and $j \neq l$.

Choose row k and column l and assign arbitrary weights to the vertices of
row k and column l. Because each row and column have n vertices and the
row and column share the vertex (v_k, w_l), there are $2n - 1$ arbitrary weights
assigned. Hence, for any vertex (v_i, w_j) such that $i \neq k$ and $j \neq l$, $w(v_i, w_j) =
w(v_i, w_l) + w(v_k, w_j) - w(v_k, w_l)$. Therefore, $wcdim(K_n \times K_n) \leq 2n - 1$. Let
$f_i, f_j : V(K_n \times K_n) \to \mathbf{F}$ where

$$f_i = \begin{cases} 1 & \text{for all vertices of column } i, \\ 0 & \text{otherwise,} \end{cases} \qquad f_j = \begin{cases} 1 & \text{for all vertices of row } j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, both f_i and f_j are well-covered weightings of $K_n \times K_n$. Let $F = \{\{f_i\}_{i=1}^n,
\{f_j\}_{j=1}^n\}$. Note that F is not a linearly independent set. However, $F^* = F \backslash \{f_{j=n}\}$
is a linearly independent set and $|F^*| = 2n - 1$. Hence, we can conclude that
$wcdim(K_n \times K_n, \mathbf{F}) \geq 2n - 1$. Therefore, $wcdim(K_n \times K_n, \mathbf{F}) = 2n - 1$.   ■

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