# COLOR ENERGY OF A UNITARY CAYLEY GRAPH 

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#### Abstract

Let $G$ be a vertex colored graph. The minimum number $\chi(G)$ of colors needed for coloring of a graph $G$ is called the chromatic number. Recently, Adiga et al. [1] have introduced the concept of color energy of a graph $E_{c}(G)$ and computed the color energy of few families of graphs with $\chi(G)$ colors. In this paper we derive explicit formulas for the color energies of the unitary Cayley graph $X_{n}$, the complement of the colored unitary Cayley graph $\overline{\left(X_{n}\right)_{c}}$ and some gcd-graphs.


Keywords: coloring of a graph, unitary Cayley graph, gcd-graph, color eigenvalues, color energy.
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## 1. Introduction

Let $G$ be a finite group and S be a subset of $G$ such that S does not contain identity of $G$. Assume $S^{-1}=\left\{s^{-1}: s \in S\right\}=S$. The Cayley graph $X=C a y(G, S)$ is an undirected graph having vertex set $V(X)=G$ and edge set $E(X)=\{\{a, b\}$ :

[^0]$\left.a b^{-1} \in S\right\}$, where $a, b \in G$. The Cayley graph $X$ is a regular graph of degree $|S|$. Its connected components are the right cosets of the subgroup generated by $S$. Therefore, if $S$ generates $G$, then $X$ is a connected graph. More information about Cayley graphs can be found in the books on algebraic graph theory by Biggs [2] and by Godsil and Royle [3].

For a positive integer $n>1$, the unitary Cayley graph $X_{n}=\operatorname{Cay}\left(Z_{n} ; U_{n}\right)$, where $U_{n}$ denotes the set of all units of the ring $Z_{n}$. Two vertices $a, b$ are adjacent if, and only if, $a-b \in U_{n}$. The unitary Cayley graph $X_{n}$ is regular of degree $\phi(n)$, where $\phi(n)$ is the Euler function. Unitary Cayley graphs are highly symmetric and have some remarkable properties connecting graph theory and number theory. These graphs have integral spectrum and play an important role in modeling quantum spin networks supporting the perfect state of transfer. In fact, it is proved in [6] that the eigenvalues of unitary Cayley graph $X_{n}$ are

$$
\lambda_{i}=\sum_{1 \leq j<n,(j, n)=1} \omega^{i j}=C(i, n), 0 \leq i \leq n-1,
$$

where $\omega$ denotes a complex primitive $n^{\text {th }}$ root of unity.
The arithmetic function $C(i, n)$ is a Ramanujan sum [5], and

$$
C(i, n)=\phi(n) \frac{\mu\left(\frac{n}{(i, n)}\right)}{\phi\left(\frac{n}{(i, n)}\right)},
$$

where $\mu$ denotes the Möbius function.
The gcd-graphs arise as a generalization of unitary Cayley graphs studied by Klotz and Sander in [6]. Let $D$ be a set of positive, proper divisors of the integer $n>1$. Define the gcd-graph $X_{n}(D)$ to have vertex set $Z_{n}=\{0,1, \ldots, n-1\}$ and edge set $E\left(X_{n}(D)\right)=\left\{\{a, b\}: a, b \in Z_{n},(a-b, n) \in D\right\}$. In [6], it was proved that these graphs have integral spectrum.

Let $G$ be a simple graph. The adjacency matrix of $G$ is the $n \times n$ matrix $A=A(G)$, whose entries $a_{i j}$ are given by $a_{i j}=1$ if $v_{i}$ and $v_{j}$ are adjacent, $a_{i j}=0$ otherwise. The eigenvalues of $A(G)$ are the eigenvalues of $G$. The energy $E(G)$ of a graph $G$ is the sum of the absolute values of the eigenvalues of $A(G)$ [4].

Recently, Sampathkumar and Sriraj [9] have introduced a new matrix $A_{L}(G)$ called $L$-matrix of a vertex labeled graph $G=(V, E)$, whose elements are defined as follows: If $\ell\left(v_{i}\right)$ is the label of the vertex $v_{i}$, then

$$
a_{i j}=\left\{\begin{aligned}
2 & \text { if } v_{i} \text { and } v_{j} \text { are adjacent with } \ell\left(v_{i}\right)=\ell\left(v_{j}\right), \\
1 & \text { if } v_{i} \text { and } v_{j} \text { are adjacent with } \ell\left(v_{i}\right) \neq \ell\left(v_{j}\right), \\
-1 & \text { if } v_{i} \text { and } v_{j} \text { are non-adjacent with } \ell\left(v_{i}\right)=\ell\left(v_{j}\right), \\
0 & \text { if } v_{i}=v_{j} \text { or } v_{i} \text { and } v_{j} \text { are non-adjacent with } \ell\left(v_{i}\right) \neq \ell\left(v_{j}\right) .
\end{aligned}\right.
$$

A coloring of graph $G$ is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring of a graph $G$ is called chromatic number and denoted by $\chi(G)$.

If we consider the vertex colored graph, then entries of the matrix $A_{L}(G)$ are as follows: If $c\left(v_{i}\right)$ is the color of $v_{i}$, then

$$
a_{i j}=\left\{\begin{aligned}
1 & \text { if } v_{i} \text { and } v_{j} \text { are adjacent with } c\left(v_{i}\right) \neq c\left(v_{j}\right), \\
-1 & \text { if } v_{i} \text { and } v_{j} \text { are non-adjacent with } c\left(v_{i}\right)=c\left(v_{j}\right), \\
0 & \text { if } v_{i}=v_{j} \text { or } v_{i} \text { and } v_{j} \text { are non-adjacent with } c\left(v_{i}\right) \neq c\left(v_{j}\right) .
\end{aligned}\right.
$$

The matrix thus obtained is the $L$-matrix of the colored graph and is $G$ denoted by $A_{c}(G)$. The eigenvalues of $A_{c}(G)$ are called color eigenvalues. If $G$ is colored with $\chi(G)$ colors, then $L$-matrix of the colored graph $G$ is denoted by $A_{\chi}(G)$. Recently in [1], Adiga, Sampathkumar, Sriraj, Shrikanth have studied the energy of the vertex colored graph, which is defined as follows. The energy of a graph with respect to a given coloring is the sum of the absolute values of the color eigenvalues of $G$ and is called the energy of a colored graph or color energy of a graph.

If we use $n$ different colors to the vertices of a graph of order $n$, then the color energy is the same as the energy of a graph. So color energy may be considered as a generalization of energy of a graph. It is possible that color energy that we are considering in this paper has similar applications in chemistry as well as in some other areas.

Recently, llić [7] obtained an explicit formula for the energy of unitary Cayley graph $X_{n}$ and also energy of the complement of $X_{n}$. The open problem posed by llić [7] about calculating energy of an arbitary integral circulant graph is completely solved by Mollahajiaghaei in [8].

Motivated by these investigations we establish formulas for color energies of the unitary Cayley graph $X_{n}$ and the complement of the colored unitary Cayley graph $X_{n}$. We also derive an explicit formula for energy of the colored gcd-graph $X_{n}(D)$, where $n=p_{1}^{\alpha_{1}} p_{2}^{\alpha_{2}} \ldots p_{i-1}{ }^{\alpha_{i-1}} p_{i}{ }^{1} p_{i+1}{ }^{\alpha_{i+1}} \ldots p_{k}{ }^{\alpha_{k}}$ and $D=\left\{1, p_{i}\right\}$.

## 2. Color Energy of Unitary Cayley Graphs

Let $X_{n}=\left(Z_{n} ; U_{n}\right)$ be the unitary Cayley graph and $E_{\chi}\left(X_{n}\right)$ denote the color energy of $X_{n}$ with $\chi\left(X_{n}\right)$ colors. In [6], Klotz and Sander proved the following theorem:

Theorem 1. If $p$ is the smallest prime divisor of $n$, then $\chi\left(X_{n}\right)=p$.
In general, an optimal coloring with $\chi(G)$ colors is not unique. So the color energy $E_{\chi}(G)$ may be different for different optimal colorings. But the unitary

Cayley graph $X_{n}$ has a unique optimal coloring, thus its color energy with respect to minimum number of colors is unique.

Theorem 2. The unitary Cayley graph $X_{n}$ has a unique optimal coloring and the color energy of $X_{n}$ with respect to minimum number of colors is unique.

Proof. Let $X_{n}$ be a unitary Cayley graph whose vertices are labeled by $0,1,2, \ldots, n-1$. Let $p$ be the smallest prime divisor of $n$. Then $\chi\left(X_{n}\right)=p$. Now we color $X_{n}$ using $p$ colors say $c_{1}, c_{2}, \ldots, c_{p}$. Consider the vertices $0,1,2, \ldots, p-1$. Let $i, j \in\{0,1,2, \ldots, p-1\}$, where $i<j$. Since $\operatorname{gcd}(j-i, n)=1, i$ and $j$ are adjacent so we cannot give the same color to $i$ and $j$. Similarly, the vertices $i p, i p+1, \ldots,(i+1) p-1, i=1,2,3, \ldots,\left(\frac{n}{p}-1\right)$ must receive different colors.

Suppose color of the vertex 0 is $c_{1}$ then color of the vertex $p$ must be equal $c_{1}$ because $p$ and $i$ are adjacent for each $i=1,2,3, \ldots, p-1$. On similar lines the color of the vertices $2 p, 3 p, \ldots,\left(\frac{n}{p}-1\right) p$ must be $c_{1}$. Thus the vertices $0, p, 2 p, 3 p, \ldots,\left(\frac{n}{p}-1\right) p$ receive the color $c_{1}$. Similarly, the colors of the vertices $1, p+1,2 p+1, \ldots,\left(\frac{n}{p}-1\right) p+1$, must be the same. In general $i, i+p, i+2 p, \ldots, i+\left(\frac{n}{p}-1\right) p, i=0,1, \ldots,(p-1)$ must receive the same color. This shows that the matrix $A_{\chi}\left(X_{n}\right)$ will not alter for different optimal colorings. Hence, the color energy of $X_{n}$ with respect to minimum number of colors is unique. This completes the proof.

Now we determine the color energy of a unitary Cayley graph with minimum number of colors.

Theorem 3. Let $X_{n}$ be the unitary Cayley graph colored with $p$ colors, where $p$ is the smallest prime divisor of $n$. Then

$$
E_{\chi}\left(X_{n}\right)=2^{\omega(n)} \phi(n)+2 \frac{n}{p} \phi(p)-2 \sum_{\mu(d)=-1}^{d \mid n} \phi(d),
$$

where $\omega(n)$ is the number of distinct prime divisors of $n$.
Proof. Let $X_{n}=\left(Z_{n} ; U_{n}\right)$ be the unitary Cayley graph colored with $p$ colors, where $p$ is the smallest prime divisor of $n$. We color the vertices $0,1,2, \ldots, n-1$ with colors $c_{1}, c_{2}, \ldots, c_{p}$ as follows: Color the vertices $k p, k p+1, k p+2, \ldots$, $(k+1) p-1$ with $c_{1}, c_{2}, c_{3}, \ldots, c_{p}$ respectively for $k=0,1,2, \ldots,\left(\frac{n}{p}-1\right)$. Then $A_{\chi}\left(X_{n}\right)$ is a circulant matrix with first row equal to $\left(a_{0}, a_{1}, a_{2}, \ldots, a_{n-1}\right)$, where

$$
a_{k}=\left\{\begin{aligned}
0 & \text { if } k=0 \\
1 & \text { if } 1 \leq k \leq n-1,(k, n)=1 \\
-1 & \text { if } k=p, 2 p, \ldots, n-p \\
0 & \text { otherwise }
\end{aligned}\right.
$$

It is well-known that for a circulant matrix of order $n$, the eigenvalues are given by the formula

$$
\lambda_{m}=\sum_{k=0}^{n-1} a_{k} e^{\frac{2 \pi i m k}{n}}, m=0,1, \ldots,(n-1)
$$

Let $\omega=e^{\frac{2 \pi i}{n}}$, then we have

$$
\begin{aligned}
\lambda_{m} & =\sum_{k=0}^{n-1} a_{k} \omega^{m k}, m=0,1, \ldots,(n-1) \\
& =\sum_{\substack{k=1 \\
k, n)=1}}^{n-1} \omega^{m k}-\sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell p m} \\
& =C(m, n)-\sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell p m}
\end{aligned}
$$

We have $\lambda_{0}=\sum_{k=0}^{n-1} a_{k}$. Note that $a_{k}=1$ if $1 \leq k \leq(n-1),(k, n)=1, a_{k}=-1$ if $k=p, 2 p \ldots, n-p$ and 0 otherwise. Thus the number of terms that are equal to 1 in the above sum is $\phi(n)$ and the number of terms that are equal to -1 is $\left(\frac{n}{p}-1\right)$ and other terms are zero. Hence,

$$
\lambda_{0}=\phi(n)-\left(\frac{n}{p}-1\right)
$$

using definition of Ramanujan's sum we have

$$
\lambda_{m}=C(m, n)-\left(\frac{n}{p}-1\right) \text { for } m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}
$$

and

$$
\lambda_{m}=C(m, n)+1, \text { for } 1 \leq m \leq n-1 \text { and } m \neq \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}
$$

Thus

$$
\begin{align*}
E_{\chi}\left(X_{n}\right) & =\sum_{m=0}^{n-1}\left|\lambda_{m}\right| \\
& =\left|\lambda_{0}\right|+\sum_{m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}\left|\lambda_{m}\right|+\sum_{\substack{1 \leq m \leq(n-1) \\
m \neq \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}}\left|\lambda_{m}\right| \\
& =\phi(n)-\left(\frac{n}{p}-1\right)+\sum_{m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}\left|C(m, n)-\left(\frac{n}{p}-1\right)\right|  \tag{1}\\
& +\sum_{\substack{1 \leq m \leq(n-1) \\
m \neq \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}}|C(m, n)+1| .
\end{align*}
$$

Now we express the last two sums in terms of Euler function $\phi(n)$. Consider

$$
\begin{align*}
& \sum_{m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}\left|C(m, n)-\left(\frac{n}{p}-1\right)\right| \\
& =\sum_{m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}\left|\phi(n) \frac{\mu\left(\frac{n}{(n, m)}\right)}{\phi\left(\frac{n}{(n, m)}\right)}-\left(\frac{n}{p}-1\right)\right|  \tag{2}\\
& =\sum_{m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}\left|\phi(n) \frac{\mu(p)}{\phi(p)}-\left(\frac{n}{p}-1\right)\right| \\
& =\phi(n)+\left(\frac{n}{p}-1\right) \phi(p) .
\end{align*}
$$

We have

$$
\begin{aligned}
& \sum_{m=1}^{n}|C(m, n)+1|=\sum_{m=1}^{n}\left|\phi(n) \frac{\mu\left(\frac{n}{(n, m)}\right)}{\phi\left(\frac{n}{(n, m)}\right)}+1\right| \\
& =\sum_{d \mid n} \phi\left(\frac{n}{d}\right)\left|\phi(n) \frac{\mu\left(\frac{n}{d}\right)}{\phi\left(\frac{n}{d}\right)}+1\right| \\
& =\sum_{\substack{d \left\lvert\, n \\
\mu\left(\frac{n}{d}\right)=1\right.}}\left|\phi(n)+\phi\left(\frac{n}{d}\right)\right| \\
& +\sum_{\substack{d \left\lvert\, n \\
\mu\left(\frac{n}{d}\right)=-1\right.}}\left|-\phi(n)+\phi\left(\frac{n}{d}\right)\right|+\sum_{\substack{d \left\lvert\, n \\
\mu\left(\frac{n}{d}\right)=0\right.}} \phi\left(\frac{n}{d}\right) \\
& =2^{\omega(n)} \phi(n)+\sum_{\substack{d \mid n=1 \\
\mu(d)=1}} \phi(d)-\sum_{\substack{d \mid n \\
\mu(d)=-1}} \phi(d) \\
& +\sum_{\mu(d)=0}^{d \mid n} \phi(d)=2^{\omega(n)} \phi(n)+n-2 \sum_{\mu(d)=-1} d \mid n d(d) .
\end{aligned}
$$

Thus
(3)

$$
\begin{aligned}
& \sum_{\substack{1 \leq m \leq(n-1) \\
m \neq \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}}|C(m, n)+1| \\
& =\sum_{m=1}^{n}|C(m, n)+1|-\sum_{m=n, \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}|C(m, n)+1| \\
& \left.=2^{\omega(n)} \phi(n)+n-2\left(\sum_{\mu(d)=-1}^{d \mid n}\right\rangle(d)\right)-2 \phi(n)+\phi(p)-1 .
\end{aligned}
$$

Substituting (2) and (3) in (1) we obtain,

$$
E_{\chi}\left(X_{n}\right)=2^{\omega(n)} \phi(n)+2 \frac{n}{p} \phi(p)-2 \sum_{\mu(d)=-1}^{d \mid n} \phi(d) .
$$

Corollary 4. If $n=p^{\alpha}$, then $E_{\chi}\left(X_{p^{\alpha}}\right)=4 \phi\left(p^{\alpha}\right)-2 \phi(p)$.

## 3. Complement of Colored Unitary Cayley Graph and Its Color Energy

Recently Adiga et al. [1] have introduced the concept of complement of the colored graph and matrix of the complement colored graph, which are defined as follows.

Definition. Let $G=(V, E)$ be a colored graph. Then the complement of colored graph $G$, denoted by $\overline{G_{c}}$, has same vertex set and same coloring of $G$ with the following properties:
(i) $v_{i}$ and $v_{j}$ are adjacent in $\overline{G_{c}}$, if $v_{i}$ and $v_{j}$ are non-adjacent in $G$ with $c\left(v_{i}\right) \neq c\left(v_{j}\right)$.
(ii) $v_{i}$ and $v_{j}$ are non-adjacent in $\overline{G_{c}}$, if $v_{i}$ and $v_{j}$ are non-adjacent in $G$ with $c\left(v_{i}\right)=c\left(v_{j}\right)$.
(iii) $v_{i}$ and $v_{j}$ are non-adjacent in $\overline{G_{c}}$, if $v_{i}$ and $v_{j}$ are adjacent in $G$.

Remark 5. In the above definition the color complement $\overline{G_{c}}$ of a graph $G$ with respect to a proper vertex coloring $c$ of $G$ is defined such that $c$ is also a proper vertex coloring of $\overline{G_{c}}$. Therefore, the matrix $A_{c}\left(\overline{G_{c}}\right)$ is well defined. We simplify the notation of $A_{c}\left(\overline{G_{c}}\right)$ to $A\left(\overline{G_{c}}\right)$.
Thus the matrix $A\left(\overline{G_{c}}\right)=\left[a_{i j}\right]$, where

$$
a_{i j}=\left\{\begin{aligned}
1 & \text { if } v_{i} \text { and } v_{j} \text { are adjacent in } \overline{G_{c}} \text { with } c\left(v_{i}\right) \neq c\left(v_{j}\right), \\
-1 & \text { if } v_{i} \text { and } v_{j} \text { are non-adjacent in } \overline{G_{c}} \text { with } c\left(v_{i}\right)=c\left(v_{j}\right), \\
0 & \text { otherwise. }
\end{aligned}\right.
$$

Theorem 6. If $\overline{\left(X_{n}\right)_{c}}$ is the complement of the colored unitary Cayley graph $X_{n}$ which is colored with minimum number of colors and $p$ is the smallest prime divisor of $n$, then

$$
E \overline{\left(X_{n}\right)_{c}}=\left(2^{\omega(n)}-2\right) \phi(n)+2(n-p+1)-2 \sum_{\mu(d)=1}^{d \mid n} \phi(d),
$$

where $\omega(n)$ is the number of distinct prime divisors $n$.

Proof. Let $\overline{\left(X_{n}\right)_{c}}$ be the complement of the colored unitary Cayley graph $X_{n}$. Then $A \overline{\left(X_{n}\right)_{c}}$ is a circulant matrix with first row equal to $\left(a_{0}, a_{1}, a_{2}, \ldots, a_{n-1}\right)$, where

$$
a_{k}=\left\{\begin{aligned}
0 & \text { if } k=0 \\
0 & \text { if } 1 \leq k \leq n-1,(k, n)=1, \\
-1 & \text { if } k=p, 2 p, \ldots, n-p, \\
1 & \text { otherwise. }
\end{aligned}\right.
$$

Thus the color eigenvalues of $\overline{\left(X_{n}\right)_{c}}$ are given by

$$
\lambda_{m}=\sum_{k=0}^{n-1} a_{k} e^{\frac{2 \pi i m k}{n}}, m=0,1, \ldots,(n-1)
$$

Let $\omega=e^{\frac{2 \pi i}{n}}$, then we have

$$
\lambda_{m}=\sum_{k=0}^{n-1} a_{k} \omega^{m k}, m=0,1, \ldots,(n-1)
$$

Thus we have

$$
\begin{aligned}
& \lambda_{m}=\sum^{n-1} \underset{\substack{k=1 \\
(k, n) \neq 1}}{ } \quad \omega^{m k}-\sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell p m}, \\
& k \neq p, 2 p, \ldots,\left(\frac{n}{p}-1\right) p \\
& =\sum_{k=1}^{n-1} \omega^{m k}-\sum_{(k, n)=1} \omega^{m k}-2 \sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell p m} \text {. }
\end{aligned}
$$

From the definition of Euler's function we have

$$
\lambda_{0}=(n-1)-\phi(n)-2\left(\frac{n}{p}-1\right)
$$

Using definition of Ramanujan's sum

$$
\lambda_{m}=-1-C(m, n)-2\left(\frac{n}{p}-1\right), m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}
$$

and

$$
\lambda_{m}=1-C(m, n), 1 \leq m \leq n-1 \text { and } m \neq \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}
$$

Thus

$$
\begin{align*}
\overline{E\left(X_{n}\right)_{c}} & =\sum_{m=0}^{n-1}\left|\lambda_{m}\right| \\
& =\left|(n-1)-\phi(n)-2\left(\frac{n}{p}-1\right)\right| \\
& +\sum_{m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}\left|1+C(m, n)+2\left(\frac{n}{p}-1\right)\right|  \tag{4}\\
& +\sum_{\substack{1 \leq m \leq(n-1) \\
m \neq \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}}|1-C(m, n)|
\end{align*}
$$

Consider

$$
\begin{align*}
& \sum_{m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}\left|1+C(m, n)+2\left(\frac{n}{p}-1\right)\right| \\
& =\phi(p)\left|1+\frac{\phi(n) \mu(p)}{\phi(p)}+2\left(\frac{n}{p}-1\right)\right|  \tag{5}\\
& =\left|\phi(p)-\phi(n)+2\left(\frac{n}{p}-1\right) \phi(p)\right|=\phi(p)\left(\frac{2 n}{p}-1\right)-\phi(n) .
\end{align*}
$$

We have

$$
\begin{aligned}
& \sum_{m \neq n, \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}^{n-1}|1-C(m, n)| \\
& =\sum_{m=1}^{n}|1-C(m, n)|-\sum_{m=n, \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}|1-C(m, n)| \\
& =\sum_{d \mid n}\left|\phi\left(\frac{n}{d}\right)-\phi(n) \mu\left(\frac{n}{d}\right)\right|-\phi(p)-2 \phi(n)+1 \\
& =\sum_{\substack{d \left\lvert\, n \\
\mu\left(\frac{n}{d}\right)=1\right.}}\left|\phi\left(\frac{n}{d}\right)-\phi(n)\right|+\sum_{\substack{d \left\lvert\, n \\
\mu\left(\frac{n}{d}\right)=-1\right.}}\left|\phi\left(\frac{n}{d}\right)+\phi(n)\right| \\
& +\sum_{\substack{d \left\lvert\, n \\
\mu\left(\frac{n}{d}\right)=0\right.}} \phi\left(\frac{n}{d}\right)-\phi(p)-2 \phi(n)+1 \\
& =2^{\omega(n)} \phi(n)+\sum_{d \mid n} \phi(d)-2\left(\sum_{\substack{d \mid n \\
\mu(d)=1}} \phi(d)\right)-\phi(p)-2 \phi(n)+1 \text {. }
\end{aligned}
$$

Substituting (5) and (6) in (4) and after some simplifications, we obtain

$$
E \overline{\left(X_{n}\right)_{c}}=\left(2^{\omega(n)}-2\right) \phi(n)+2(n-p+1)-2 \sum_{\mu(d)=1}^{d \mid n} \phi(d) .
$$

Corollary 7. If $n=p^{\alpha}$, then $E \overline{\left(X_{p^{\alpha}}\right)_{c}}=2\left(p^{\alpha}-p\right)$.
Remark 8. $E_{\chi}\left(X_{p^{\alpha}}\right)>E \overline{\left(X_{p^{\alpha}}\right)_{c}}$.

## 4. Color Energy of Some gcd-graphs

Klotz and Sander extended the class of unitary Cayley graphs. Let $D$ be a set of positive, proper divisors of the integer $n>1$. We recall the definition of the gcd-graph. Define the gcd-graph $X_{n}(D)$ to have vertex set $Z_{n}=\{0,1, \ldots, n-1\}$ and edge set $E\left(X_{n}(D)\right)=\left\{\{a, b\}: a, b \in Z_{n},(a-b, n) \in D\right\}$. So [10] proved that a circulant graph is integral if, and only if, it is a gcd-graph.

Theorem 9. Let $n=p_{1}{ }^{\alpha_{1}} p_{2}{ }^{\alpha_{2}} \ldots p_{k}{ }^{\alpha_{k}}$ and $X_{n}(D)$ be the gcd-graph, where $D=$ $\left\{1, p_{i}\right\}, i>1, \alpha_{i}=1$. If $p\left(=p_{1}\right)$ is the smallest prime divisor of $n$, then the gcd-graph $X_{n}(D)$ has a unique optimal coloring.

Proof. Let $X_{n}(D)$ be the gcd-graph where $D=\left\{1, p_{i}\right\}, i>1$ and $\alpha_{i}=1$ whose vertices are labelled by $0,1,2, \ldots,(n-1)$. Let $p\left(=p_{1}\right)$ be the smallest prime divisor of n . Now we color $X_{n}(D)$ using $p$ colors say $c_{1}, c_{2}, \ldots, c_{p}$. Using similar arguments as in the proof of Theorem 2, the vertices $i, i+p, i+2 p, \ldots, i+\left(\frac{n}{p}-1\right) p$, $i=0,1, \ldots,(p-1)$ must receive the same color. Also observe that, if $x$ and $y$ are two vertices of $X_{n}(D)$ such that $\operatorname{gcd}(x-y, n)=p_{i}$, then $x$ and $y$ must receive different colors. For, if $x$ and $y$ receive the same color then $p \mid(x-y)$ and hence $p \mid p_{i}$ which is a contradiction.

Theorem 10. Let $n=p_{1}{ }^{\alpha_{1}} p_{2}{ }^{\alpha_{2}} \ldots p_{k}{ }^{\alpha_{k}}$ and $X_{n}(D)$ be the gcd-graph, where $D=\left\{1, p_{i}\right\}, i>1, \alpha_{i}=1$. If $p\left(=p_{1}\right)$ is the smallest prime divisor of $n$ and $X_{n}(D)$ be colored with $p$ colors, then

$$
\begin{aligned}
E_{\chi}\left[X_{n}(D)\right] & =n-\frac{n}{p_{i}}+2^{\omega(n)-1}\left(\phi(n)+\phi\left(\frac{n}{p_{i}}\right)\right) \\
& +\frac{n}{p} \phi(p)-\frac{n}{p}+\sum_{\ell \left\lvert\,\left(\frac{n}{p_{i}}\right)\right.} \mu\left(\frac{n}{\ell p_{i}}\right) \phi\left(\frac{n}{\ell p_{i}}\right)
\end{aligned}
$$

Proof. Let $X_{n}(D)$ be the gcd-graph, where $D=\left\{1, p_{i}\right\}$. We color the vertices $0,1,2, \ldots, n-1$ with colors $c_{1}, c_{2}, \ldots, c_{p}$ as follows. Color the vertices $k p, k p+1, k p+2, \ldots,(k+1) p-1$ with $c_{1}, c_{2}, c_{3}, \ldots, c_{p}$ respectively for $k=$ $0,1,2, \ldots,\left(\frac{n}{p}-1\right)$. Then $A_{\chi}\left[X_{n}(D)\right]$ is a circulant matrix with first row equal to $\left(a_{0}, a_{1}, a_{2}, \ldots, a_{n-1}\right)$, where

$$
a_{k}=\left\{\begin{aligned}
1 & \text { if }(k, n)=1 \\
1 & \text { if }(k, n)=p_{i} \\
-1 & \text { if } k=p, 2 p, \ldots, n-p \\
0 & \text { otherwise }
\end{aligned}\right.
$$

Thus the color eigenvalues of $X_{n}(D)$ are given by

$$
\lambda_{m}=\sum_{k=0}^{n-1} a_{k} \omega^{m k}, m=0,1, \ldots,(n-1)
$$

Let $\omega=e^{\frac{2 \pi i}{n}}$. Then we have,

$$
\begin{aligned}
\lambda_{m} & =\sum_{\substack{k, n)=1}}^{n-1} \omega^{m k}+\sum_{(k, n)=p_{i}}^{n-1} \omega^{m k}-\sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell p m}, m=0,1, \ldots,(n-1) \\
& =C(m, n)+C\left(m, \frac{n}{p_{i}}\right)-\sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell p m} .
\end{aligned}
$$

We note that

$$
\lambda_{m}= \begin{cases}\phi(n)+\phi\left(\frac{n}{p_{i}}\right)-\left(\frac{n}{p}-1\right), & m=0, \\ C(m, n)+C\left(m, \frac{n}{p_{i}}\right)+1, & 1 \leq m \leq n-1, \\ & m \neq \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}, \\ C(m, n)+C\left(m, \frac{n}{p_{i}}\right)-\left(\frac{n}{p}-1\right), & 1 \leq m \leq n-1, \\ & m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p} .\end{cases}
$$

Observe that

$$
\begin{equation*}
\left|\lambda_{0}\right|=\phi(n)+\phi\left(\frac{n}{p_{i}}\right)-\left(\frac{n}{p}-1\right), \tag{7}
\end{equation*}
$$

and

$$
\begin{align*}
& \sum_{m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}\left|C(m, n)+C\left(m, \frac{n}{p_{i}}\right)-\left(\frac{n}{p}-1\right)\right|  \tag{8}\\
& =\phi(n)+\phi\left(\frac{n}{p_{i}}\right)+\phi(p)\left(\frac{n}{p}-1\right) .
\end{align*}
$$

Further,

$$
\begin{aligned}
& \sum_{m \neq \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}^{n}\left|C(m, n)+C\left(m, \frac{n}{p_{i}}\right)+1\right| \\
& =\sum_{m=1}^{n}\left|C(m, n)+C\left(m, \frac{n}{p_{i}}\right)+1\right| \\
& -\sum_{m=n, \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}\left|C(m, n)+C\left(m, \frac{n}{p_{i}}\right)+1\right|} \begin{array}{l}
n \\
=\sum_{m=1}^{n}\left|1+\frac{\phi(n) \mu\left(\frac{n}{(m, n)}\right)}{\phi\left(\frac{n}{(m, n)}\right)}+\frac{\phi\left(n / p_{i}\right) \mu\left(\frac{n}{\left(p_{i} m, n\right)}\right)}{\phi\left(\frac{n}{\left(p_{i} m, n\right)}\right)}\right| \\
-2 \phi(n)-2 \phi\left(\frac{n}{p_{i}}\right)+\phi(p)-1 .
\end{array} .
\end{aligned}
$$

But, $\left(m, p_{i}\right)=1$ or $p_{i}$. If $\left(m, p_{i}\right)=1$, then $\left(m p_{i}, n\right)=p_{i}(m, n)$ and if $\left(m, p_{i}\right)=p_{i}$,
then $(m, n)=\left(m p_{i}, n\right)$. Thus the above sum is equal to

$$
\begin{align*}
& \sum_{\left(m, p_{i}\right)=1}^{n}\left|1+\frac{\phi(n) \mu\left(\frac{n}{(m, n)}\right)}{\phi\left(\frac{n}{(m, n)}\right)}+\frac{\phi\left(\frac{n}{p_{i}}\right) \mu\left(\frac{n}{\left(p_{i} m, n\right)}\right)}{\phi\left(\frac{n}{\left(p_{i} m, n\right)}\right)}\right| \\
& +\sum_{\substack{m=1 \\
\left(m, p_{i}\right)=p_{i}}}^{n}\left|1+\frac{\phi(n) \mu\left(\frac{n}{(m, n)}\right)}{\phi\left(\frac{n}{(m, n)}\right)}+\frac{\phi\left(\frac{n}{p_{i}}\right) \mu\left(\frac{n}{\left(p_{i} m, n\right)}\right)}{\phi\left(\frac{n}{\left(p_{i} m, n\right)}\right)}\right|  \tag{9}\\
& -2 \phi(n)-2 \phi\left(\frac{n}{p_{i}}\right)+\phi(p)-1 .
\end{align*}
$$

After some simplification the first sum is equal to

$$
\begin{align*}
& \sum_{\substack{d \neq p_{i}, 2 p_{i}, \ldots, \frac{n}{p_{i}} p_{i}}} \phi\left(\frac{n}{d}\right)\left|\frac{\phi(n) \mu\left(\frac{n}{d}\right)}{\phi\left(\frac{n}{d}\right)}+\frac{\phi\left(\frac{n}{p_{i}}\right) \mu\left(\frac{n}{p_{i} d}\right)}{\phi\left(\frac{n}{p_{i} d}\right)}+1\right| \\
& =\sum_{d \neq p_{i}, 2 p_{i}, \ldots, \frac{n}{p_{i}} p_{i}}^{d \mid n}\left|\phi(n) \mu\left(\frac{n}{d}\right)+\phi(n) \mu\left(\frac{n}{p_{i} d}\right)+\phi\left(\frac{n}{d}\right)\right|  \tag{10}\\
& \left.=\sum_{d \neq p_{i}, 2 p_{i}, \ldots, \frac{n}{p_{i}} p_{i}}^{d \mid n}\right\rangle\left(\frac{n}{d}\right)=n-\frac{n}{p_{i}} .
\end{align*}
$$

The second sum is equal to

$$
\begin{aligned}
& \sum_{\left(m, p_{i}\right)=p_{i}}^{n}\left|1+\frac{\phi(n) \mu\left(\frac{n}{(m, n)}\right)}{\phi\left(\frac{n}{(m, n)}\right)}+\frac{\phi\left(\frac{n}{p_{i}}\right) \mu\left(\frac{n}{(m, n)}\right)}{\phi\left(\frac{n}{(m, n)}\right)}\right| \\
& =\sum_{\ell=1}^{\frac{n}{p_{i}}}\left|\frac{\mu\left(\frac{n}{\left.\ell \ell p_{i}, n\right)}\right)}{\phi\left(\frac{n}{\left.\ell \ell p_{i}, n\right)}\right)}\left(\phi(n)+\phi\left(\frac{n}{p_{i}}\right)\right)+1\right| \\
& =\sum_{\ell \left\lvert\, \frac{n}{p_{i}}\right.} \phi\left(\frac{n}{\ell p_{i}}\right)\left|\frac{\mu\left(\frac{n}{\left(\ell p_{i}\right)}\right)}{\phi\left(\frac{n}{\left.\ell \ell p_{i}\right)}\right)}\left(\phi(n)+\phi\left(\frac{n}{p_{i}}\right)\right)+1\right| \\
& =\sum_{\ell \left\lvert\, \frac{n}{p_{i}}\right.} \mu\left(\frac{n}{p_{i}}\right)\left(\phi(n)+\phi\left(\frac{n}{p_{i}}\right)\right)+\phi\left(\frac{n}{\ell p_{i}}\right) \\
& =\sum_{\ell \left\lvert\, \frac{n}{p_{i}}\right.} \phi(n)+\phi\left(\frac{n}{p_{i}}\right)+\mu\left(\frac{n}{\ell p_{i}}\right) \phi\left(\frac{n}{\ell p_{i}}\right) \\
& =2^{\omega(n)-1}\left(\phi(n)+\phi\left(\frac{n}{p_{i}}\right)\right)+\sum_{\ell \left\lvert\, \frac{n}{p_{i}}\right.} \mu\left(\frac{n}{\ell p_{i}}\right) \phi\left(\frac{n}{\ell p_{i}}\right) .
\end{aligned}
$$

Substituting (10) and (11) in (9) we get

$$
\begin{align*}
& \sum_{m \neq \frac{n}{p}, \frac{m}{p}, \ldots, \frac{(p-1) n}{p}}^{n}\left|C(m, n)+C\left(m, \frac{n}{p_{i}}\right)+1\right| \\
& =n-\frac{n}{p_{i}}+2\left(2^{\omega(n)-2}-1\right)\left(\phi(n)+\phi\left(\frac{n}{p_{i}}\right)\right)  \tag{12}\\
& +\phi(p)-1+\sum_{\ell \backslash\left(\frac{n}{p_{i}}\right)} \mu\left(\frac{n}{\ell p_{i}}\right) \phi\left(\frac{n}{\ell p_{i}}\right) .
\end{align*}
$$

Adding (7), (8) and (12) and simplifying we obtain

$$
\begin{aligned}
E_{\chi}\left[X_{n}(D)\right] & =n-\frac{n}{p_{i}}+2^{\omega(n)-1}\left(\phi(n)+\phi\left(\frac{n}{p_{i}}\right)\right)+\frac{n}{p} \phi(p) \\
& -\frac{n}{p}+\sum_{\ell \left\lvert\,\left(\frac{n}{p_{i}}\right)\right.} \mu\left(\frac{n}{\ell p_{i}}\right) \phi\left(\frac{n}{\ell p_{i}}\right) .
\end{aligned}
$$

This completes the proof.
Theorem 11. Let $n=p_{1} p_{2} \ldots p_{k}$ be a square-free number and $X_{n}(D)$ be the gcd-graph, where $D=\left\{p_{i}, p_{j}\right\}$. If $p\left(=p_{1}\right) \neq p_{i}, p_{j}$ is the smallest prime divisor of $n$ and $X_{n}(D)$ can be colored with $p$ colors, then

$$
\begin{aligned}
E_{\chi}\left[X_{n}(D)\right] & =\phi\left(\frac{n}{p_{i}}\right)+\phi\left(\frac{n}{p_{j}}\right)-\left(\frac{n}{p}-1\right) \\
& +\sum_{m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}^{n}\left|C\left(m, \frac{n}{p_{i}}\right)+C\left(m, \frac{n}{p_{j}}\right)-\left(\frac{n}{p}-1\right)\right| \\
& +\sum_{m \neq \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}^{n}\left|C\left(m, \frac{n}{p_{i}}\right)+C\left(m, \frac{n}{p_{j}}\right)+1\right| .
\end{aligned}
$$

Proof. Let $X_{n}(D)$ be the gcd-graph, where $D=\left\{p_{i}, p_{j}\right\}$. We color the vertices $0,1,2, \ldots, n-1$ with colors $c_{1}, c_{2}, \ldots, c_{p}$ as follows: Color the vertices $k p, k p+1, k p+2, \ldots,(k+1) p-1$ with $c_{1}, c_{2}, c_{3}, \ldots, c_{p}$ respectively for $k=$ $0,1,2, \ldots,\left(\frac{n}{p}-1\right)$. Then $A_{\chi}\left[X_{n}(D)\right]$ is a circulant matrix with first row equal to ( $a_{0}, a_{1}, a_{2}, \ldots, a_{n-1}$ ), where

$$
a_{k}=\left\{\begin{aligned}
1 & \text { if }(k, n)=p_{i} \\
1 & \text { if }(k, n)=p_{j}, \\
-1 & \text { if } k=p, 2 p, \ldots, n-p, \\
0 & \text { otherwise. }
\end{aligned}\right.
$$

The color eigenvalues of $X_{n}(D)$ are given by

$$
\begin{aligned}
\lambda_{m} & =\sum_{k=0}^{n-1} a_{k} \omega^{m k}, m=0,1, \ldots,(n-1) \\
& =\sum_{\substack{k=1 \\
(k, n)=p_{i}}}^{n-1} \omega^{m k}+\sum_{\substack{k=1 \\
(k, n)=p_{j}}}^{n-1} \omega^{m k}-\sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell p m} \\
& =C\left(m, \frac{n}{p_{i}}\right)+C\left(m, \frac{n}{p_{j}}\right)-\sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell p m} .
\end{aligned}
$$

Thus we have

$$
\lambda_{m}= \begin{cases}\phi\left(\frac{n}{p_{i}}\right)+\phi\left(\frac{n}{p_{j}}\right)-\left(\frac{n}{p}-1\right) & m=0, \\ C\left(m, \frac{n}{p_{i}}\right)+C\left(m, \frac{n}{p_{j}}\right)+1 & (1 \leq m \leq n-1) \text { and } \\ & m \neq \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p} \\ C\left(m, \frac{n}{p_{i}}\right)+C\left(m, \frac{n}{p_{j}}\right)-\left(\frac{n}{p}-1\right) & m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p} .\end{cases}
$$

Therefore,

$$
\begin{aligned}
E_{\chi}\left[X_{n}(D)\right] & =\phi\left(\frac{n}{p_{i}}\right)+\phi\left(\frac{n}{p_{j}}\right)-\left(\frac{n}{p}-1\right) \\
& +\sum_{m=\frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}^{n}\left|C\left(m, \frac{n}{p_{i}}\right)+C\left(m, \frac{n}{p_{j}}\right)-\left(\frac{n}{p}-1\right)\right| \\
& +\sum_{\substack{m=1 \\
m \neq \frac{n}{p}, \frac{2 n}{p}, \ldots, \frac{(p-1) n}{p}}}^{n}\left|C\left(m, \frac{n}{p_{i}}\right)+C\left(m, \frac{n}{p_{j}}\right)+1\right|
\end{aligned}
$$

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