COLOR ENERGY OF A UNITARY CAYLEY GRAPH

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Abstract

Let G be a vertex colored graph. The minimum number $\chi(G)$ of colors needed for coloring of a graph G is called the *chromatic number*. Recently, Adiga *et al.* [1] have introduced the concept of color energy of a graph $E_c(G)$ and computed the color energy of few families of graphs with $\chi(G)$ colors. In this paper we derive explicit formulas for the color energies of the unitary Cayley graph X_n, the complement of the colored unitary Cayley graph $\overline{(X_n)_c}$ and some gcd-graphs.

Keywords: coloring of a graph, unitary Cayley graph, gcd-graph, color eigenvalues, color energy.

2010 Mathematics Subject Classification: 05C15, 05C25, 05C50.

1. Introduction

Let G be a finite group and S be a subset of G such that S does not contain identity of G. Assume $S^{-1} = \{s^{-1} : s \in S\} = S$. The *Cayley graph* $X = Cay(G, S)$ is an undirected graph having vertex set $V(X) = G$ and edge set $E(X) = \{\{a, b\} :$

$ab^{-1} \in S\}$, where $a, b \in G$. The Cayley graph X is a regular graph of degree $|S|$. Its connected components are the right cosets of the subgroup generated by S. Therefore, if S generates G, then X is a connected graph. More information about Cayley graphs can be found in the books on algebraic graph theory by Biggs [2] and by Godsil and Royle [3].

For a positive integer $n > 1$, the *unitary Cayley graph* $X_n = Cay(Z_n; U_n)$, where U_n denotes the set of all units of the ring Z_n. Two vertices a, b are adjacent if, and only if, $a - b \in U_n$. The unitary Cayley graph X_n is regular of degree $\phi(n)$, where $\phi(n)$ is the Euler function. Unitary Cayley graphs are highly symmetric and have some remarkable properties connecting graph theory and number theory. These graphs have integral spectrum and play an important role in modeling quantum spin networks supporting the perfect state of transfer. In fact, it is proved in [6] that the eigenvalues of unitary Cayley graph X_n are

$$\lambda_i = \sum\nolimits_{1 \leq j < n, \ (j,n)=1} \omega^{ij} = C(i,n), \ 0 \leq i \leq n-1,$$

where ω denotes a complex primitive n^{th} root of unity.

The arithmetic function $C(i, n)$ is a Ramanujan sum [5], and

$$C(i,n) = \phi(n) \frac{\mu\left(\frac{n}{(i,n)}\right)}{\phi\left(\frac{n}{(i,n)}\right)},$$

where μ denotes the Möbius function.

The gcd-graphs arise as a generalization of unitary Cayley graphs studied by Klotz and Sander in [6]. Let D be a set of positive, proper divisors of the integer $n > 1$. Define the *gcd-graph* $X_n(D)$ to have vertex set $Z_n = \{0, 1, \ldots, n-1\}$ and edge set $E(X_n(D)) = \{\{a, b\} : a, b \in Z_n, (a - b, n) \in D\}$. In [6], it was proved that these graphs have integral spectrum.

Let G be a simple graph. The *adjacency matrix* of G is the $n \times n$ matrix $A = A(G)$, whose entries a_{ij} are given by $a_{ij} = 1$ if v_i and v_j are adjacent, $a_{ij} = 0$ otherwise. The eigenvalues of $A(G)$ are the eigenvalues of G. The *energy* $E(G)$ of a graph G is the sum of the absolute values of the eigenvalues of $A(G)$ [4].

Recently, Sampathkumar and Sriraj [9] have introduced a new matrix $A_L(G)$ called *L-matrix of a vertex labeled graph* $G = (V, E)$, whose elements are defined as follows: If $\ell(v_i)$ is the label of the vertex v_i, then

$$a_{ij} = \begin{cases} 2 & \text{if } v_i \text{ and } v_j \text{ are adjacent with } \ell(v_i) = \ell(v_j), \\ 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent with } \ell(v_i) \neq \ell(v_j), \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } \ell(v_i) = \ell(v_j), \\ 0 & \text{if } v_i = v_j \text{ or } v_i \text{ and } v_j \text{ are non-adjacent with } \ell(v_i) \neq \ell(v_j). \end{cases}$$

A *coloring* of graph G is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring of a graph G is called *chromatic number* and denoted by $\chi(G)$.

If we consider the vertex colored graph, then entries of the matrix $A_L(G)$ are as follows: If $c(v_i)$ is the color of v_i, then

$$
a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j), \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j), \\ 0 & \text{if } v_i = v_j \text{ or } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) \neq c(v_j). \end{cases}
$$

The matrix thus obtained is the *L-matrix of the colored graph* and is G denoted by $A_c(G)$. The eigenvalues of $A_c(G)$ are called *color eigenvalues*. If G is colored with $\chi(G)$ colors, then L-matrix of the colored graph G is denoted by $A_\chi(G)$. Recently in [1], Adiga, Sampathkumar, Sriraj, Shrikanth have studied the energy of the vertex colored graph, which is defined as follows. The energy of a graph with respect to a given coloring is the sum of the absolute values of the color eigenvalues of G and is called the *energy of a colored graph* or *color energy* of a graph.

If we use n different colors to the vertices of a graph of order n, then the color energy is the same as the energy of a graph. So color energy may be considered as a generalization of energy of a graph. It is possible that color energy that we are considering in this paper has similar applications in chemistry as well as in some other areas.

Recently, llić [7] obtained an explicit formula for the energy of unitary Cayley graph X_n and also energy of the complement of X_n. The open problem posed by llić [7] about calculating energy of an arbitary integral circulant graph is completely solved by Mollahajiaghaei in [8].

Motivated by these investigations we establish formulas for color energies of the unitary Cayley graph X_n and the complement of the colored unitary Cayley graph X_n. We also derive an explicit formula for energy of the colored gcd-graph $X_n(D)$, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{i-1}^{\alpha_{i-1}} p_i^{1} p_{i+1}^{\alpha_{i+1}} \ldots p_k^{\alpha_k}$ and $D = \{1, p_i\}$.

2. COLOR ENERGY OF UNITARY CAYLEY GRAPHS

Let $X_n = (Z_n; U_n)$ be the unitary Cayley graph and $E_\chi(X_n)$ denote the color energy of X_n with $\chi(X_n)$ colors. In [6], Klotz and Sander proved the following theorem:

Theorem 1. *If p is the smallest prime divisor of n, then $\chi(X_n) = p$.*

In general, an optimal coloring with $\chi(G)$ colors is not unique. So the color energy $E_\chi(G)$ may be different for different optimal colorings. But the unitary

Cayley graph X_n has a unique optimal coloring, thus its color energy with respect to minimum number of colors is unique.

Theorem 2. *The unitary Cayley graph X_n has a unique optimal coloring and the color energy of X_n with respect to minimum number of colors is unique.*

Proof. Let X_n be a unitary Cayley graph whose vertices are labeled by $0, 1, 2, \ldots, n-1$. Let p be the smallest prime divisor of n. Then $\chi(X_n) = p$. Now we color X_n using p colors say $c_1, c_2, \ldots, c_p$. Consider the vertices $0, 1, 2, \ldots, p-1$. Let $i, j \in \{0, 1, 2, \ldots, p-1\}$, where $i < j$. Since $gcd(j - i, n) = 1$, i and j are adjacent so we cannot give the same color to i and j. Similarly, the vertices $ip, ip+1, \ldots, (i+1)p - 1$, $i = 1, 2, 3, \ldots, (\frac{n}{p} - 1)$ must receive different colors.

Suppose color of the vertex 0 is c_1 then color of the vertex p must be equal c_1 because p and i are adjacent for each $i = 1, 2, 3, \ldots, p-1$. On similar lines the color of the vertices $2p, 3p, \ldots, (\frac{n}{p} - 1)p$ must be c_1. Thus the vertices $0, p, 2p, 3p, \ldots, (\frac{n}{p} - 1)p$ receive the color c_1. Similarly, the colors of the vertices $1, p + 1, 2p + 1, \ldots, (\frac{n}{p} - 1)p + 1$, must be the same. In general $i, i + p, i + 2p, \ldots, i + (\frac{n}{p} - 1)p$, $i = 0, 1, \ldots, (p - 1)$ must receive the same color. This shows that the matrix $A_\chi(X_n)$ will not alter for different optimal colorings. Hence, the color energy of X_n with respect to minimum number of colors is unique. This completes the proof. ∎

Now we determine the color energy of a unitary Cayley graph with minimum number of colors.

Theorem 3. *Let X_n be the unitary Cayley graph colored with p colors, where p is the smallest prime divisor of n. Then*

$$E_\chi(X_n) = 2^{\omega(n)}\phi(n) + 2\frac{n}{p}\phi(p) - 2\sum_{\substack{d|n \\ \mu(d)=-1}} \phi(d),$$

where $\omega(n)$ is the number of distinct prime divisors of n.

Proof. Let $X_n = (Z_n; U_n)$ be the unitary Cayley graph colored with p colors, where p is the smallest prime divisor of n. We color the vertices $0, 1, 2, \ldots, n-1$ with colors $c_1, c_2, \ldots, c_p$ as follows: Color the vertices $kp, kp + 1, kp + 2, \ldots, (k+1)p - 1$ with $c_1, c_2, c_3, \ldots, c_p$ respectively for $k = 0, 1, 2, \ldots, \left(\frac{n}{p} - 1\right)$. Then $A_\chi(X_n)$ is a circulant matrix with first row equal to $(a_0, a_1, a_2, \ldots, a_{n-1})$, where

$$a_k = \begin{cases} 0 & \text{if } k = 0, \\ 1 & \text{if } 1 \leq k \leq n - 1, \ (k, n) = 1, \\ -1 & \text{if } k = p, 2p, \ldots, n - p, \\ 0 & \text{otherwise.} \end{cases}$$

It is well-known that for a circulant matrix of order n, the eigenvalues are given by the formula

$$\lambda_m = \sum_{k=0}^{n-1} a_k e^{\frac{2\pi imk}{n}}, \ m = 0, 1, \ldots, (n-1).$$

Let $\omega = e^{\frac{2\pi i}{n}}$, then we have

$$\lambda_m = \sum_{k=0}^{n-1} a_k \omega^{mk}, \ m = 0, 1, \ldots, (n-1)$$
$$= \sum_{\substack{k=1 \\ (k,n)=1}}^{n-1} \omega^{mk} - \sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell pm}$$
$$= C(m, n) - \sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell pm}.$$

We have $\lambda_0 = \sum_{k=0}^{n-1} a_k$. Note that $a_k = 1$ if $1 \leq k \leq (n-1)$, $(k, n) = 1$, $a_k = -1$ if $k = p, 2p \ldots, n - p$ and 0 otherwise. Thus the number of terms that are equal to 1 in the above sum is $\phi(n)$ and the number of terms that are equal to -1 is $\left(\frac{n}{p} - 1\right)$ and other terms are zero. Hence,

$$\lambda_0 = \phi(n) - \left(\frac{n}{p} - 1\right),$$

using definition of Ramanujan's sum we have

$$\lambda_m = C(m, n) - \left(\frac{n}{p} - 1\right) \ \text{for} \ m = \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p},$$

and

$$\lambda_m = C(m, n) + 1, \ \text{for} \ 1 \leq m \leq n - 1 \ \text{and} \ m \neq \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}.$$

Thus

(1)
$$
\begin{aligned}
E_\chi(X_n) &= \sum_{m=0}^{n-1} |\lambda_m| \\
&= |\lambda_0| + \sum_{m = \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}} |\lambda_m| + \sum_{\substack{1 \leq m \leq (n-1) \\ m \neq \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}}} |\lambda_m| \\
&= \phi(n) - \left(\frac{n}{p} - 1\right) + \sum_{m = \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}} \left| C(m, n) - \left(\frac{n}{p} - 1\right) \right| \\
&\quad + \sum_{\substack{1 \leq m \leq (n-1) \\ m \neq \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}}} |C(m, n) + 1|.
\end{aligned}
$$

Now we express the last two sums in terms of Euler function $\phi(n)$. Consider

$$
\sum_{m=\frac{n}{p},\frac{2n}{p},...,\frac{(p-1)n}{p}} \left| C(m,n) - \left(\frac{n}{p} - 1\right) \right|
$$

(2)
$$
= \sum_{m=\frac{n}{p},\frac{2n}{p},...,\frac{(p-1)n}{p}} \left| \phi(n)\frac{\mu\left(\frac{n}{(n,m)}\right)}{\phi\left(\frac{n}{(n,m)}\right)} - \left(\frac{n}{p} - 1\right) \right|
$$

$$
= \sum_{m=\frac{n}{p},\frac{2n}{p},...,\frac{(p-1)n}{p}} \left| \phi(n)\frac{\mu(p)}{\phi(p)} - \left(\frac{n}{p} - 1\right) \right|
$$

$$
= \phi(n) + \left(\frac{n}{p} - 1\right)\phi(p).
$$

We have

$$
\sum_{m=1}^{n} |C(m,n) + 1| = \sum_{m=1}^{n} \left| \phi(n)\frac{\mu\left(\frac{n}{(n,m)}\right)}{\phi\left(\frac{n}{(n,m)}\right)} + 1 \right|
$$

$$
= \sum_{d|n} \phi\left(\frac{n}{d}\right) \left| \phi(n)\frac{\mu\left(\frac{n}{d}\right)}{\phi\left(\frac{n}{d}\right)} + 1 \right|
$$

$$
= \sum_{\substack{d|n \\ \mu\left(\frac{n}{d}\right)=1}} \left| \phi(n) + \phi\left(\frac{n}{d}\right) \right|
$$

$$
+ \sum_{\substack{d|n \\ \mu\left(\frac{n}{d}\right)=-1}} \left| -\phi(n) + \phi\left(\frac{n}{d}\right) \right| + \sum_{\substack{d|n \\ \mu\left(\frac{n}{d}\right)=0}} \phi\left(\frac{n}{d}\right)
$$

$$
= 2^{\omega(n)}\phi(n) + \sum_{\substack{d|n \\ \mu(d)=1}} \phi(d) - \sum_{\substack{d|n \\ \mu(d)=-1}} \phi(d)
$$

$$
+ \sum_{\substack{d|n \\ \mu(d)=0}} \phi(d) = 2^{\omega(n)}\phi(n) + n - 2\sum_{\substack{d|n \\ \mu(d)=-1}} \phi(d).
$$

Thus

$$
\sum_{\substack{1 \le m \le (n-1) \\ m \ne \frac{n}{p},\frac{2n}{p},...,\frac{(p-1)n}{p}}} |C(m,n) + 1|
$$

(3)
$$
= \sum_{m=1}^{n} |C(m,n) + 1| - \sum_{m=n,\frac{n}{p},\frac{2n}{p},...,\frac{(p-1)n}{p}} |C(m,n) + 1|
$$

$$
= 2^{\omega(n)}\phi(n) + n - 2\left(\sum_{\substack{d|n \\ \mu(d)=-1}} \phi(d)\right) - 2\phi(n) + \phi(p) - 1.
$$

Substituting (2) and (3) in (1) we obtain,

$$E_\chi(X_n) = 2^{\omega(n)}\phi(n) + 2\frac{n}{p}\phi(p) - 2\sum_{\substack{d|n \\ \mu(d)=-1}} \phi(d).$$

∎

Corollary 4. *If* $n = p^\alpha$, *then* $E_\chi(X_{p^\alpha}) = 4\phi(p^\alpha) - 2\phi(p)$.

3. Complement of Colored Unitary Cayley Graph and Its Color Energy

Recently Adiga *et al.* [1] have introduced the concept of complement of the colored graph and matrix of the complement colored graph, which are defined as follows.

Definition. Let $G = (V, E)$ be a colored graph. Then the *complement of colored graph* G, denoted by $\overline{G_c}$, has same vertex set and same coloring of G with the following properties:
  (i) v_i and v_j are adjacent in $\overline{G_c}$, if v_i and v_j are non-adjacent in G with $c(v_i) \neq c(v_j)$.
  (ii) v_i and v_j are non-adjacent in $\overline{G_c}$, if v_i and v_j are non-adjacent in G with $c(v_i) = c(v_j)$.
  (iii) v_i and v_j are non-adjacent in $\overline{G_c}$, if v_i and v_j are adjacent in G.

Remark 5. In the above definition the color complement $\overline{G_c}$ of a graph G with respect to a proper vertex coloring c of G is defined such that c is also a proper vertex coloring of $\overline{G_c}$. Therefore, the matrix $A_c(\overline{G_c})$ is well defined. We simplify the notation of $A_c(\overline{G_c})$ to $A(\overline{G_c})$.

Thus the matrix $A(\overline{G_c}) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent in } \overline{G_c} \text{ with } c(v_i) \neq c(v_j), \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent in } \overline{G_c} \text{ with } c(v_i) = c(v_j), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6. *If* $\overline{(X_n)_c}$ *is the complement of the colored unitary Cayley graph* X_n *which is colored with minimum number of colors and* p *is the smallest prime divisor of* n, *then*

$$E\overline{(X_n)_c} = (2^{\omega(n)} - 2)\phi(n) + 2(n - p + 1) - 2\sum_{\substack{d|n \\ \mu(d)=1}} \phi(d),$$

where $\omega(n)$ *is the number of distinct prime divisors* n.

Proof. Let $\overline{(X_n)_c}$ be the complement of the colored unitary Cayley graph X_n. Then $A\overline{(X_n)_c}$ is a circulant matrix with first row equal to $(a_0, a_1, a_2, \ldots, a_{n-1})$, where

$$a_k = \begin{cases} 0 & \text{if } k = 0, \\ 0 & \text{if } 1 \leq k \leq n-1, \ (k,n) = 1, \\ -1 & \text{if } k = p, 2p, \ldots, n-p, \\ 1 & \text{otherwise.} \end{cases}$$

Thus the color eigenvalues of $\overline{(X_n)_c}$ are given by

$$\lambda_m = \sum_{k=0}^{n-1} a_k e^{\frac{2\pi i m k}{n}}, \ m = 0, 1, \ldots, (n-1).$$

Let $\omega = e^{\frac{2\pi i}{n}}$, then we have

$$\lambda_m = \sum_{k=0}^{n-1} a_k \omega^{mk}, \ m = 0, 1, \ldots, (n-1).$$

Thus we have

$$\lambda_m = \sum_{\substack{k=1 \\ (k,n) \neq 1 \\ k \neq p, 2p, \ldots, \left(\frac{n}{p}-1\right)p}}^{n-1} \omega^{mk} - \sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell pm},$$

$$= \sum_{k=1}^{n-1} \omega^{mk} - \sum_{(k,n)=1} \omega^{mk} - 2\sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell pm}.$$

From the definition of Euler's function we have

$$\lambda_0 = (n-1) - \phi(n) - 2\left(\frac{n}{p} - 1\right).$$

Using definition of Ramanujan's sum

$$\lambda_m = -1 - C(m,n) - 2\left(\frac{n}{p} - 1\right), \ m = \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}$$

and

$$\lambda_m = 1 - C(m,n), 1 \leq m \leq n-1 \text{ and } m \neq \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}.$$

Thus

$$\begin{aligned} E\overline{(X_n)_c} &= \sum_{m=0}^{n-1} |\lambda_m| \\ &= \left|(n-1) - \phi(n) - 2\left(\frac{n}{p} - 1\right)\right| \\ &\quad + \sum_{m=\frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}} \left|1 + C(m,n) + 2\left(\frac{n}{p} - 1\right)\right| \\ &\quad + \sum_{\substack{1 \leq m \leq (n-1) \\ m \neq \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}}} |1 - C(m,n)|. \end{aligned} \tag{4}$$

Consider

$$\sum_{m=\frac{n}{p},\frac{2n}{p},\ldots,\frac{(p-1)n}{p}} \left| 1 + C(m,n) + 2\left(\frac{n}{p}-1\right) \right|$$

(5)
$$= \phi(p)\left| 1 + \frac{\phi(n)\mu(p)}{\phi(p)} + 2\left(\frac{n}{p}-1\right) \right|$$

$$= \left| \phi(p) - \phi(n) + 2\left(\frac{n}{p}-1\right)\phi(p) \right| = \phi(p)\left(\frac{2n}{p}-1\right) - \phi(n).$$

We have

$$\sum_{\substack{m \,\neq\, n,\, \frac{n}{p},\frac{2n}{p},\ldots,\frac{(p-1)n}{p}}}^{\substack{m=1 \\ n-1}} |1 - C(m,n)|$$

$$= \sum_{m=1}^{n} |1 - C(m,n)| - \sum_{m \,=\, n,\frac{n}{p},\frac{2n}{p},\ldots,\frac{(p-1)n}{p}} |1 - C(m,n)|$$

$$= \sum_{d|n} \left| \phi\left(\frac{n}{d}\right) - \phi(n)\mu\left(\frac{n}{d}\right) \right| - \phi(p) - 2\phi(n) + 1$$

(6)
$$= \sum_{\substack{d|n \\ \mu\left(\frac{n}{d}\right)=1}} \left| \phi\left(\frac{n}{d}\right) - \phi(n) \right| + \sum_{\substack{d|n \\ \mu\left(\frac{n}{d}\right)=-1}} \left| \phi\left(\frac{n}{d}\right) + \phi(n) \right|$$

$$+ \sum_{\substack{d|n \\ \mu\left(\frac{n}{d}\right)=0}} \phi\left(\frac{n}{d}\right) - \phi(p) - 2\phi(n) + 1$$

$$= 2^{\omega(n)}\phi(n) + \sum_{d|n} \phi(d) - 2\left(\sum_{\substack{d|n \\ \mu(d)=1}} \phi(d) \right) - \phi(p) - 2\phi(n) + 1.$$

Substituting (5) and (6) in (4) and after some simplifications, we obtain

$$E\overline{(X_n)_c} = (2^{\omega(n)} - 2)\phi(n) + 2(n - p + 1) - 2\sum_{\substack{d|n \\ \mu(d)=1}} \phi(d).$$

∎

Corollary 7. *If* $n = p^\alpha$, *then* $E\overline{(X_{p^\alpha})_c} = 2(p^\alpha - p)$.

Remark 8. $E_\chi(X_{p^\alpha}) > E\overline{(X_{p^\alpha})_c}$.

4.   Color Energy of Some gcd-graphs

Klotz and Sander extended the class of unitary Cayley graphs. Let D be a set of positive, proper divisors of the integer $n > 1$. We recall the definition of the gcd-graph. Define the gcd-graph $X_n(D)$ to have vertex set $Z_n = \{0, 1, \ldots, n-1\}$ and edge set $E(X_n(D)) = \{\{a, b\} : a, b \in Z_n, (a - b, n) \in D\}$. So [10] proved that a circulant graph is integral if, and only if, it is a gcd-graph.

Theorem 9. *Let $n = p_1{}^{\alpha_1} p_2{}^{\alpha_2} \ldots p_k{}^{\alpha_k}$ and $X_n(D)$ be the gcd-graph, where $D = \{1, p_i\}$, $i > 1$, $\alpha_i = 1$. If $p \,(= p_1)$ is the smallest prime divisor of n, then the gcd-graph $X_n(D)$ has a unique optimal coloring.*

Proof. Let $X_n(D)$ be the gcd-graph where $D = \{1, p_i\}$, $i > 1$ and $\alpha_i = 1$ whose vertices are labelled by $0, 1, 2, \ldots, (n-1)$. Let $p \,(= p_1)$ be the smallest prime divisor of n. Now we color $X_n(D)$ using p colors say $c_1, c_2, \ldots, c_p$. Using similar arguments as in the proof of Theorem 2, the vertices $i, i+p, i+2p, \ldots, i+(\frac{n}{p}-1)p$, $i = 0, 1, \ldots, (p-1)$ must receive the same color. Also observe that, if x and y are two vertices of $X_n(D)$ such that $gcd(x - y, n) = p_i$, then x and y must receive different colors. For, if x and y receive the same color then $p|(x - y)$ and hence $p|p_i$ which is a contradiction. ∎

Theorem 10. *Let $n = p_1{}^{\alpha_1} p_2{}^{\alpha_2} \ldots p_k{}^{\alpha_k}$ and $X_n(D)$ be the gcd-graph, where $D = \{1, p_i\}$, $i > 1$, $\alpha_i = 1$. If $p \,(= p_1)$ is the smallest prime divisor of n and $X_n(D)$ be colored with p colors, then*

$$E_\chi[X_n(D)] = n - \frac{n}{p_i} + 2^{\omega(n)-1}\left(\phi(n) + \phi\left(\frac{n}{p_i}\right)\right)$$
$$+ \frac{n}{p}\phi(p) - \frac{n}{p} + \sum_{\ell\mid\left(\frac{n}{p_i}\right)} \mu\left(\frac{n}{\ell p_i}\right)\phi\left(\frac{n}{\ell p_i}\right).$$

Proof. Let $X_n(D)$ be the gcd-graph, where $D = \{1, p_i\}$. We color the vertices $0, 1, 2, \ldots, n - 1$ with colors $c_1, c_2, \ldots, c_p$ as follows. Color the vertices $kp, kp + 1, kp + 2, \ldots, (k + 1)p - 1$ with $c_1, c_2, c_3, \ldots, c_p$ respectively for $k = 0, 1, 2, \ldots, \left(\frac{n}{p} - 1\right)$. Then $A_\chi[X_n(D)]$ is a circulant matrix with first row equal to $(a_0, a_1, a_2, \ldots, a_{n-1})$, where

$$a_k = \begin{cases} 1 & \text{if } (k, n) = 1, \\ 1 & \text{if } (k, n) = p_i, \\ -1 & \text{if } k = p, 2p, \ldots, n - p, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the color eigenvalues of $X_n(D)$ are given by

$$\lambda_m = \sum_{k=0}^{n-1} a_k \omega^{mk}, \; m = 0, 1, \ldots, (n - 1).$$

Let $\omega = e^{\frac{2\pi i}{n}}$. Then we have,

$$\lambda_m = \sum_{\substack{k=1 \\ (k,n)=1}}^{n-1} \omega^{mk} + \sum_{\substack{k=1 \\ (k,n)=p_i}}^{n-1} \omega^{mk} - \sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell pm}, \; m = 0, 1, \ldots, (n - 1)$$
$$= C(m, n) + C\left(m, \frac{n}{p_i}\right) - \sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell pm}.$$

We note that

$$
\lambda_m =
\begin{cases}
\phi(n) + \phi\left(\frac{n}{p_i}\right) - \left(\frac{n}{p} - 1\right), & m = 0, \\[2mm]
C(m,n) + C\left(m, \frac{n}{p_i}\right) + 1, & 1 \le m \le n-1, \\
& m \ne \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}, \\[2mm]
C(m,n) + C\left(m, \frac{n}{p_i}\right) - \left(\frac{n}{p} - 1\right), & 1 \le m \le n-1, \\
& m = \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}.
\end{cases}
$$

Observe that

$$
(7) \qquad |\lambda_0| = \phi(n) + \phi\left(\frac{n}{p_i}\right) - \left(\frac{n}{p} - 1\right),
$$

and

$$
(8) \qquad
\begin{aligned}
&\sum_{m=\frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}} \left| C(m,n) + C\left(m, \frac{n}{p_i}\right) - \left(\frac{n}{p} - 1\right) \right| \\
&= \phi(n) + \phi\left(\frac{n}{p_i}\right) + \phi(p)\left(\frac{n}{p} - 1\right).
\end{aligned}
$$

Further,

$$
\begin{aligned}
&\sum_{\substack{m=1 \\ m \ne \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}}}^{n} \left| C(m,n) + C\left(m, \frac{n}{p_i}\right) + 1 \right| \\
&= \sum_{m=1}^{n} \left| C(m,n) + C\left(m, \frac{n}{p_i}\right) + 1 \right| \\
&\quad - \sum_{m= n,\, \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}} \left| C(m,n) + C\left(m, \frac{n}{p_i}\right) + 1 \right| \\
&= \sum_{m=1}^{n} \left| 1 + \frac{\phi(n)\mu\left(\frac{n}{(m,n)}\right)}{\phi\left(\frac{n}{(m,n)}\right)} + \frac{\phi(n/p_i)\mu\left(\frac{n}{(p_i m, n)}\right)}{\phi\left(\frac{n}{(p_i m, n)}\right)} \right| \\
&\quad - 2\phi(n) - 2\phi\left(\frac{n}{p_i}\right) + \phi(p) - 1.
\end{aligned}
$$

But, $(m, p_i) = 1$ or p_i. If $(m, p_i) = 1$, then $(m p_i, n) = p_i(m,n)$ and if $(m, p_i) = p_i$,

then $(m, n) = (mp_i, n)$. Thus the above sum is equal to

$$
\sum_{\substack{m=1 \\ (m,p_i)=1}}^{n} \left| 1 + \frac{\phi(n)\mu\left(\frac{n}{(m,n)}\right)}{\phi\left(\frac{n}{(m,n)}\right)} + \frac{\phi\left(\frac{n}{p_i}\right)\mu\left(\frac{n}{(p_im,n)}\right)}{\phi\left(\frac{n}{(p_im,n)}\right)} \right|
$$

(9)
$$
+ \sum_{\substack{m=1 \\ (m,p_i)=p_i}}^{n} \left| 1 + \frac{\phi(n)\mu\left(\frac{n}{(m,n)}\right)}{\phi\left(\frac{n}{(m,n)}\right)} + \frac{\phi\left(\frac{n}{p_i}\right)\mu\left(\frac{n}{(p_im,n)}\right)}{\phi\left(\frac{n}{(p_im,n)}\right)} \right|
$$

$$
- 2\phi(n) - 2\phi\left(\frac{n}{p_i}\right) + \phi(p) - 1.
$$

After some simplification the first sum is equal to

$$
\sum_{\substack{d|n \\ d\neq p_i, 2p_i, \ldots, \frac{n}{p_i}p_i}} \phi\left(\frac{n}{d}\right) \left| \frac{\phi(n)\mu\left(\frac{n}{d}\right)}{\phi\left(\frac{n}{d}\right)} + \frac{\phi\left(\frac{n}{p_i}\right)\mu\left(\frac{n}{p_id}\right)}{\phi\left(\frac{n}{p_id}\right)} + 1 \right|
$$

(10)
$$
= \sum_{\substack{d|n \\ d\neq p_i, 2p_i, \ldots, \frac{n}{p_i}p_i}} \left| \phi(n)\mu\left(\frac{n}{d}\right) + \phi(n)\mu\left(\frac{n}{p_id}\right) + \phi\left(\frac{n}{d}\right) \right|
$$

$$
= \sum_{\substack{d|n \\ d\neq p_i, 2p_i, \ldots, \frac{n}{p_i}p_i}} \phi\left(\frac{n}{d}\right) = n - \frac{n}{p_i}.
$$

The second sum is equal to

$$
\sum_{\substack{m=1 \\ (m,p_i)=p_i}}^{n} \left| 1 + \frac{\phi(n)\mu\left(\frac{n}{(m,n)}\right)}{\phi\left(\frac{n}{(m,n)}\right)} + \frac{\phi\left(\frac{n}{p_i}\right)\mu\left(\frac{n}{(m,n)}\right)}{\phi\left(\frac{n}{(m,n)}\right)} \right|
$$

$$
= \sum_{\ell=1}^{\frac{n}{p_i}} \left| \frac{\mu\left(\frac{n}{(\ell p_i,n)}\right)}{\phi\left(\frac{n}{(\ell p_i,n)}\right)} \left(\phi(n) + \phi\left(\frac{n}{p_i}\right)\right) + 1 \right|
$$

(11)
$$
= \sum_{\ell|\frac{n}{p_i}} \phi\left(\frac{n}{\ell p_i}\right) \left| \frac{\mu\left(\frac{n}{(\ell p_i)}\right)}{\phi\left(\frac{n}{(\ell p_i)}\right)} \left(\phi(n) + \phi\left(\frac{n}{p_i}\right)\right) + 1 \right|
$$

$$
= \sum_{\ell|\frac{n}{p_i}} \mu\left(\frac{n}{p_i}\right) \left(\phi(n) + \phi\left(\frac{n}{p_i}\right)\right) + \phi\left(\frac{n}{\ell p_i}\right)
$$

$$
= \sum_{\ell|\frac{n}{p_i}} \phi(n) + \phi\left(\frac{n}{p_i}\right) + \mu\left(\frac{n}{\ell p_i}\right)\phi\left(\frac{n}{\ell p_i}\right)
$$

$$
= 2^{\omega(n)-1}\left(\phi(n) + \phi\left(\frac{n}{p_i}\right)\right) + \sum_{\ell|\frac{n}{p_i}} \mu\left(\frac{n}{\ell p_i}\right)\phi\left(\frac{n}{\ell p_i}\right).
$$

Substituting (10) and (11) in (9) we get

$$\sum_{\substack{m=1 \\ m \neq \frac{n}{p}, \frac{2n}{p}, \dots, \frac{(p-1)n}{p}}}^{n} \left| C(m,n) + C\left(m, \frac{n}{p_i}\right) + 1 \right|$$

(12)
$$= n - \frac{n}{p_i} + 2(2^{\omega(n)-2} - 1)\left(\phi(n) + \phi\left(\frac{n}{p_i}\right)\right)$$

$$+ \phi(p) - 1 + \sum_{\ell \mid \left(\frac{n}{p_i}\right)} \mu\left(\frac{n}{\ell p_i}\right) \phi\left(\frac{n}{\ell p_i}\right).$$

Adding (7), (8) and (12) and simplifying we obtain

$$E_\chi[X_n(D)] = n - \frac{n}{p_i} + 2^{\omega(n)-1}\left(\phi(n) + \phi\left(\frac{n}{p_i}\right)\right) + \frac{n}{p}\phi(p)$$

$$- \frac{n}{p} + \sum_{\ell \mid \left(\frac{n}{p_i}\right)} \mu\left(\frac{n}{\ell p_i}\right) \phi\left(\frac{n}{\ell p_i}\right).$$

This completes the proof. ∎

Theorem 11. *Let $n = p_1 p_2 \dots p_k$ be a square-free number and $X_n(D)$ be the gcd-graph, where $D = \{p_i, p_j\}$. If $p\ (= p_1) \neq p_i$, p_j is the smallest prime divisor of n and $X_n(D)$ can be colored with p colors, then*

$$E_\chi[X_n(D)] = \phi\left(\frac{n}{p_i}\right) + \phi\left(\frac{n}{p_j}\right) - \left(\frac{n}{p} - 1\right)$$

$$+ \sum_{\substack{m=1 \\ m = \frac{n}{p}, \frac{2n}{p}, \dots, \frac{(p-1)n}{p}}}^{n} \left| C\left(m, \frac{n}{p_i}\right) + C\left(m, \frac{n}{p_j}\right) - \left(\frac{n}{p} - 1\right) \right|$$

$$+ \sum_{\substack{m=1 \\ m \neq \frac{n}{p}, \frac{2n}{p}, \dots, \frac{(p-1)n}{p}}}^{n} \left| C\left(m, \frac{n}{p_i}\right) + C\left(m, \frac{n}{p_j}\right) + 1 \right|.$$

Proof. Let $X_n(D)$ be the gcd-graph, where $D = \{p_i, p_j\}$. We color the vertices $0, 1, 2, \dots, n-1$ with colors $c_1, c_2, \dots, c_p$ as follows: Color the vertices $kp, kp+1, kp+2, \dots, (k+1)p-1$ with $c_1, c_2, c_3, \dots, c_p$ respectively for $k = 0, 1, 2, \dots, \left(\frac{n}{p} - 1\right)$. Then $A_\chi[X_n(D)]$ is a circulant matrix with first row equal to $(a_0, a_1, a_2, \dots, a_{n-1})$, where

$$a_k = \begin{cases} 1 & \text{if } (k,n) = p_i, \\ 1 & \text{if } (k,n) = p_j, \\ -1 & \text{if } k = p, 2p, \dots, n-p, \\ 0 & \text{otherwise.} \end{cases}$$

The color eigenvalues of $X_n(D)$ are given by

$$
\begin{aligned}
\lambda_m &= \sum_{k=0}^{n-1} a_k \omega^{mk}, \ m = 0, 1, \ldots, (n-1) \\
&= \sum_{\substack{k=1 \\ (k,n)=p_i}}^{n-1} \omega^{mk} + \sum_{\substack{k=1 \\ (k,n)=p_j}}^{n-1} \omega^{mk} - \sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell p m} \\
&= C\left(m, \frac{n}{p_i}\right) + C\left(m, \frac{n}{p_j}\right) - \sum_{\ell=1}^{\frac{n}{p}-1} \omega^{\ell p m}.
\end{aligned}
$$

Thus we have

$$
\lambda_m = \begin{cases}
\phi\left(\frac{n}{p_i}\right) + \phi\left(\frac{n}{p_j}\right) - \left(\frac{n}{p} - 1\right) & m = 0, \\
C\left(m, \frac{n}{p_i}\right) + C\left(m, \frac{n}{p_j}\right) + 1 & (1 \le m \le n-1) \text{ and} \\
& m \ne \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}. \\
C\left(m, \frac{n}{p_i}\right) + C\left(m, \frac{n}{p_j}\right) - \left(\frac{n}{p} - 1\right) & m = \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}.
\end{cases}
$$

Therefore,

$$
\begin{aligned}
E_\chi[X_n(D)] = {}& \phi\left(\frac{n}{p_i}\right) + \phi\left(\frac{n}{p_j}\right) - \left(\frac{n}{p} - 1\right) \\
& + \sum_{\substack{m=1 \\ m=\frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}}}^{n} \left| C\left(m, \frac{n}{p_i}\right) + C\left(m, \frac{n}{p_j}\right) - \left(\frac{n}{p} - 1\right) \right| \\
& + \sum_{\substack{m=1 \\ m \ne \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}}}^{n} \left| C\left(m, \frac{n}{p_i}\right) + C\left(m, \frac{n}{p_j}\right) + 1 \right|.
\end{aligned}
$$

$\blacksquare$

Acknowledgements

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