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Note

THE SMALLEST NONEVASIVE GRAPH PROPERTY

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Abstract

A property of n-vertex graphs is called evasive if every algorithm testing this property by asking questions of the form "is there an edge between vertices u and v" requires, in the worst case, to ask about all pairs of vertices. Most "natural" graph properties are either evasive or conjectured to be such, and of the few examples of nontrivial nonevasive properties scattered in the literature the smallest one has $n = 6$.

We exhibit a nontrivial, nonevasive property of 5-vertex graphs and show that it is essentially the unique such with $n \leq 5$.

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Evasiveness of graph properties is a classical complexity-theoretic concept defined via the following combinatorial game. Two players, Alice and Bob, first fix a number n and a property $\mathcal{P}$ of n-vertex graphs. Bob wants to find out if some unknown graph G, secretly chosen by Alice, has the property $\mathcal{P}$, by asking Alice one by one if a particular pair of vertices forms an edge. Alice wins if she can force Bob to ask about all the $\binom{n}{2}$ pairs before he knows if $G \in \mathcal{P}$. Bob wins if he can decide the membership of G in $\mathcal{P}$ after at most $\binom{n}{2} - 1$ questions. Of course there is no reason why Alice should fix any particular graph in advance — she can adapt her answers so as to force Bob to ask the maximal number of questions. We say $\mathcal{P}$ is *evasive* (or *elusive*) if Alice has a winning strategy; it is *nonevasive* if Bob does. For example, the simple property of "being the complete graph" is evasive. Alice's strategy is to say "Yes" to Bob's first $\binom{n}{2} - 1$ questions, at which point he is still not sure if G is complete or not.

Evasiveness is a classical notion which arose as a way of measuring the decision-tree complexity of Boolean functions. The lecture notes [6] are an excellent introduction to this general topic. Here it suffices to say that most "natural"

graph properties, for example connectedness, planarity, triangle-freeness, perfectness, existence of an isolated vertex and many more are all evasive. A major conjecture, attributed to Karp, claims that every nontrivial *monotone* property, that is a property closed under inserting new edges, is evasive. Its proof when n is a prime power [3] is one of the celebrated applications of topological methods in combinatorics.

Unsurprisingly, the known constructions of nonevasive properties are rare and to some extent artificial (see [1, 7] for the original papers and [2], [5, Chapter 3], [4, Chapter 13] for surveys). The example usually presented in the literature involves classes of graphs called *scorpions*, for which Bob can solve the membership problem after at most $6n - 13$ questions, which is better than $\binom{n}{2}$ for any $n \geq 11$. An optimized example of similar kind can be found in [5, Figure 3.10] and it requires $n = 6$ vertices. The purpose of this note is to establish the existence of a nonevasive graph property with $n = 5$, prove its uniqueness, and make some related observations.

Definition. For a fixed natural number n let $\mathcal{G}_n$ be the set of isomorphism classes of n-vertex simple, unlabeled graphs. A *property* of n-vertex graphs is an arbitrary subset $\mathcal{P} \subseteq \mathcal{G}_n$. A property $\mathcal{P}$ is *nontrivial* if $\mathcal{P} \neq \emptyset$ and $\mathcal{P} \neq \mathcal{G}_n$. The *complement* of $\mathcal{P}$ is $\overline{\mathcal{P}} = \{\overline{G} \ : \ G \in \mathcal{P}\}$, where $\overline{G}$ denotes the complement of a graph G.

Note that one can equivalently think of isomorphism-invariant properties of unlabeled graphs. If $\mathcal{P}$ is a property of n-vertex graphs then there is another property which deserves to be called the complement of $\mathcal{P}$, namely $\mathcal{G}_n \setminus \mathcal{P}$. However, in this note we reserve the name "complement" for $\overline{\mathcal{P}}$. Observe that the trivial properties are nonevasive, as Bob wins without asking any questions at all.

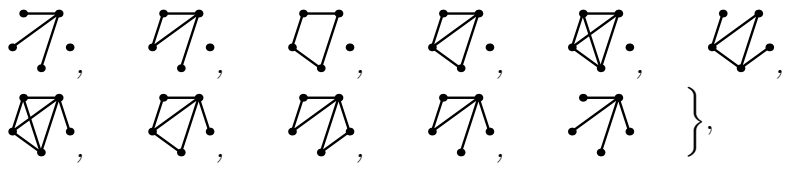
Lemma 1. *If $\mathcal{P}$ is a nonevasive property of n-vertex graphs, then the properties $\mathcal{G}_n \setminus \mathcal{P}$ and $\overline{\mathcal{P}}$ are also nonevasive.*

Proof. The claim about $\mathcal{G}_n \setminus \mathcal{P}$ is obvious — Bob uses the same strategy he has for $\mathcal{P}$. We will now describe Bob's winning strategy for $\overline{\mathcal{P}}$. We first modify Bob's perception, so that every time Alice says "Yes" he understands "No" and vice versa. This way if Alice is thinking of a graph G, Bob is reconstructing the graph $\overline{G}$. With this modification, we now make Bob play the $\overline{\mathcal{P}}$-game using the strategy he has for $\mathcal{P}$. Since that is a winning strategy, regardless of Alice's answers, one question before the end Bob can determine if $\overline{G} \in \mathcal{P}$, which is equivalent to $G \in \overline{\mathcal{P}}$. ∎

We recorded the easy statement about $\overline{\mathcal{P}}$ since it seems not to have appeared in the literature. The usual formulation appearing in topological contexts [3, 9] relates $\mathcal{P}$ directly to $\overline{\mathcal{G}_n \setminus \mathcal{P}}$ (if $\mathcal{P}$ is a simplicial complex then $\overline{\mathcal{G}_n \setminus \mathcal{P}}$ is its Alexander dual).

Next comes the main result of this note.

Theorem 2. *If $\mathcal{E} \subseteq \mathcal{G}_5$ is defined as*

$\mathcal{E} = \{$  $\}$,

then $\mathcal{E} = \overline{\mathcal{E}}$ and, moreover, $\mathcal{E}$ and $\mathcal{G}_5 \setminus \mathcal{E}$ are the only nontrivial, nonevasive properties of 5*-vertex graphs.*

Furthermore, if $n \leq 4$ then every nontrivial property $G \subseteq \mathcal{P}_n$ is evasive.

Proof. The assertion $\mathcal{E} = \overline{\mathcal{E}}$ is verified by a direct check — in the statement each graph is displayed above its complement and one graph is self-complementary. The proof of uniqueness is computer assisted (for $n \leq 3$ the nonexistence of a nonevasive property is an easy exercise which can be solved by hand).

Define a *position* in the game as the complete graph on n vertices whose edges are labeled with either "present", "absent" or "unknown", and such that at least one edge is "unknown". The first two labels indicate the status of an edge already discovered by Bob. The edges labeled "unknown" are those Bob hasn't asked about yet. A position with just one unknown edge is winning for Bob if the two graphs obtained by declaring this edge present or absent are either both in $\mathcal{P}$ or both not in $\mathcal{P}$. To find the winning player and winning moves for other positions we find and precompute the recursive dependencies between them.

For $n = 5$ there are 34 graphs and 758 isomorphism classes of positions[1]. Evaluating the initial position, with all edges unknown, against all $2^{|\mathcal{G}_5|} = 2^{34}$ graph properties, is a matter of at most one day on any reasonably modern personal computer. For $n \leq 4$ the same thing is immediate. That verifies the theorem. ∎

To make $\mathcal{E}$ more accessible we present Bob's winning strategy for $\mathcal{E}$ in a human-readable form. We first describe a special feature of $\mathcal{E}$ which reduces the number of positions we have to consider.

We first extend the definition of a complement to positions in the game. If P is a position, we define $\overline{P}$ by renaming all edges labeled "present" to "absent" and vice-versa. The edges unknown in P remain unknown in $\overline{P}$. The self-complementarity of $\mathcal{E}$ together with an easy inductive argument imply that in the $\mathcal{E}$-game a position P is winning for Bob if and only if its complement $\overline{P}$ is. It is also easy to read off the strategy for P from the strategy for $\overline{P}$. It follows

[1]The number of positions is $|\mathcal{H}_n| - |\mathcal{G}_n|$, where $\mathcal{H}_n$ is the set of isomorphism classes of edge-3-colorings of the complete graph K_n. See A004102 in [8].

that in our analysis we can identify a position with its complement. We can, for example, choose to work only with positions that have at least as many "present" as "absent" edges.

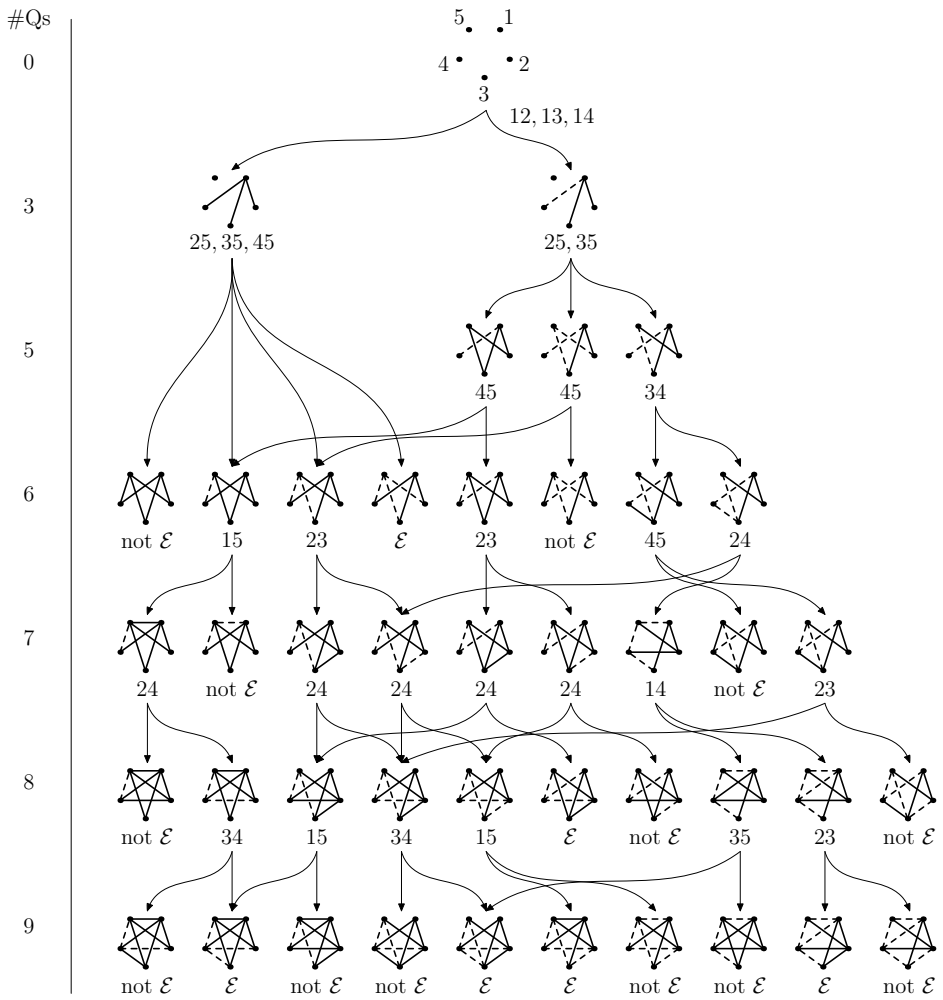


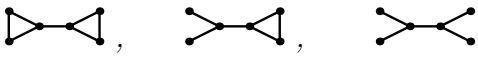
Figure 1. Bob's winning strategy in the $\mathcal{E}$-game. For convenience the vertices are labeled $1, \ldots, 5$ as indicated at the top. In the first step Bob asks about the three edges $12, 13, 14$. The eight possible outcomes fall into two equivalence classes of positions under isomorphism and complementation, both depicted in the second row. For Bob's remaining questions the same interpretation applies.

Bob's strategy in the $\mathcal{E}$-game is depicted in Figure 1. In each position the solid lines denote the present edges and the dashed lines are the absent edges (the edges not shown are unknown). The label under a position is either the next

question(s) to ask or an indication that membership in $\mathcal{E}$ is already decided. The arrows lead to possible outcomes depending on Alice's answers, subject to applying isomorphism and complementation of positions along the way. In each case Bob wins after the 9th question at the latest.

The property $\mathcal{E}$ is minimal in terms of the number of vertices n, but not in terms of its cardinality $|\mathcal{E}|$. Since every graph property $\mathcal{P}$ with $|\mathcal{P}| \in \{1, 2\}$ is evasive [2], the proposition below classifies the smallest examples which minimize the cardinality.

Proposition 3. *The only* 6*-vertex, nonevasive graph properties of cardinality* 3 *are* $\mathcal{S}$ *and* $\overline{\mathcal{S}}$*, where*

$\mathcal{S} = \{$  $\} \subseteq \mathcal{G}_6\,.$

Proof. The property $\mathcal{S}$ appears in [5, Figure 3.10], where showing its nonevasiveness is left as an exercise. The proof of uniqueness is again a computer search through all $\binom{|\mathcal{G}_6|}{3} = \binom{156}{3}$ properties. A single game has 25350 positions. ■

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