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SOME REMARKS ON THE STRUCTURE OF STRONG k-TRANSITIVE DIGRAPHS[1]

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Abstract

A digraph D is k-transitive if the existence of a directed path $(v_0, v_1, \ldots, v_k)$, of length k implies that $(v_0, v_k) \in A(D)$. Clearly, a 2-transitive digraph is a transitive digraph in the usual sense. Transitive digraphs have been characterized as compositions of complete digraphs on an acyclic transitive digraph. Also, strong 3 and 4-transitive digraphs have been characterized.

In this work we analyze the structure of strong k-transitive digraphs having a cycle of length at least k. We show that in most cases, such digraphs are complete digraphs or cycle extensions. Also, the obtained results are used to prove some particular cases of the Laborde-Payan-Xuong Conjecture.

Keywords: digraph, transitive digraph, k-transitive digraph, quasi-transitive digraph, k-quasi-transitive digraph, Laborde-Payan-Xuong Conjecture.

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1. Introduction

In this work, $D = (V(D), A(D))$ will denote a finite digraph without loops or multiple arcs in the same direction, with vertex set $V(D)$ and arc set $A(D)$. For general concepts and notation we refer the reader to [1] and [2], particularly we

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will use the notation of [2] for walks. If $W = (x_0, x_1, \ldots, x_n)$ is a walk and $i < j$ then $x_i W x_j$ will denote the subwalk $(x_i, x_{i+1}, \ldots, x_{j-1}, x_j)$ of W. Union of walks will be denoted by concatenation or with $\cup$. For a vertex $v \in V(D)$, we define the *out-neighborhood* of v in D as the set $N^+_D(v) = \{u \in V(D) \colon (v,u) \in A(D)\}$; when there is no possibility of confusion we will omit the subscript D. The elements of $N^+(v)$ are called the *out-neighbors* of v, and the *out-degree* of v, $d^+_D(v)$, is the number of out-neighbors of v. Definitions of *in-neighborhood*, *in-neighbors* and *in-degree* of v are analogously given. We say that a vertex u *reaches* a vertex v in D if a directed uv-directed path exists in D. An arc $(u,v) \in A(D)$ is called *asymmetrical* (resp. *symmetrical*) if $(v,u) \notin A(D)$ (resp. $(v,u) \in A(D)$).

If D is a digraph and $X, Y \subseteq V(D)$, an XY-arc is an arc with initial vertex in X and terminal vertex in Y. If $X \cap Y = \emptyset$, $X \to Y$ will denote that $(x,y) \in A(D)$ for every $x \in X$ and $y \in Y$. Again, if X and Y are disjoint, $X \Rightarrow Y$ will denote that there are no YX-arcs in D. When $X \to Y$ and $X \Rightarrow Y$ we will simply write $X \mapsto Y$. If D_1, D_2 are subdigraphs of D, we will abuse notation to write $D_1 \to D_2$ or D_1D_2-arc, instead of $V(D_1) \to V(D_2)$ or $V(D_1)V(D_2)$-arc, respectively. Also, if $X = \{v\}$, we will abuse notation to write $v \to Y$ or vY-arc instead of $\{v\} \to Y$ or $\{v\}Y$-arc, respectively. Analogously, if $Y = \{v\}$.

A digraph is *strongly connected* (or *strong*) if for every $u, v \in V(D)$, there exists a uv-directed path, *i.e.*, a directed path with initial vertex u and terminal vertex v. A *strong component* (or *component*) of D is a maximal strong subdigraph of D. The *condensation* of D is the digraph $D^\star$ with $V(D^\star)$ equal to the set of all strong components of D, and $(S,T) \in A(D^\star)$ if and only if there is an ST-arc in D. Clearly, $D^\star$ is an acyclic digraph (a digraph without directed cycles), and thus, it has both vertices of out-degree equal to zero and vertices of in-degree equal to zero. A *terminal component* of D is a strong component T of D such that $d^+_{D^\star}(T) = 0$. An *initial component* of D is a strong component S of D such that $d^-_{D^\star}(S) = 0$. A strong component that is neither initial nor terminal is called an *intermediate component*.

A *biorientation* of the graph G is a digraph D obtained from G by replacing each edge $\{x,y\} \in E(G)$ by either the arc (x,y) or the arc (y,x) or the pair of arcs (x,y) and (y,x). A *semicomplete* digraph is a biorientation of a complete graph. An *orientation* of a graph G is an asymmetrical biorientation of G; thus, an *oriented graph* is an asymmetrical digraph. The *complete biorientation* of a graph G is the digraph obtained by replacing each edge $xy \in E(G)$ by the arcs (x,y) and (y,x). A *complete digraph* is a complete biorientation of a complete graph, and a *complete bipartite digraph* is a complete biorientation of a complete bipartite graph. A digraph D is *cyclically k-partite* if there exists a partition $\{V_0, V_1, \ldots, V_{k-1}\}$ of $V(D)$ such that every arc of D is a V_iV_{i+1}-arc (mod k).

Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and $H_1, H_2, \ldots, H_n$ a family of vertex disjoint digraphs. The *composition* of digraphs $D[H_1, H_2, \ldots,$

$H_n]$ is the digraph having $\bigcup_{i=1}^{n} V(H_i)$ as its vertex set and arc set $\bigcup_{i=1}^{n} A(H_i) \cup \{(u,v)\colon\ u \in V(H_i), v \in V(H_j), (v_i, v_j) \in A(D)\}$. If $D = H[S_1, S_2, \ldots, S_n]$ and none of the digraphs $S_1, \ldots, S_n$ has an arc, then D is an *extension* of H. The dual (or converse) of D, $\overleftarrow{D}$ is the digraph with vertex set $V(\overleftarrow{D}) = V(D)$ and such that $(u,v) \in A(\overleftarrow{D})$ if and only if $(v,u) \in A(D)$.

We will often consider cycles with set of vertices $\mathbb{Z}_n$, the ring of integers modulo n. Let us recall that for any integer $r \in \mathbb{Z}_n$, we have $r+\mathbb{Z}_n = \{r+m\colon\ m \in \mathbb{Z}_n\}$, and $r\mathbb{Z}_n = \{rm\colon\ m \in \mathbb{Z}_n\}$

A classical result states that a digraph D, with an acyclic ordering $D_1, \ldots, D_p$ of its strong components, is transitive if and only if each D_i is a complete digraph for $1 \leq i \leq n$, the digraph T obtained from D by contraction of $D_1, \ldots, D_p$ followed by deletion of multiple arcs is a transitive digraph and $D = T[D_1, D_2, \ldots, D_p]$, where $p = |V(T)|$. Using this characterization theorem it can be proved, *e.g.*, that every transitive digraph has a (k,l)-kernel for every pair of integers $k \geq 2$, $l \geq 1$; or that the Laborde-Payan-Xuong conjecture holds for every transitive digraph. The family of k-transitive digraphs was introduced in [3]. Recently, strong 3-transitive digraphs have been characterized in [5]. A strong 3-transitive digraph is either complete, complete bipartite or a directed 3-cycle with none, one or two symmetrical arcs. Also, a thorough description of the interaction between strong components of 3-transitive digraphs has been given, so the structure of 3-transitive digraphs is very well determined by now. Additional work on the subject includes [6], where strong 4-transitive digraphs are characterized.

As usual, (a,b) will denote the greatest common divisor of a and b.

In [6] it was conjectured that if $k-1$ is a prime and D a strong k-transitive digraph such that $|V(D)| \geq k+1$, D contains an n-directed cycle with $n \geq k$, $(n, k-1) = 1$, and D is not a symmetrical $(k+1)$-cycle, then D is a complete digraph. In [8], R. Wang proved this conjecture to be true. Further observations of the results in [5] and [6] brought us to think that every strong k-transitive digraph that has a "large enough" directed cycle is either a complete digraph or an n-cycle extension where n is a divisor of $k-1$. Considering that every digraph of order less than or equal to k is a k-transitive digraph, we aim to characterize the k-transitive digraphs of order greater than k. Towards the general characterization, our principal results are condensed in the following theorems.

Theorem 1. *Let k be an integer, $k \geq 2$. Let D be a strong k-transitive digraph. Suppose that D contains a cycle of length n such that $(n, k-1) = d$ and $n \geq k+1$. Then the following hold.*

(1) *If $d = 1$, then D is a complete digraph.*

(2) *If $d \geq 2$, then D is either a complete digraph, a complete bipartite digraph, or a d-cycle extension.*

Theorem 2. *Let k be an integer, $k \geq 2$. Let D be a strong k-transitive digraph of order at least $k+1$. If D contains a cycle of length k, then D is a complete digraph.*

Theorems 1 and 2 for $k = 3$ and $k = 4$ are immediate consequences of the results in [5] and [6], respectively. The case $k = 2$ is trivial since every strong transitive digraph is complete. This work will focus on the results for the case $k \geq 5$.

So, in Section 2 we will prove some preliminary results concerning the existence of cycles in k-transitive digraphs. The principal result of the section states that if an n-cycle exists in a k-transitive digraph D with $n \geq k+2$, then D has a $(k-1)$-cycle, a k-cycle or a $(k+1)$-cycle. In Section 3, the necessary lemmas for Theorem 1 are proved. Finally, we devote Section 4 to prove some consequences of Theorem 1, including the Laborde-Payan-Xoung Conjecture for particular cases of k-transitive digraphs.

2. Basic Tools

The following pair of propositions can be found in [6] and will be very useful through this work.

Proposition 3. *Let $k \geq 2$ be an integer, D a k-transitive digraph and $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$ a directed cycle in D with $n \geq k$. If $v \in V(D) \setminus V(C)$ and $(v, x_1, \ldots, x_{m-1}, v_0)$ is a vv_0-directed path in D, then $v \to S = \{v_i \colon i \in (k-1)\mathbb{Z}_n + (k-m)\}$.*

Proposition 4. *Let $k \geq 2$ be an integer, D a k-transitive digraph and C an n-cycle with $n \geq k$ and $(n, k-1) = 1$. If $v \in V(D) \setminus V(C)$ is such that a vC-directed path exists in D, then $v \to C$.*

Through the following lemmas we will prove that if a cycle C of a k-transitive digraph has length at least $k+2$, then we can find a shorter cycle in $D[V(C)]$. This will be done by considering all the possibilities for the length of C.

The following lemma will be used to prove Lemma 7.

Lemma 5. *Let k, i, j, r be integers such that $0 \leq j < i$, $k = r(i+1) + i - j$ and D be a k-transitive digraph. We define $x_0 = 2j+1$ and, for $s \geq 0$, $x_{s+1} = 2x_s + 3$. If $C = (0, 1, \ldots, n-1, 0)$ is an n-cycle in D such that $n = k+i$, then $(0, 2i - x_0) \in A(D)$. Also, if $x_s + 1 < i$, then $(0, 2i - x_{s+1}) \in A(D)$.*

Proof. Let $V(C) = \mathbb{Z}_n$ and $A(C) = \{(0,1), (1,2), \ldots, (n-1, 0)\}$. By the k-transitivity of D, we have that $(j, j+k) \in A(D)$ for any $j \in V(C)$, which also implies $(j, j-i) \in A(D)$.

Clearly, $P = \bigcup_{m=0}^{r-1} \big((k-(m-1)i-m, k-mi-m) \cup (k-mi-m)C(k-(m-1)i-(m+1)) \cup (k-(m-1)i-(m+1), k-mi-(m+1))\big) \cup (k-(r-1)i-r, k-ri-r) \cup (k-ri-r)C(k-(r-1)i-r-j-1)$ is a directed path in D. The length of each of the segments of the union is $2+k-(m-1)i-(m+1)-(k-mi-m) = i+1$ and the length of the last segment of P (outside the union) is $1+k-(r-1)i-r-j-1-(k-ri-r) = i-j$. Since there are r segments in the first part of P, $\ell(P) = r(i+1)+i-j = k$. Let us recall that D is k-transitive, hence $(0, k-(r-1)i-r-j-1) \in A(D)$. But $k-(r-1)i-r-j-1 = r(i+1)+i-j-(r-1)i-r-j-1 = 2i-(2j+1) = 2i-x_0$. Thus, $(0, 2i-x_0) \in A(D)$, which can be used as the basis of induction to proceed inductively on s.

If $(0, 2i-x_s) \in A(D)$ and $x_s+1 < i$, we will prove that $(0, 2i-x_{s+1}) \in A(D)$. Observe that, by symmetry, we have the existence of the arc $(\alpha, \alpha+2i-x_s) \pmod n$ in D for every $0 \leq \alpha \leq n-1$. In particular, recalling that $n = k+i$, we have $(k+i-1, 2i-x_s-1) \in A(D)$. Let us consider $P' = (0, 2i-x_s) \cup (2i-x_s)C(k+i-1) \cup (k+i-1, 2i-x_s-1, i-x_s-1) \cup (i-x_s-1)C(2i-x_{s+1})$. We can observe that $0 < i-x_s-1$ since $x_s+1 < i$. Also, $2i-x_{s+1}-(i-x_s-1) = i-x_s-2 \geq 0$, $i-x_s-1 \leq 2i-x_{s+1}$. Since clearly $0 < 2i-x_{s+1} < 2i-x_s-1$, P' is a directed path in D. But $\ell(P') = 3+(k+i-1)-(2i-x_s)+(2i-x_{s+1})-(i-x_s-1) = 3+(k-i+x_s-1)+(i-x_s-2) = k$. By the k-transitivity of D, we have that $(0, 2i-x_{s+1}) \in A(D)$. ■

Our next result will be useful to prove Lemma 7, but it will be also used in further results. This kind of result is very important for the present subject because once the behavior of "long" cycles in a k-transitive digraph has been studied, we wish to focus on the interactions between "'short" cycles and when the existence of various short cycles in the same strong component implies the existence of a long cycle, like in this case.

Lemma 6. *Let $k \geq 2$ be an integer and D a k-transitive digraph. If $C_1 = (x_0, x_1, \ldots, x_{n-1}, x_0)$ and $C_2 = (y_0, y_1, \ldots, y_{k-2}, y_0)$ are disjoint directed cycles in D such that $n \leq k-1$ and $(x_0, y_0) \in A(D)$, then $x_0 \to \{y_i\colon\ i \in (k-1-n)\mathbb{Z}_{k-1}\}$.*

Proof. We will prove by induction on i that $(x_0, y_i) \in A(D)$ for $i \in (k-1-n)\mathbb{Z}_{k-1}$. We assume without loss of generality that $V(C_2) = \mathbb{Z}_{k-1}$. Thus, we have that $(x_0, 0) \in A(D)$. If $(x_0, r(k-1-n)) \in A(D)$, then $(x_1)C_1(x_0) \cup (x_0, r(k-1-n)) \cup (r(k-1-n))C_2((r+1)(k-1-n)+1)$ is a directed path in D of length $n+(k-1-n)+1 = k$, hence $(x_1, (r+1)(k-1-n)+1) \in A(D)$. But then $(x_0, x_1, (r+1)(k-1-n)+1) \cup ((r+1)(k-1-n)+1)C_2((r+1)(k-1-n))$ is a directed path in D of length $2+(k-2) = k$. By the k-transitivity of D we can conclude that $(x_0, (r+1)(k-1-n)) \in A(D)$. ■

The following lemmas have a similar structure. We will consider a cycle C of length $k+i$, with $2 \leq i \leq k-1$, and we will exhibit a cycle C' in $D[V(C)]$ such that $k-1 \leq \ell(C') < \ell(C)$. In order to do this, we will consider different cases for the possible values of i.

Lemma 7. *Let $k \geq 2$ be an integer and D a k-transitive digraph. If C is an n-cycle in D such that $n = k+i$, $k \neq r(i+1)+i+2$ for every $r \in \mathbb{N}$ and $2 \leq i < \frac{k}{2}$, then there is a directed cycle C' in $D[V(C)]$ such that $k \leq \ell(C') < n$. If $k = r(i+1)+i+2$, then a directed cycle C' exists in $D[V(C)]$ such that $\ell(C') = k-1$.*

Proof. Let $V(C) = \mathbb{Z}_n$ and $A(C) = \{(0,1),(1,2),\ldots,(n-1,0)\}$. By the k-transitivity of D, we have that $(j, j+k) \in A(D)$ for any $j \in V(C)$, which also implies $(j, j-i) \in A(D)$.

Let $r = \max\{s\colon\ k - si - s > i\}$.

If $i \neq \lfloor \frac{k}{2} \rfloor$, then $0 < r$ and $P = (0, k, k+1, \ldots, k+i-2, k-2, k-1, k-i-1, \ldots, k-3, \ldots, k-ri-r+i-1, k-ri-r+i, k-ri-r, k-ri-r+1, \ldots, k-ri-r+i-2)$ is a path of order $r(i+1)+i$ containing every vertex of $(k-ri-r)C(0)-(k+i-2)$. By the definition of r, $k-(r+1)i-(r+1) \leq i < k-ri-r$.

If $i = k-ri-r-2$, then $k = r(i+1)+i+2$ and $P \cup (k-ri-r+i-2, k-ri-r-2, 0)$ is a cycle of length $k-1$ in D. If $i = k-ri-r-1$, then $k = r(i+1)+i+1$ and $P \cup (k-ri-r+i-2, k-ri-r-2, k-ri-r-1, 0)$ is a cycle of length $k+1$ in D.

Thus, we will assume that $k-(r+1)i-(r+1) \leq i \leq k-ri-r-3$. Therefore, $i = k-(r+1)i-(r+1)+j$, $0 \leq j \leq i-2$, and $k = (r+1)(i+1)+i-j$. The last equality is true also for $i = \lfloor \frac{k}{2} \rfloor$ (because $r = 0$ and hence $j = 0$), so we will include this case in the following argument.

Let us apply Lemma 5 and consider the largest s such that $x_s + 1 < i$. Then $x_{s+1} + 1 \geq i$ and $(0, 2i - x_{s+1}) \in A(D)$. If $1 < 2i - x_{s+1}$, let us consider the directed cycle

$$C' = (0, 2i - x_{s+1}) \cup (2i - x_{s+1})C(k+i)$$

of length $1 + k + i - 2i + x_{s+1} = k - i + x_{s+1} + 1$. But $k - i + x_{s+1} + 1 \geq k$ if and only if $x_{s+1} + 1 \geq i$, which we have as hypothesis.

If $2i - x_{s+1} = 1$, then $\frac{x_{s+1}+1}{2} = i$ and $\frac{2x_s+4}{2} = i$, hence $x_s = i-2$. But $(0, 2i - x_s) = (0, i+2) \in A(D)$, so we can consider the directed cycles $C_1 = (i+1, 1) \cup (1)C(i+1)$ and $C_2 = (0, i+2) \cup (i+2)C(0)$ of lengths $i+1$ and $k-1$, respectively. Since $i+1 < k-1$ and $(i+1, i+2) \in A(D)$, if we rename the vertices of C_2 in such a way that $i+2 = y_0, i+3 = y_1, \ldots, n-1 = y_{k-3}$, then we can conclude from Lemma 6 that $i+1 \to \{y_i\colon\ i \in (k-i-2)\mathbb{Z}_{k-1}\}$. If we let $t = r+1$, then $k = t(i+1)+i-j$, hence $i+1 \to \{y_i\colon\ i \in (t(i+1)-j-2)\mathbb{Z}_{k-1}\}$. In particular, $(i+1, t(t(i+1)-j-2)) \in A(D)$. But $t(t(i+1)-j-2) = t^2(i+1)-tj-2t+t(i-j-1)-t(i-j-1) = t(t(i+1)+i-j-1)-tj-2t-t(i-j-1) \equiv -tj-2t-t(i-j-1) = -t(i+1) = -(t(i+1)+i-j-1)+(i-j-1) \equiv (i-j-1) \pmod{k-1 = t(i+1)+i-j-1}$.

Thus we can conclude that $(i+1, y_{i-j-1}) = (i+1, 2i+1-j) \in A(D)$. So,

$$C' = (0)C(i+1) \cup (i+1, 2i+1-j) \cup (2i+1-j)C(k+i)$$

is a directed cycle in D of length $i+1+1+(k+i)-(2i+1-j) = k+j+1 > k$. ■

Lemma 8. *Let $k \geq 5$ be an integer and D a k-transitive digraph. If C is an n-cycle in D such that $n = k+i$ and $\frac{k}{2} \leq i < \frac{2k-5}{3}$, then there is a directed cycle C' in $D[V(C)]$ such that $k \leq \ell(C') < n$.*

Proof. Let $V(C) = \mathbb{Z}_n$ and $A(C) = \{(0,1),(1,2),\ldots,(n-1,0)\}$. By the k-transitivity of D, we have that $(j, j+k) \in A(D)$ for any $j \in V(C)$, which also implies $(j, j-i) \in A(D)$.

Clearly,

$$C' = (0, k, \ldots, 2i{+}2, i{+}2, \ldots, k{-}1, k{-}1{-}i, \ldots, i{+}1, 1, \ldots, k{-}i{-}2, 2k{-}i{-}2, \ldots, 0)$$

is a directed cycle in $D[V(C)]$ of length $\ell(C') = 5 + (2i+2-k) + (k-1) - (i+2) + (i+1) - (k-1-i) + (k-i-2) - 1 + (k+i) - (2k-i-2) = 4i-k+5$. Since $i < \frac{2k-5}{3}$, the vertex $2k-i-3 \notin V(C')$, so $\ell(C') < n$. Since $i \geq \frac{k}{2}$, we have $\ell(C') \geq k$. For $\frac{k}{2} < \frac{2k-5}{3}$ to happen, we need that $10 < k$, so, this construction works for $k \geq 11$. Thus, the only cases not covered by this construction and Lemma 7 are when $k \leq 10$ and $i = \frac{k}{2}$. But k must be even to satisfy $i = \frac{k}{2}$, and by hypothesis $k \geq 5$ (strong 2 and 4-transitive digraphs satisfying the hypothesis of the theorem are complete digraphs), hence we only need to consider the cases $k \in \{6, 8, 10\}$.

For $k = 6$, $n = 9$ and $(0,6,7,4,5,2,3,0)$ is the needed cycle. For $k = 8$, $n = 12$ and $(0,8,9,10,6,7,3,4,0)$ is the cycle we have been looking for. For $k = 10$, $n = 15$ and $(0,10,11,6,7,2,3,13,8,9,4,14,0)$ is the desired cycle. These cycles are depicted in Figure 1. ■

Lemma 9. *Let $k \geq 5$ be an integer and D a k-transitive digraph. If C is an n-cycle in D such that $n = k+i$ and $\frac{k}{2} < i = \frac{2k-5}{3}$, then there is a directed cycle C' in $D[V(C)]$ such that $k \leq \ell(C') < n$.*

Proof. Let $V(C) = \mathbb{Z}_n$ and $A(C) = \{(0,1),(1,2),\ldots,(n-1,0)\}$. By the k-transitivity of D, we have that $(j, j+k) \in A(D)$ for any $j \in V(C)$, which also implies $(j, j-i) \in A(D)$.

If $i = \frac{2k-5}{3}$, then $k \equiv 1 \pmod 3$, thus, $k = 3d+1$ for some $d \in \mathbb{N}$. Hence, $i = 2d-1$, $n = k+i = 5d$ and we can observe that $(n, k-1) = d$. We have two cases.

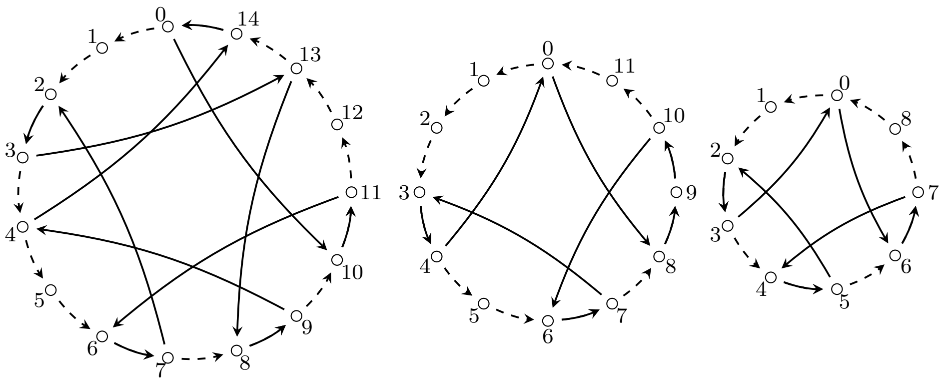

Figure 1. Exceptional cases considered in Lemma 8: $k = 10$, $k = 8$ and $k = 6$.

If d is even, then $C' = (0, 3d+1, 3d+2, \ldots, 3d+\frac{d}{2}+1, d+\frac{d}{2}+2, d+\frac{d}{2}+3, \ldots, 2d+2, 3, 4, \ldots, \frac{d}{2}+3, 3d+\frac{d}{2}+4, 3d+\frac{d}{2}+5, \ldots, 4d+4, 2d+5, 2d+6, \ldots, 2d+\frac{d}{2}+5, \frac{d}{2}+6, \frac{d}{2}+7, \ldots, d+6, 4d+7, 4d+8, \ldots, 5d)$. Thus, $\ell(C') = 7 + 6\frac{d}{2} + 5d - 4d - 7 = 4d > 3d+1 = k$.

If d is odd, then $C' = (0, 3d+1, 3d+2, \ldots, 3d+\frac{d-1}{2}+1, d+\frac{d-1}{2}+2, d+\frac{d-1}{2}+3, \ldots, 2d+1, 2, 3, \ldots, \frac{d-1}{2}+2, 3d+\frac{d-1}{2}+3, 3d+\frac{d-1}{2}+4, \ldots, 4d+2, 2d+3, 2d+4, \ldots, 2d+\frac{d-1}{2}+3, \frac{d-1}{2}+4, \frac{d-1}{2}+5, \ldots, d+3, 4d+4, 4d+5, \ldots, 5d)$. Thus, $\ell(C') = 7 + 6\frac{d-1}{2} + 5d - 4d - 4 = 4d > 3d+1 = k$.

Again, we need that $k > 10$ for $\frac{k}{2} < \frac{2k-5}{3}$, thus, $d \geq 4$. But in order for C' to be a cycle in the even case, we need that $3 < \frac{d}{2}+3 < \frac{d}{2}+6 < d+6 < d+\frac{d}{2}+2 < 2d+2 < 2d+5 < 2d+\frac{d}{2}+5 < 3d+1 < 3d+\frac{d}{2}+1 < 3d+\frac{d}{2}+4 < 4d+4 < 4d+7 \leq 5d$, otherwise, there would be repeated vertices. But all inequalities hold for $d \geq 10$. In order for C' to be a cycle in the odd case, we need that $2 < \frac{d-1}{2}+2 < \frac{d-1}{2}+4 < d+3 < d+\frac{d-1}{2}+2 < 2d+1 < 2d+3 < 2d+\frac{d-1}{2}+3 < 3d+1 < 3d+\frac{d-1}{2}+1 < 3d+\frac{d-1}{2}+3 < 4d+2 < 4d+4 \leq 5d$, otherwise, there would be repeated vertices. But all inequalities hold for $d \geq 5$. So, the construction of C' always works for the odd case, but for the even case we need to propose another cycle for $d \in \{4, 6, 8\}$.

For $d = 4$ we consider the cycle (0, 13, 14, 7, 8, 1, 2, 15, 16, 9, 10, 3, 4, 17, 18, 19, 0). For $d = 6$ we consider the cycle (0,19, 20, 21, 10, 11, 12, 1, 2, 3, 22, 23, 24, 13, 14, 15, 4, 5, 6, 25, 26, 27, 28, 29, 0). For $d = 8$ we consider the cycle (0, 25, 26, 27, 28, 13, 14, 15, 16, 1, 2, 3, 4, 29, 30, 31, 32, 17, 18, 19, 20, 5, 6, 7, 8, 33, 34, 35, 36, 37, 38, 39, 0). These cycles also have length $4d$. The cycles for this exceptional cases can be observed in Figures 2 and 3; it is worth observing that, although these cycles could not be considered in the general case, their structure is very similar to the cycle C' of this proof. ■

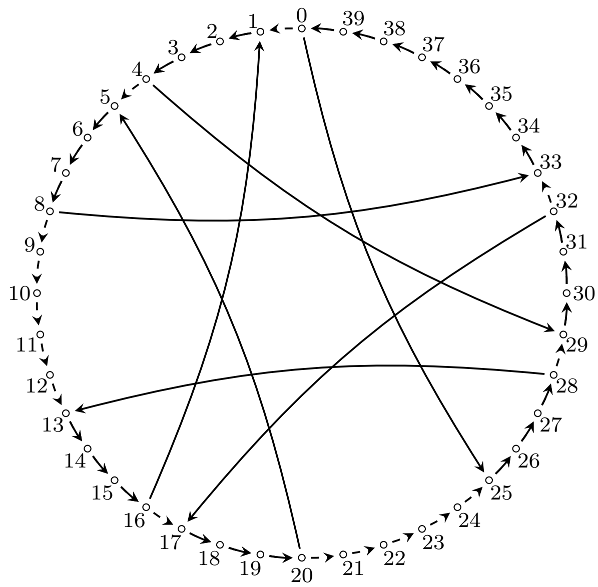

Figure 2. Exceptional case considered in Lemma 9: $d = 8$.

Lemma 10. *Let $k \geq 5$ be an integer and D a k-transitive digraph. If C is an n-cycle in D such that $n = k + i$ and $\frac{2k-2}{3} \leq i$, then there is a directed cycle C' in $D[V(C)]$ such that $k \leq \ell(C') < n$.*

Proof. Let $V(C) = \mathbb{Z}_n$ and $A(C) = \{(0,1), (1,2), \ldots, (n-1, 0)\}$. By the k-transitivity of D, we have that $(j, j+k) \in A(D)$ for any $j \in V(C)$, which also implies $(j, j-i) \in A(D)$.

We will consider two cases. If $i \geq k-1$, then $C' = (0, k, k+1, \ldots, n-1, 0)$. Otherwise, $\frac{2k-4}{3} \leq i \leq k-2$ and, clearly,

$$C' = (0, k, \ldots, 2i+1, i+1, \ldots, k-1, k-1-i, \ldots, i-1, i, 0)$$

is a directed cycle in $D[V(C)]$ of length $\ell(C') = 5 + (2i+1-k) + (k-1) - (i+1) + (i-1) - (k-1-i) = 3i - k + 4 \geq k$, by the choice of i. Also, we may observe that $2i + 2 \leq (2(k-3)+2) = 2k-4$ and then, $2i+2 \notin V(C')$, so $\ell(C') < n$. ■

Theorem 11. *Let $k \geq 5$ and $i \geq 2$ be integers and D a k-transitive digraph. If $n = k + i$ with $k \neq r(i+1) + i + 2$ for every $r \in \mathbb{N}$ and C is an n-cycle in D, then there is a directed cycle C' in $D[V(C)]$ such that $k \leq \ell(C') < n$. If $k = r(i+1) + i + 2$ for some $r \in \mathbb{N}$, then there is a directed cycle C' in $D[V(C)]$ such that $\ell(C') = k - 1$.*

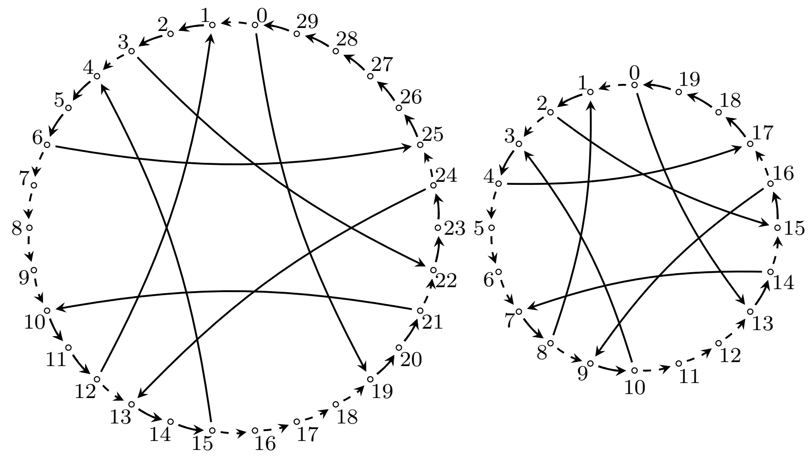

Figure 3. Exceptional cases considered in Lemma 9: $d = 4$ and $d = 6$.

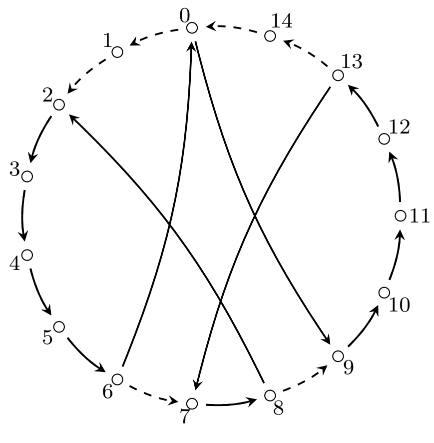

Figure 4. Example of the cycle C' of Lemma 10 with $k = 9$ and $i = 6$.

Proof. If $\ell(C) \geq 2k-1$ and $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$, then $(v_0, v_k, v_{k+1}, \ldots, v_{n-1}, v_0)$ is a cycle in $D[V(C)]$ shorter than C, so we may assume that $\ell(C) \leq 2k-2$. The result follows from Lemmas 7, 8, 9 and 10. ■

Corollary 12. *Let $k \geq 5$ be an integer and D a k-transitive digraph. If C is an n-cycle in D with $n \geq k+2$, then D contains a $(k-1)$-cycle, a k-cycle or a $(k+1)$-cycle.*

Proof. By induction on $\ell(C)$. If $\ell(C)$ is $k-1$, k or $k+1$, there is nothing to prove. So, suppose that $\ell(C) \geq k+2$. It follows from Theorem 11 that there is a directed cycle C' in $D[V(C)]$ such that either $\ell(C') = k-1$ or $k \leq \ell(C') < n$, and thus we can apply the induction hypothesis to C'. ■

3. Preliminary Results

In this section we will prove the statements of Theorem 1 through a series of lemmas considering different cases for the existence of a cycle of length at least $k+1$ in a k-transitive digraph.

Lemma 13. *Let $k \geq 5$ be an integer and D a strong k-transitive digraph. If D contains a k-cycle C, and $|V(D)| \geq k+1$, then D is a complete digraph.*

Proof. Let $V(C) = \mathbb{Z}_k$. It is clear from Proposition 4 that $v \to V(C)$ and $V(C) \to v$ for every $v \in V(D) \setminus V(C)$. Since $|V(D)| \geq k+1$, there is at least one vertex $v \in V(D) \setminus V(C)$. If $v \neq u \in V(D) \setminus V(C)$, then $(v, 0, 1, \ldots, k-2, u)$ is a directed path of length k in D. If follows from the k-transitivity of D that $(v, u) \in A(D)$. Hence, $v \to V(D) \setminus \{v\}$ for every vertex $v \in V(D) \setminus V(C)$. Also, clearly $(v, 1, 2, \ldots, k-1, v)$ is a k-cycle in D not containing 0. Thus, $0 \to V(D) \setminus \{0\}$. It follows from the symmetries of C that D is a complete digraph. ■

Lemma 14. *Let $k \geq 5$ be an integer and D a strong k-transitive digraph. If D contains a $(k+1)$-cycle C, and $|V(D)| \geq k+2$, then:*

(1) *If $k \equiv 0 \,(\mathrm{mod}\ 2)$, then D is a complete digraph.*

(2) *If $k \equiv 1 \,(\mathrm{mod}\ 2)$, then D is a complete digraph or a complete bipartite digraph.*

Proof. Let $V(C) = \mathbb{Z}_{k+1}$. If $k \equiv 0 \,(\mathrm{mod}\ 2)$, then $(k-1, k+1) = 1$. Again, it is clear from Proposition 4 that $v \to V(C)$ and $V(C) \to v$ for every $v \in V(D) \setminus V(C)$. Since $|V(D)| \geq k+2$, then there is at least one vertex $v \in V(D) \setminus V(C)$. The same argument as in Lemma 13 can be used from this point to conclude that D is a complete digraph.

If $k \equiv 1 \,(\mathrm{mod}\ 2)$, then $(k-1, k+1) = 2$. It follows that $(k-1)\mathbb{Z}_{k+1} = 2\mathbb{Z}_{k+1}$. Observe that $(X = 2\mathbb{Z}_{k+1}, Y = 1 + 2\mathbb{Z}_{k+1})$ is a bipartition of C. If D is bipartite, then Proposition 3 and the fact that D is strong imply that for every $v \in V(D)$ either $X \to v$ and $v \to X$, or $Y \to v$ and $v \to Y$ hold. Arguments similar to those of Lemma 13 can be used to conclude that D is a complete bipartite digraph.

If D is not bipartite, then Proposition 3 and the fact that D is strong imply the existence of $v \in V(D)$ such that $v \to Y$ and $X \to v$ (or $Y \to v$ and $v \to X$). Hence, $(v, 1, \ldots, k-1, v)$ is a k-cycle in D, and Lemma 13 implies that D is a complete digraph. ■

Lemma 15. *Let $d, p, k \geq 2$ be integers such that $k - 1 = d(p-1)$ and D a k-transitive digraph. If C is a dp-cycle in D, then C has a d-cycle extension as a spanning subdigraph. Moreover, if $V(C) = (v_0, v_1, \ldots, v_{dp-1}, 0)$, $\{V_i\}_{i=0}^{d-1}$ is the cyclical partition of $V(C)$, where $V_i = \{v_j \colon\ j \equiv i \pmod d\}$.*

Proof. If we let $i = d-1$ and $r = p-2$, it is clear that $k = r(i+1) + i + 2$, so we may consider the cycle C' of Lemma 7 that has length $k-1$. Such cycle is

$$\begin{aligned} C' &= \bigcup_{m=0}^{r-1} \big((v_{k-(m-1)i-m}, v_{k-mi-m}) \cup (v_{k-mi-m} C v_{k+(1-m)i-(m+2)}) \\ &\cup (v_{k+(1-m)i-(m+2)}, v_{k-mi-(m+2)}) \cup (v_{k-mi-(m+2)}, v_{k-mi-(m+1)}) \big) \\ &\cup (v_{2i+2}, v_{i+2}) \cup (v_{i+2} C v_{2i}) \cup (v_{2i}, v_i, 0). \end{aligned}$$

Let us observe that $v_{k+i-1} \notin V(C')$. So, we may consider the directed path $P = (v_{k+i-1}, v_0) \cup (v_0 C' v_i) \cup (v_i, v_{i+1})$ of length $\ell(P) = 1 + \ell(v_0 C' v_i) + 1 = 1 + k - 2 + 1 = k$. Thus, $(v_{k+i-1}, v_{i+1}) \in A(D)$. By symmetry of C, we have that $(v_0, v_{i+2}) \in A(D)$.

We may then consider the cycles $C_1 = (v_{i+1}, v_1) \cup (v_1 C v_{i+1})$ and $C_2 = (v_0, v_{i+2}) \cup (v_{i+2} C v_0)$ of lengths $i+1$ and $k-1$ respectively. Since $i + 1 \leq k - 1$ and $(v_{i+1}, v_{i+2}) \in A(D)$, if we rename the vertices of C_2 in such way that $v_{i+2} = y_0, v_{i+3} = y_1, \ldots, v_{k+i-1} = y_{k-3}$, then we can conclude from Lemma 6 that $v_{i+1} \to \{y_i \colon\ i \in (k-i-2)\mathbb{Z}_{k-1}\}$. Let us recall that $k = r(i+1) + i + 2$, hence $v_{i+1} \to \{y_i \colon\ i \in (r(i+1))\mathbb{Z}_{k-1}\}$. But $r(i+1) = d(p-2)$ and $k - 1 = d(p-1)$, thus $v_{i+1} \to \left\{y_i \colon\ i \in (d(p-2))\mathbb{Z}_{d(p-1)}\right\}$. Also, $(d(p-2), d(p-1)) = d$, thus $(d(p-2))\mathbb{Z}_{d(p-1)} = d\mathbb{Z}_{d(p-1)} = \{0, d, 2d, \ldots, d(p-2)\}$. Thus, $v_{i+1} = v_d \to \{y_{sd}\}_{s=0}^{p-2} = \{v_{d+1}, v_{2d+1}, \ldots, v_{(p-2)d+1}\}$. We already knew that $(v_d, v_1) \in A(D)$, thus $v_d \to V_1 = \{v_j \colon\ j \equiv 1 \pmod d\}$.

By the symmetries of C, the desired result is then obtained. ■

If D is a (possibly infinite) digraph and $k \geq 2$ is an integer, we can recursively define a family of digraphs as follows.

- $D_0 = D$.
- $D_{i+1} = D_i + \{(u, v) \colon\ d_{D_i}(u, v) = k\}$.

Clearly, $D \subseteq D_i \subseteq D_{i+1}$ for every $i \in \mathbb{N}$. We can define the *k-transitive closure* of D as $\mathcal{C}^k(D) = \bigcup_{i \in \mathbb{N}} D_i$. It is direct to observe that $\mathcal{C}^k(D)$ is a k-transitive digraph containing D. As a matter of fact, it is the minimal k-transitive digraph containing D. Thus, if D is k-transitive, $D = \mathcal{C}^k(D)$. We will use these simple observations to prove our next lemma.

Lemma 16. *Let $k \geq 2$ be an integer. If $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$ is an n-cycle with $n \geq k+2$ and $(n, k-1) = d$, with $d \geq 2$, then $\mathcal{C}^k(C)$ is a d-cycle extension with cyclical partition $\{V_i\}_{i=0}^{d-1}$ of $V(C)$, where $V_i = \{v_j : \ j \equiv i \pmod d\}$.*

Proof. Let $D = \mathcal{C}^k(C)$ and let us consider the family $\{C_i\}_{i\in\mathbb{N}}$ used to define $\mathcal{C}^k(C)$. We will prove by induction on i that C_i is a cyclically d-partite digraph with cyclical partition $\{V_i\}_{i=0}^{d-1}$. Clearly, $C_0 = C$ fulfills the desired property.

Let us consider an arc $(u, v) \in A(C_{i+1}) \setminus A(C_i)$. By the definition of C_{i+1}, a uv-directed path $P = (u = u_0, u_1, \ldots, u_k = v)$ of length k exists in C_i. We can assume without loss of generality that $u_0 \in V_0$, thus, $v_i \in V_i \pmod d$, because C_i is a cyclically d-partite digraph. But $(n, k-1) = d$, implying that $d|k-1$, thus $k \equiv 1 \pmod d$. Hence, $v = u_k \in V_1$. Every arc of C_{i+1} is then a V_jV_{j+1}-arc $\pmod d$.

The Principle of Mathematical Induction and the definition of $\mathcal{C}^k(C)$ imply that $\mathcal{C}^k(C)$ is a cyclically d-partite digraph. It remains to show that $\mathcal{C}^k(C)$ is a d-cycle extension.

If $k-1 = (p-1)d$ and $n = pd$ for some integer $p \geq 2$, then the result follows directly from Lemma 15. Otherwise, we have that $k-1 = (p-1)d$ and $n = qd$ for some pair of integers $2 \leq p < q$. Since C is an n-cycle in the k-transitive digraph $\mathcal{C}^k(C)$, we know from Corollary 12 that $\mathcal{C}^k(C)$ contains a $(k-1)$-cycle, a k-cycle or a $(k+1)$-cycle. But $\mathcal{C}^k(C)$ is a cyclically d-partite digraph, thus, it cannot contain a k-cycle.

If $\mathcal{C}^k(C)$ contains a $(k+1)$-cycle, then $d\big|k+1$, hence $d = 2$ and $k \equiv 1 \pmod 2$. Thus, by Lemma 14, $\mathcal{C}^k(C)$ is a complete 2-partite digraph and thus a cyclically 2-partite digraph.

So, let us suppose that $\mathcal{C}^k(C)$ contains a $(k-1)$-cycle. It can be observed from Lemmas 7, 8, 9, and 10 that the existence of a $(k-1)$-cycle is the outcome of only one case, when $k = r(i+1) + i + 2$ and a $(k+i)$-cycle is considered. If we let $i+1 = d$ and $r = p-1$, we necessarily have the existence of a pd-cycle in $\mathcal{C}^k(C)$. Let C_1 be such a cycle. Then, by Lemma 15, $\mathcal{C}^k(C)[V(C_1)]$ is a d-cycle extension. If follows from the fact that $\mathcal{C}^k(C)$ is a cyclically d-partite digraph that the d-cyclical partition $\{V'_0, \ldots, V'_{d-1}\}$ of $V(C_1)$ is such that $V'_i \subseteq V_i$ for $0 \leq i \leq d-1$.

Also, $\mathcal{C}^k(C)$ is a strong k-transitive digraph, thus, since $\ell(C_1) \geq k+2$, for every vertex $v \in V(\mathcal{C}^k(C)) \setminus V(C_1)$ there exist a vC_1-arc and a C_1v-arc in $\mathcal{C}^k(C)$. As we have already observed, $\mathcal{C}^k(C)$ is a cyclically d-partite digraph, thus, every vC_1-arc is a V_iV_{i+1}-arc and every C_1v-arc is a $V_{i-1}V_i$-arc for some $0 \leq i \leq d-1$. Also, since $d\big|k-1$, it follows from the k-transitivity of $\mathcal{C}^k(C)$ that $V'_{i-1} \to v \to$

V'_{i+1} for some $0 \leq i \leq d-1$. From here it is easy to observe that if $v \in V_i \setminus V'_i$ and $u \in V_{i+1} \setminus V'_{i+1}$, a uv-directed path of length k can be found in $\mathcal{C}^k(C)$, thus, $V_i \to V_{i+1}$. Hence $\mathcal{C}^k(C)$ is a d-cycle extension. ∎

Lemma 14 describes the situation of a strong k-transitive digraph containing a $(k+1)$-cycle and at least one vertex outside the cycle. It is easy to observe that a $(k+1)$-cycle with all its arcs symmetrical is a k-transitive digraph, and the existence of diagonals of the cycle cannot be derived. But, as the following lemma states, if a single diagonal exists, it is easy to show the existence of many more of them.

Lemma 17. *Let $k \geq 2$ be an integer and D a k-transitive digraph. If $C = (v_0, v_1, \ldots, v_k, v_0)$ is a $(k+1)$-directed cycle in D and (v_i, v_j) is a diagonal of C, then*

(i) *If $k \equiv 0 \pmod 2$, then $D[V(C)]$ is a complete digraph.*

(ii) *If $k \equiv 1 \pmod 2$, we have two cases:*

  (i) *If $i \equiv j \pmod 2$, then $D[V(C)]$ is a complete digraph.*

  (ii) *If $i \not\equiv j \pmod 2$, then $D[V(C)]$ is a complete bipartite digraph with bipartition (V_{2i}, V_{2i+1}), where V_{2i} is the set of vertices with even index and V_{2i+1} the complement of V_{2i}.*

Proof. First, let us observe that it follows from the k-transitivity of D that every arc of C is symmetrical, thus, the directed cycle $C^{-1} = \overleftarrow{C}$ is a directed cycle in D. We will assume without loss of generality that the diagonal (v_i, v_j) is of the form (v_0, v_j), thus, $2 \leq j \leq k-1$. Let us make an observation that works for every case.

If $j > 2$, then the directed paths $(v_{j-1}C^{-1}v_0) \cup (v_0, v_j) \cup (v_jCv_k)$ and $(v_{j+1}Cv_0) \cup (v_0, v_j) \cup (v_jC^{-1}v_1)$ have length k. Thus, $(v_{j-1}, v_k), (v_{j+1}, v_1) \in A(D)$. Also, the directed path $(v_0, v_j, v_{j-1}, v_k) \cup (v_kC^{-1}v_{j+1}) \cup (v_{j+1}, v_1) \cup (v_1Cv_{j-2})$ is a k-directed path in D, thus, $(v_0, v_{j-2}) \in A(D)$. Clearly, we can make an inductive proof.

Thus, if $j = 2$, then $(v_0, v_2) \cup (v_2Cv_0)$ is a k-cycle, and the result follows from Lemma 13. Also, if $(v_0, v_j) \in A(D)$ with $j \geq 4$, then the previous observation give us that $(v_0, v_{j-2}) \in A(D)$. For the case when $k \equiv 1 \pmod 2$ but $j \equiv 0 \pmod 2$ we are done, we already have the base case and the inductive step.

For the cases $k \equiv 0 \pmod 2$ and $k \equiv 1 \equiv j \pmod 2$ we need to consider also $j = 3$. But it is easy to observe that $(v_0, v_3) \cup (v_3Cv_0)$ is a $(k-1)$-directed cycle in D. Also $(v_2, v_3), (v_3, v_2) \in A(D)$. In the former case, $(k-1, k-3) = 1$, thus, $v_1 \to V(C) \setminus \{v_1\} \to v_1$ and $v_2 \to V(C) \setminus \{v_2\} \to v_2$. Also, $(v_1, v_2), (v_2, v_1) \in A(D)$, so it can be easily observed that for every pair of distinct vertices in $V(C)$ a k-directed path between them can be found. Thus, the base case for $k \equiv 0 \pmod 2$

is done. Since the inductive step is a trivial consequence of the observation at the beginning of the proof, this case is finished.

If $k \equiv 1 \equiv j \,(\mathrm{mod}\ 2)$, then it follows from Lemma 6 and $(k-1, k-3) = 2$ that $v_2 \to \{v_i\colon\ i \in 2\mathbb{Z}_{k+1} + 1\}$ and $\{v_i\colon\ i \in 2\mathbb{Z}_{k+1} + 1\} \to v_2$. Analogously, since $(v_0, v_1), (v_1, v_0) \in A(D)$,we have $v_1 \to \{v_i\colon\ i \in 2\mathbb{Z}_{k+1}\}$ and $\{v_i\colon\ i \in 2\mathbb{Z}_{k+1}\} \to v_1$. Finally, if $0 \leq n \leq k-1$ is even, $3 \leq k$ is odd, $2 \neq n \neq m\pm1$, and $n < m$, then $P = (v_n C^{-1} v_2) \cup (v_2, v_{n+1}) \cup (v_{n+1} C v_{m-1}) \cup (v_{m-1}, v_1) \cup (v_1 C^{-1} v_m)$ is a k-directed path in D. Clearly, $\overleftarrow{P}$ is also a k-directed path in D, thus, $(v_n, v_m), (v_m, v_n) \in A(D)$. If $m < n$, then $P = (v_m C^{-1} v_2) \cup (v_2, v_{n-1}) \cup (v_{n-1} C^{-1} v_{m+1}) \cup (v_{m+1}, v_1) \cup (v_1 C^{-1} v_m)$ is a k-directed path in D. Clearly, $\overleftarrow{P}$ is also a k-directed path in D, thus, $(v_n, v_m), (v_m, v_n) \in A(D)$. It follows that $V_{2i} \to V_{2i+1}$ and $V_{2i+1} \to V_{2i}$. Thus, $D[V(C)]$ is a complete bipartite digraph. This complete the base case. The inductive step is trivial considering the initial observation of the proof. ■

Lemma 18. *Let $k \geq 2$ be an integer and D a k-transitive digraph. If $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$ is an n-cycle in D with $n \geq k+2$, then the following assertions hold.*

(i) *If $(n, k-1) = 1$, then $D[V(C)]$ is a complete digraph.*

(ii) *If $(n, k-1) = d$, with $d \geq 2$, then $D[V(C)]$ contains a d-cycle extension as a spanning subdigraph. Moreover, $\{V_i\}_{i=0}^{d-1}$ is the cyclical partition of $V(C)$, where $V_i = \{v_j\colon\ j \equiv i \,(\mathrm{mod}\ d)\}$.*

Proof. For $k = 2$ the assertion is trivial. For $3 \leq k \leq 4$, the result follows from the characterization theorems in [5] and [6]. For $k \geq 5$, it follows from Corollary 12 that a $(k-1)$, k or $(k+1)$-cycle exists in $D[V(C)]$. If a k-cycle exists in $D[V(C)]$, Lemma 13 give us the desired result. If a $(k+1)$-cycle exists in $D[V(C)]$ we can consider two cases. If $k \equiv 0 \,(\mathrm{mod}\ 2)$, then $D[V(C)]$ is a complete digraph by virtue of Lemma 14. If $k \equiv 1 \,(\mathrm{mod}\ 2)$, using again Lemma 14, then $D[V(C)]$ contains a complete bipartite spanning subdigraph $D' = (X, Y)$. But $(n, k-1) = 1$, so $D[V(C)]$ is not a bipartite digraph. We can assume without loss of generality that a XX-arc exists in D. But D' is complete bipartite, so a k-cycle can be easily constructed in D. Applying again Lemma 13 we obtain the desired result.

If a $(k-1)$-cycle exists in $D[V(C)]$, then the same argument as used in the proof of Lemma 16 shows that there exist integers $d, p \geq 2$ such that a dp-cycle C_1 exists in $D[V(C)]$ and $k - 1 = d(p-1)$. Moreover, it can be deduced from Lemma 16 that $D[V(C_1)]$ has a d-cycle extension as a spanning subdigraph. Let us suppose that the d-cyclical partition of $D[V(C_1)]$ is $\{V_i\}_{i=0}^{d-1}$. It follows from Proposition 3, the k-transitivity of D and the fact that D is strong, that for every vertex $v \in V(C) \setminus V(C_1)$, there exist $i, j \in \{0, 1, \ldots, d-1\}$ (not necessarily distinct) such that $v \to V_i$ and $V_j \to v$.

But $(n, k-1) = 1$, so there must exist a vertex $v \in V(C) \setminus V(C_1)$ and integers $i, j \in \{0, 1, \ldots, d-1\}$ such that $i \not\equiv j + 2 \,(\mathrm{mod}\ d)$, $v \to V_i$ and $V_j \to v$, otherwise $D[V(C)]$ would be cyclically d-partite, which is impossible. If $i = j$, then the existence of a $(k+1)$-cycle can be easily deduced. But in this case we have already showed that the desired result is reached.

Thus, we can assume without loss of generality that $0 \leq i < j \leq d-1$. Since $|V_i| = p$ for each $0 \leq i \leq d-1$ and $V[(C_1)]$ has a d-cycle extension as a spanning subdigraph, directed paths of length k passing through v can be found from every vertex of V_j to every vertex of V_{i-1}, thus, $V_j \to V_{i-1}$ (it can be the case that $j = i-1$). Thus, we can consider a directed path P of length $k-1$ with initial and terminal vertex in V_{i-1}. Let us observe that only $p-1$ vertices of each V_i have been used in the path P, thus, we can consider a directed path P' disjoint with P such that the initial vertex of P' is the terminal vertex of P and the terminal vertex of P' is in V_j. The walk $P \cup P'$ together with the arc joining the terminal vertex of P' to the initial vertex of P is a directed cycle of length $k - 1 < k + (j-i) + 1 < pd$. If we name such cycle as C_2, then either $\ell(C_2) \in \{k, k+1\}$ or $k + 2 \leq \ell(C_2) < pd$. In the latter case, since $\ell(C_2) < pd$, Corollary 12 implies that a k-cycle or a $(k+1)$-cycle exists in $D[V(C_2)]$, because the only case that has a $k-1$ cycle as outcome cannot occur. In either case, we have already proved that $D[V(C)]$ is a complete digraph.

For the second part of the result, we know by Lemma 16 that $\{V_i\}_{i=0}^{d-1}$ is the cyclical partition of $\mathcal{C}^k(C) \subseteq D[V(C)]$, where $V_i = \{v_j \colon\ j \equiv i \,(\mathrm{mod}\ d)\}$. Also $\mathcal{C}^k(C)$ is a d-cycle extension, which completes the proof. ■

Lemma 19. *Let $k \geq 2$ be an integer and D a strong k-transitive digraph such that D has at least one directed cycle of length greater than or equal to $k+2$, but D does not contain any directed cycle C such that $(\ell(C), k-1) = 1$. If $\{C_i\}_{i \in I}$ is the family of all the directed cycles of D of length greater than or equal to $k+1$ and d is the g.c.d of $\{(\ell(C_i), k-1)\}_{i \in I}$, then $d = \min_{i \in I}\{(\ell(C_i), k-1)\}$ and D is a d-cycle extension.*

Proof. Let $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$ be an n-cycle with $(n, k-1) = d = \min_{i \in I}\{(\ell(C_i), k-1)\}$. By Lemma 18, $D[V(C)]$ has a d-cycle extension as a spanning subdigraph with cyclical partition $\{V'_i\}_{i=0}^{d-1}$, where $V'_i = \{v_j \colon\ j \equiv i\}$ (mod d). Since $\ell(C) \geq k+2$ and D is strong, for every vertex $v \in V(D) \setminus V(C)$ there exist integers $0 \leq i \leq j \leq d-1$ such that $V'_i \to v \to V'_j$. If $i \equiv j+2$ (mod d) for every $v \in V(D) \setminus V(C)$, then D has a d-cycle extension as a spanning subdigraph with d-cyclical partition $\{V_i\}_{i=0}^{d-1}$ such that $V'_i \subseteq V_i$ for $0 \leq i \leq d-1$.

If there is a $V'_i V'_j$-arc in $D[V(C)]$ with $j \not\equiv i + 1 \,(\mathrm{mod}\ d)$, then we can assume without loss of generality that $i = 0$. Since $\ell(C) \geq k+1$, each V'_i has at least p vertices, where $k - 1 = (p-1)d$, thus, if (u, v) is the $V'_0 V'_j$-arc in D, we can consider a directed path of length $k-1$ beginning at $u' \in V'_0$, ending at u and not using v. If we add (u,v) to the end of such path, we have a $u'v$-directed path of length k and we can conclude that $(u',v) \in A(D)$. Analogously, for every $v' \in V_j$, we can consider a directed path of length $k-1$ with initial vertex v, end vertex v' and without $V_0'V_j'$-arcs. If we add (u',v) to the beginning of such path, for any $u' \in V_0'$, then we have a $u'v'$-directed path in D. It follows from the k-transitivity of D that $(u',v') \in A(D)$. Since u' and v' were arbitrarily chosen, $V_0' \to V_j'$. Also, we can consider a directed path P with initial vertex in V_1' of length $k-2$, using arcs of the form $V_iV_{i+1} \pmod{d}$ only. Thus, the endpoint of P belongs to V_0', and we can add a $V_0'V_j'$-arc and a $V_j'V_{j+1}'$ arc to the end of P in such way that a $V_1'V_{j+1}'$-directed path of length k is obtained. Again, a $V_1'V_{j+1}'$-arc exists in D by the k-transitivity of D. Inductively, it can be shown that $V_i' \to V_{i+j}' \pmod{d}$. We have two cases.

If $(d,j) = d' > 1$, then $P_1 = (v_0, v_j, v_{2j}, \ldots, v_{(\frac{d}{d'}-1)j}, v)$ with $v_i \in V_i'$ and $v \in V_0'$ is a V_0V_0-directed path of length $\frac{d}{d'}$. Once again, we can consider a vv_0-directed path of length $k-1$ in D, say P_2, disjoint with P_1. Thus, $P = P_1 \cup P_2$ is a directed cycle in D of length $m = (k-1) + \frac{d}{d'} \geq k+1$. But $m-(k-1) = \frac{d}{d'} < d$, and hence $(m, k-1) < d$, which results in a contradiction by the minimality of d.

If $(d,j) = 1$, then by the Division Algorithm, there are positive integers q, r such that $0 \leq r < j$ and $d = qj + r$. If we consider P_1, a V_0V_0-directed path of length $k-1$ with initial vertex u and terminal vertex v, we can consider the directed path $P_2 = (u, v_j, v_{2j}, \ldots, v_{qj}, v_{qj+1}, \ldots, v_{qj+r-1}, v)$ disjoint with P_1 and of length $q + r$. Now, $P_1 \cup P_2$ is a directed cycle of length $m = (k-1) + q + r$. Recalling that $(d,j) = 1$ and $j < d$, we have that $q, r > 0$. It follows from this observation that $q + r > 1$, therefore $m \geq k+1$. We can observe that $d = qj + r > q(j-1) + r \geq q + r$. But, $m - (k-1) = q + r$, and hence, $(m, k-1) \leq q + r < d$, contradicting the minimality of d.

Thus, every arc of $D[V(C)]$ is a V_iV_{i+1}-arc, and $D[V(C)]$ is a d-cycle extension.

If there is a vertex $v \in V(D) \setminus V(C)$ and integers $0 \leq i \leq j \leq k-1$ such that $i \not\equiv j+2 \pmod{d})$ and $V_i' \to v \to V_j'$ a similar argument can be used to show that there is a $V_i'V_j'$-arc in $D[V(C)]$ with $j \not\equiv i+1 \pmod{d}$. We just have to consider a directed path P of length $k-2$ beginning at V_{i+1}' and ending at V_i'. We can add to P a $V_i'v$-arc and a vV_j'-arc to obtain a directed path of length k. The k-transitivity of D implies that a $V_{i+1}'V_j'$-arc exists in D and $j \not\equiv i+2 \pmod{d}$.

Thus, it follows that D is a d-cycle extension. ■

4. Main Results and Consequences

Now we are ready to prove our main results.

***Proof of Theorem* 1.** It follows from the lemmas in the previous section. ■

***Proof of Theorem* 2.** It follows from the lemmas in the previous section. ■

Corollary 20. *Let $k \geq 2$ be an integer and D a strong k-transitive digraph with at least one directed cycle of length greater than or equal to $k+2$. Then D is either a complete digraph or a cycle extension.*

Proof. It is straightforward from Theorem 1. ■

Now, the case when $k-1$ is a prime greater than 2 can be easily studied.

Corollary 21. *Let $k-1 \geq 3$ be a prime integer, D a strong k-transitive digraph and C a directed cycle of length at least $k+1$. Exactly one of the following possibilities hold:*

(i) *D is a symmetrical $(k+1)$-cycle.*

(ii) *D is a $(k-1)$-cycle extension.*

(iii) *D is a complete digraph.*

Proof. It follows from Theorem 1 and the fact that $k-1$ is a prime (and hence k is even). ■

Another easy observation can be done for k-transitive digraphs with directed cycles of length greater than k. We define an extension $D[E_1, \ldots, E_n]$ of a digraph D to be r-regular if $|E_i| = r$ for every $1 \leq i \leq n$; an extension will be regular if it is r-regular for some $r \in \mathbb{Z}^+$. An extension of a digraph will be non-regular if it is not regular.

Corollary 22. *Let $k \geq 2$ be an integer and D a strong k-transitive digraph with at least one directed cycle of length greater than or equal to k. Then D is hamiltonian if and only if D is not a non-regular cycle extension.*

Proof. If D is hamiltonian and it is a cycle extension, then D must be a regular cycle extension since it visits each class of the cyclical partition the same number of times. If D is not a non-regular cycle extension, then it is either a complete digraph, a regular cycle extension, a hamiltonian digraph on k vertices or a symmetrical $(k+1)$-cycle. It is clear that all these four families are hamiltonian. ■

4.1. The Laborde-Payan-Xuong Conjecture

The Laborde-Payan-Xuong Conjecture (LPX), [7], states that for every digraph D, there exists an independent set $I \subseteq V(D)$ such that I intersects every directed path of maximum length in D. We will prove LPX to be true for some particular cases of k-transitive digraphs.

Proposition 23. *Let $k \geq 2$ be an integer and D a strong k-transitive digraph. If a cycle of length greater than or equal to k exists in D, then LPX is valid for D.*

Proof. Observing Theorem 1 there are only four cases to consider. If D is a complete digraph, then the independent set $\{v\}$ will work for every $v \in V(D)$. If $D = (X, Y)$ is complete bipartite, then we can choose either X or Y. If D is a d-cycle extension with cyclical partition $\{V_i\}_{i=0}^{d-1}$, then any V_i will work as our independent set. Finally, if a k-cycle exists in D but $|V(D)| = k$, then D is hamiltonian and once again, $\{v\}$ will work for every $v \in V(D)$.

As a final observation, whenever D has a $(k+1)$-cycle and contains a complete bipartite digraph as a spanning subdigraph, then D is either complete or complete bipartite. ■

Theorem 24. *Let $k \geq 2$ be an integer and D a k-transitive digraph. If a cycle of length greater than or equal to k exists in every terminal strong component of D, then LPX is valid for D.*

Proof. It is a well known result that every non-terminal strong component of D reaches at least one terminal component. Hence, applying Lemma 3 is easy to observe that every vertex of D dominates a vertex in a terminal component and thus, every longest path must have its terminal vertex in a terminal component.

In the case that a longest path of D has its terminal vertex in a terminal component T that is complete or hamiltonian, then this longest path must visit every vertex of T. Therefore, the independent set $\{v\}$ will work for every $v \in V(T)$.

If a longest path P of D has its terminal vertex in a cycle extension (which includes a complete bipartite digraph), then P must visit every vertex in the smallest class of the cyclical partition, so that class is the independent set we were looking for. ■

As a matter of fact, by the simple observation that D is k-transitive if and only if $\overleftarrow{D}$ is k-transitive, we have also proved the following corollary.

Corollary 25. *Let $k \geq 2$ be an integer and D a k-transitive digraph. If a cycle of length greater than or equal to k exist in every initial strong component of D, then LPX is valid for D.*

Also, it is easy to observe that besides finding an independent set I intersecting every longest path, a longest path of D can be found having any vertex of I as its initial vertex, which is a stronger version of LPX. It has been proved that LPX is true for 2 and 3-transitive digraphs. We think that an analysis of the behaviour between strong components of a k-transitive digraph may be useful to improve the results of this section.

5. Conclusions and Further Problems

We have successfully analyzed the structure of strong k-transitive digraphs with a directed cycle of length at least $k+1$. We have also observed that those k-transitive digraphs containing a directed cycle of length k which are not complete, at least are hamiltonian. Also, we used those results to prove some interesting consequences, like the Laborde-Payan-Xuong Conjecture for k-transitive digraphs with cycles of length at least k.

But it is more important to notice that the family of k-transitive digraphs has a lot of structure. We hope that this work will encourage others to work in this fascinating family of digraphs, and also with the very related family of k-quasi-transitive digraphs, also defined in [3]. A digraph is k-quasi-transitive if the existence of a directed path $(v_0, \ldots, v_k)$ implies that $(v_0, v_k) \in A(D)$ or $(v_k, v_0) \in A(D)$. As some initial problems, we think that k-quasi-transitive digraphs with cycles of length at least $k+2$ have a very similar behavior as k-transitive digraphs, but, in most cases, the obtained results will reduce to semicomplete instead of complete digraphs. Also, the generalization of a classical result relating asymmetrical transitive and quasi-transitive digraph remains unexplored. Is it true that a graph can receive a k-transitive orientation if and only if it can receive a k-quasi-transitive orientation? In [4] this question received an affirmative answer for $k=2$ and also in [9] R. Wang and S. Wang prove the case $k=3$.

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References

[1] J. Bang-Jensen and G. Gutin, Digraphs. Theory, Algorithms and Applications (Springer-Verlag Berlin, Heidelberg New York, 2002).

[2] R. Diestel, Graph Theory 3rd Edition (Springer-Verlag Berlin, Heidelberg New York, 2005).

[3] H. Galeana-Sánchez and C. Hernández-Cruz, *k-kernels in k-transitive and k-quasi-transitive digraphs*, Discrete Math. **312** (2012) 2522–2530. doi:10.1016/j.disc.2012.05.005

[4] A. Ghouila-Houri, *Caractérisation des graphes non orientés dont on peut orienterles arrêtes de manière* à *obtenir le graphe d'une relation d'ordre*, C. R. Acad. Sci. Paris **254** (1962) 1370–1371.

[5] C. Hernández-Cruz, 3-*transitive digraphs*, Discuss. Math. Graph Theory **32** (2012) 205–219.
doi:10.7151/dmgt.1613

[6] C. Hernández-Cruz, 4-*transitive digraphs* I: *The structure of strong* 4-*transitive digraphs*, Discuss. Math. Graph Theory **33** (2013) 247–260.
doi:10.7151/dmgt.1645

[7] J.M. Laborde, C. Payan and N.H. Xuong, *Independent sets and longest directed paths in digraphs*, in: Graphs and other Combinatorial Topics, Prague, M. Fiedler (Ed(s)), (Teubner, Leipzig, 1983) 173–177.

[8] R. Wang, *A conjecture on k-transitive digraphs*, Discrete Math. **312** (2012) 1458–1460.
doi:0.1016/j.disc.2012.01.011

[9] R. Wang and S. Wang, *Underlying graphs of* 3-*quasi-transitive digraphs and* 3-*transitive digraphs*, Discuss. Math. Graph Theory **33** (2013) 429–436.
doi:10.7151/dmgt.1680

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