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ON SUPER EDGE-ANTIMAGIC TOTAL LABELING OF SUBDIVIDED STARS[1]

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Abstract

In 1980, Enomoto *et al.* proposed the conjecture that every tree is a super $(a, 0)$-edge-antimagic total graph. In this paper, we give a partial support for the correctness of this conjecture by formulating some super (a, d)-edge-antimagic total labelings on a subclass of subdivided stars denoted by $T(n, n+1, 2n+1, 4n+2, n_5, n_6, \ldots, n_r)$ for different values of the edge-antimagic labeling parameter d, where $n \geq 3$ is odd, $n_m = 2^{m-4}(4n+1)+1$, $r \geq 5$ and $5 \leq m \leq r$.

Keywords: super (a, d)-EAT labeling, subdivision of star.

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1. Introduction

All graphs in this paper are finite, simple and undirected. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively. A (v, e)-graph G is a graph such that $|V(G)| = v$ and $|E(G)| = e$. Moreover, the theoretic ideas of graphs can be seen in [22]. A *labeling* (or *valuation*) of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). In this paper, the domain will be the set of all vertices and edges and such a labeling is called a *total labeling*. Some labelings use the vertex set only or the edge set only and we shall call them *vertex-labelings* or *edge-labelings*, respectively.

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There are many types of graph labelings, for example harmonius, cordial, graceful and antimagic. The most complete recent survey of graph labelings can be found in [6]. In this paper, we focus on an antimagic total labeling. More details on an antimagic total labeling can be found in [4]. The subject of edge-magic total labeling of graphs has its origin in the works of Kotzig and Rosa [13, 14] on what they called magic valuations of graphs.

Definition 1.1. An (s,d)-*edge-antimagic vertex* ((s,d)-EAV) *labeling* of a graph G is a bijective function $\lambda : V(G) \to \{1, 2, \ldots, v\}$ such that the set of edge-sums of all edges in G, $\{w(xy) = \lambda(x) + \lambda(y) : xy \in E(G)\}$, forms an arithmetic progression $\{s, s+d, s+2d, \ldots, s+(e-1)d\}$, where $s > 0$ and $d \geq 0$ are two fixed integers.

Simanjuntak *et al.* [21] proved that the odd cycle C_{2n+1}, the odd path P_{2n+1} and the even path P_{2n} have a $(n+2,1)$-EAV labeling, where $n \geq 1$. They also proved that the odd path P_{2n+1} has a $(n+3,1)$-EAV labeling and the path P_n admits a $(3,2)$-EAV labeling for $n \geq 1$. Moreover, Bača, Miller, Simanjuntak, Lin and Bertault [2, 21] proved the following results.

- If a non-tree connected graph G has an (a,d)-EAV labeling then $d = 1$.
- The cycle C_n has no (a,d)-EAV labeling for $d > 1$ and $n \geq 3$.
- The complete graph K_n has no (a,d)-EAV labeling, where $n > 3$.
- The symmetric complete bipartite graph $K_{n,n}$ has no (a,d)-EAV labeling, where $n > 1$.

Definition 1.2. An (a,d)-*edge-antimagic total* ((a,d)-EAT) *labeling* of a graph G is a bijective function $\lambda : V(G) \cup E(G) \to \{1, 2, \ldots, v+e\}$ such that the set of edge-weights of all edges in G, $\{w(xy) = \lambda(x) + \lambda(xy) + \lambda(y) : xy \in E(G)\}$, forms an arithmetic progression $\{a, a+d, a+2d, \ldots, a+(e-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. If such a labeling exists, then G is said to be an (a,d)-EAT *graph*.

Definition 1.3. An (a,d)-EAT labeling λ is called a *super* (a,d)-*edge-antimagic total* (*super* (a,d)-EAT) *labeling* of G if $\lambda(V(G)) = \{1, 2, \ldots, v\}$. Thus, a *super* (a,d)-EAT *graph* is a graph that admits a super (a,d)-EAT labeling.

In the above definition, if $d = 0$, then a super $(a,0)$-EAT labeling is called a *super edge-magic total* (SEMT) *labeling* and a is called a *magic constant*. For $d \neq 0$, a is called *minimum edge-weight*. The definition of an (a,d)-EAT labeling was introduced by Simanjuntak, Bertault and Miller in [21] as a natural extension of an edge-magic total labeling defined by Kotzig and Rosa. A super (a,d)-EAT labeling is a natural extension of the notion of a super $(a,0)$-EAT labeling defined by Enomoto, Lladó, Nakamigawa and Ringel in [5]. They also proposed the conjecture that every tree is a super $(a,0)$-EAT graph. In the favour of this conjecture, many authors have derived different results on a super (a,d)-EAT labeling for many particular classes of trees, for example path-like trees [3], banana trees [7], w-trees [11], extended w-trees [10, 12], subdivided stars [8, 9, 18, 19, 16, 17], subdivided w-trees [8] and caterpillars [20]. Lee and Shah [15] verified this conjecture by a computer search for trees with at most 17 vertices. However, this conjecture is still open.

Definition 1.4. For $n_i \geq 1$, $r \geq 2$ and $1 \leq i \leq r$, let $T(n_1, n_2, \ldots, n_r)$ be a subdivided star obtained by inserting $n_i - 1$ vertices to each of the i-th edge of the star $K_{1,r}$. Thus, the subdivided star $T\underbrace{(1,1,\ldots,1)}_{r-times}$ is the star $K_{1,r}$.

A star is a particular type of trees and many authors have investigated antimagicness for subdivided stars under certain conditions. Lu [16, 17] called the subdivided star $T(m,n,k)$ a *three-path tree* and proved that it is a super $(a,0)$-EAT if n, m are odd and $k = n+1$ or $k = n+2$. Ngurah *et al.* [18] proved that $T(m,n,k)$ is also a super $(a,0)$-EAT graph if n, m are odd and $k = n+3$ or $k = n+4$. Salman *et al.* [19] proved the existence of a super $(a,0)$-EAT labeling on a particular subclass of the subdivided stars denoted by S_r^1 and S_r^2, where $S_r^1 \cong T\underbrace{(2,2,\ldots,2)}_{r-times}$ and $S_r^2 \cong T\underbrace{(3,3,\ldots,3)}_{r-times}$. Javaid *et al.* [8] investigated some results related to a super $(a,0)$-EAT labeling on the subdivision of the star $K_{1,4}$ and the w-tree $WT(n,k)$. Javaid *et al.* [9] proved that a particular subclass of the subdivided stars in its generalized form denoted by $T(n,n,n+2,n+2,n_5,\ldots,n_r)$ admits a super (a,d)-EAT labeling for different values of d. Some of the results are as follows.

Theorem 1.5 [9]**.** *For any odd $n \geq 3$, $T(n,n,n+2,n+2,2n+3)$ admits a super (a,d)-EAT labeling for $d \in \{0,2\}$.*

Theorem 1.6 [9]**.** *For any odd $n \geq 3$, $T(n,n,n+2,n+2,2n+3)$ admits a super $(a,1)$-EAT labeling.*

Theorem 1.7 [9]**.** *For any odd $n \geq 3$, $T(n,n,n+2,n+2,2n+3,4n+5)$ admits a super (a,d)-EAT labeling for $d \in \{0,2\}$.*

Theorem 1.8 [9]**.** *For any $r \geq 5$ and odd $n \geq 3$, $T(n,n,n+2,n+2,n_5,\ldots,n_r)$ admits a super (a,d)-EAT labeling, where $n_m = 1+(n+1)2^{m-4}$, $5 \leq m \leq r$ and $d \in \{0,2\}$.*

Theorem 1.9 [9]**.** *For any $r \geq 5$ and odd $n \geq 3$, $T(n,n,n+2,n+2,n_5,\ldots,n_r)$ admits a super $(a,1)$-EAT labeling if $|T(n,n,n+2,n+2,n_5,\ldots,n_r)|$ is even, where $n_m = 1+(n+1)2^{m-4}$ for $5 \leq m \leq r$.*

In this paper, we construct another generalized subclass of subdivided stars denoted by $T(n, n+1, 2n+1, 4n+2, n_5, n_6 \ldots, n_r)$, where $n_m = 2^{m-4}(4n+1)+1$, $5 \leq m \leq r$ and $r \geq 5$. Moreover, it is proved that this subclass also admits some super (a, d)-EAT labelings for different values of d. Let us consider the following proposition which we will use in the main results.

Proposition 1.10 [2]. *If a (v, e)-graph G has an (s, d)-EAV labeling, then*

(i) *G has a super $(s+v+1, d+1)$-*EAT *labeling,*

(ii) *G has a super $(s+v+e, d-1)$-*EAT *labeling.*

1.1. Bounds for the magic constant a

Ngurah *et al.* [18] found lower and upper bounds of the magic constant a for a particular family of subdivided stars which are stated as follows.

Lemma 1.11. *If $T(m, n, k)$ is a super $(a, 0)$-*EAT *graph, then $\frac{1}{2l}(5l^2+3l+6) \leq a \leq \frac{1}{2l}(5l^2+11l-6)$, where $l = m+n+k$.*

The lower and upper bounds of the magic constant a proved by Salman *et al.* [19] are as follows.

Lemma 1.12. *If $T\underbrace{(n, n, \ldots, n)}_{n-times}$ is a super $(a, 0)$-*EAT *graph, then $\frac{1}{2l}\left(5l^2+(9-2n)l+n^2-n\right) \leq a \leq \frac{1}{2l}\left(5l^2+(2n+5)l+n-n^2\right)$, where $l = n^2$.*

Now we find lower and upper bounds of the magic constant a for the most extended family of the subdivided stars denoted by $T(n_1, n_2, n_3, \ldots, n_r)$ with any $n_i \geq 1$ for $1 \leq i \leq r$.

Lemma 1.13. *If $T(n_1, n_2, n_3, \ldots, n_r)$ is a super $(a, 0)$-*EAT *graph, then $\frac{1}{2l}\left(5l^2+(9-2r)l+(r^2-r)\right) \leq a \leq \frac{1}{2l}\left(5l^2+(5+2r)l-(r^2-r)\right)$, where $l = \sum_{i=1}^{r} n_i$.*

Proof. Suppose that $T(n_1, n_2, n_3, \ldots, n_r)$ admits a super $(a, 0)$-EAT labeling with magic constant a and $l = \sum_{i=1}^{r} n_i$. Then "la" cannot be smaller than the sum obtained by assigning the smallest label 1 to the vertex of the degree r, the labels from 2 to $l+1-r$ to the vertices of degree 2 and the labels from $l+2-r$ to $l+1$ to the next r vertices of degree 1 as

$$la \geq r + 2\sum_{i=2}^{l-r+1} i + \sum_{i=l-r+2}^{l+1} i + \sum_{i=l+2}^{2l+1} i.$$

Consider $\sum_{i=2}^{l-r+1} i = \frac{l-r}{2}(l-r+3)$, $\sum_{i=l-r+2}^{l+1} i = \frac{1}{2}(2lr-r^2+3r)$ and $\sum_{i=l+2}^{2l+1} i = \frac{l}{2}(3l+3)$. Consequently, we have $la \geq \frac{1}{2}(5l^2+r^2-2lr+9l-r)$ or

$$a \geq \frac{1}{2l}\left(5l^2+r^2-2lr+9l-r\right) \tag{1}$$

Similarly, the upper bound of "la" is obtained by assigning the largest label $l+1$ to the vertex of the degree r, the labels from $r+1$ to l to the vertices of degree 2 and the labels from 1 to r to the next r vertices of degree 1 as

$$la \leq r(l+1) + 2\sum_{i=r+1}^{l} i + \sum_{i=1}^{r} i + \sum_{i=l+2}^{2l+1} i.$$

Consider $\sum_{i=r+1}^{l} i = \frac{3l}{2}(l+1)$, $\sum_{i=1}^{r} i = \frac{r}{2}(r+1)$ and $\sum_{i=l+2}^{2l+1} i = \frac{l-r}{2}(l+r+1)$. Consequently, we have $la \leq \frac{1}{2l}(5l^2 - r^2 + 2lr + 5l + r)$ or

$$a \leq \frac{1}{2l}\left(5l^2 - r^2 + 2lr + 5l + r\right) \tag{2}$$

Combining (1) and (2), we get

$$\frac{1}{2l}\left(5l^2 + (9-2r)l + (r^2 - r)\right) \leq a \leq \frac{1}{2l}\left(5l^2 + (5+2r)l - (r^2 - r)\right).$$

■

1.2. Strategy of construction for labeling schemes

Before presenting the main results, let us consider the overall strategy which is applied to find the results related to super (a,d)-EAT labelings on the particular subclasses of the subdivided stars for different values of the labeling parameter d. It is important to know about three terms edge-label, edge-sum and edge-weight. Let xy be an edge with end vertices x and y. Suppose that the assigned labels to the edge is $\lambda(xy)$ and to the vertices are $\lambda(x)$ and $\lambda(y)$. Thus, $\lambda(xy)$, $\lambda(x)+\lambda(y)$ and $\lambda(x)+\lambda(xy)+\lambda(y)$ are called *edge-label, edge-sum* and *edge-weight*, respectively.

In order to construct a super (a,d)-EAT labeling for $d = 0, 2$ on the graph G, the following steps have been performed:

1.2.1. Working steps for super $(a, 0)$-EAT labeling

- Define a bijection $\lambda : V(G) \to \{1, 2, \ldots, v\}$ in such a way that the set of edge-sums $\{\lambda(x)+\lambda(y)\colon xy \in E(G)\}$ forms a sequence of consecutive integers with minimum edge-sum, say, s.
- It follows that the graph G admits an $(s,1)$-EAV labeling.
- After getting an $(s,1)$-EAV labeling on the graph G, the goal is to extend it to a super $(a,0)$-EAT labeling with the help of the magic constant a.
- The magic constant can be calculated as $a = s + v + e$.
- Using set of edge-sums and the value of magic constant, the set of edge-labels can be obtained as $\{a - (\lambda(x)+\lambda(y))\colon xy \in E(G)\}$.

Consequently, the graph G admits a super $(a,0)$-EAT labeling.

1.2.2. Working steps for super $(a',2)$-EAT labeling

- Define a bijection $\lambda: V(G) \to \{1,2,\ldots,v\}$ in such a way that the set of edge-sums $\{\lambda(x)+\lambda(y)\colon\, xy \in E(G)\}$ forms a sequence of consecutive integers with minimum edge-sum, say, s.
- It follows that the graph G admits an $(s,1)$-EAV labeling.
- After getting an $(s,1)$-EAV labeling on the graph G, the goal is to extend it to a super $(a',2)$-EAT labeling with the help of the minimum edge-weight a'.
- The minimum edge-weight is calculated as $a' = s+v+1$.
- Define the set of edge-weights as $\{a'-2+2i : 1 \le i \le e\}$.
- Define the set of edge-sums as $H = \{h_i : 1 \le i \le e\}$.
- Using a' and the set H, the set of edge-labels can be obtained as $\{(a'-2+2i)-h_i : 1 \le i \le e\}$.

Consequently, the graph G admits a super $(a',2)$-EAT labeling.

In this paper, a super $(a,1)$-EAT labeling is formulated if the order of the graph G is even. Thus, for the construction of a super $(a,1)$-EAT labeling scheme, we proceed as follows.

1.2.3. Working steps for a super $(a,1)$-EAT labeling

- Define a bijection $\lambda: V(G) \to \{1,2,\ldots,v\}$ in such a way that the set of edge-sums $\{\lambda(x)+\lambda(y)\colon\, xy \in E(G)\}$ forms a sequence of consecutive integers with minimum edge-sum, say, s.
- Define the set of edge-sums as $A = \{a_i : 1 \le i \le e\}$.
- The set of edges-labels is $B = \{b_j : 1 \le j \le e\} = \{v_j + 1 : 1 \le j \le e\}$.
- The set of edge-weights can be obtained as
  $C = \{\lambda(x)+\lambda(xy)+\lambda(y)\colon\, xy \in E(G)\}$
  $= \{a_{2i-1} + b_{e-i+1} :\ 1 \le i \le \frac{e+1}{2}\} \cup \{a_{2j} + b_{\frac{e-1}{2}-j+1} : 1 \le j \le \frac{e+1}{2} - 1\}$.
- Thus, the minimum edge-weight is $a = s + \frac{3v}{2}$.

Consequently, the graph G admits a super $(a,1)$-EAT labeling.

2. Main Results

In this section, we present the main results related to a super (a,d)-EAT labeling on a subclass of the subdivided stars for different values of the labeling parameter d.

Theorem 2.1. *For any odd $n \ge 3$, $G \cong T(n,n+1,2n+1,4n+2,8n+3)$ admits a super $(a,0)$-EAT labeling with $a = s+v+e$ and a super $(a',2)$-EAT labeling with $a' = s+v+1$, where $v = |V(G)|$ and $s = 8n+7$.*

Proof. Let us denote the vertices and edges of G as follows.

$V(G) = \{c\} \cup \left\{x_i^{l_i} : 1 \le i \le 5, 1 \le l_i \le n_i\right\}$,

$E(G) = \{cx_i^1 : 1 \le i \le 5\} \cup \left\{x_i^{l_i} x_i^{l_i+1} : 1 \le i \le 5, 1 \le l_i \le n_i - 1\right\}$.

If $v = |V(G)|$ and $e = |E(G)|$, then $v = 16n + 8$ and $e = v - 1$.

Now, we define $\lambda : V(G) \to \{1, 2, \dots, v\}$ as follows: $\lambda(c) = 8n + 6$.

For $1 \le i \le 5$, $1 \le l_i \le n_i$ and l_i odd, we define:

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+1) - \frac{l_2-1}{2}, & \text{for } u = x_2^{l_2}, \\ (2n+2) - \frac{l_3-1}{2}, & \text{for } u = x_3^{l_3}, \\ (4n+3) - \frac{l_4-1}{2}, & \text{for } u = x_4^{l_4}, \\ (8n+5) - \frac{l_5-1}{2}, & \text{for } u = x_5^{l_5}, \end{cases}$$

and for l_i even, we construct:

$$\lambda(u) = \begin{cases} (8n+6) + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (9n+7) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (10n+7) - \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (12n+8) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4}, \\ (16n+9) - \frac{l_5}{2}, & \text{for } u = x_5^{l_5}. \end{cases}$$

The set of all edge-sums generated by the above formulas is $\{\lambda(x) + \lambda(y) \colon xy \in E(G)\}$=$\{8n + 6 + i : 1 \le i \le e\}$. It forms a sequence of consecutive integers starting from the minimum edge-sum $s = 8n + 7$. Thus, by Definition 1.1, λ is a $(8n+7, 1)$-EAV labeling. As a consequence of Proposition 1.10, λ can be extended to a super $(a, 0)$-EAT labeling with magic constant $a = s + v + e = 40n + 22$. The set of edge-labels is $\{a - (8n + 6 + i) : 1 \le i \le e\}$. Similarly, by Proposition 1.10, λ can be extended to a super $(a', 2)$-EAT labeling with the minimum edge-weight $a' = s + v + 1 = 24n + 16$. The set of edge-labels can be obtained by $\{a' - (8n + 6 + i) : 1 \le i \le e\}$. ■

As a consequence of the labeling which is formulated in Theorem 2.1, Figure 1(a) gives the set of edge-sums $\{31, 32, 33, \dots, 85\}$ as a sequence of consecutive integers starting from $s = 31$. Thus, the subdivided star $T(3, 4, 7, 14, 27)$ admits a $(31, 1)$-EAV labeling. The magic constant can be obtained by $c = v + e + s = 56 + 55 + 31 = 142$. The set of edge-labels is $\{(142 - 31), (142 - 32), (142 - 33), \dots, (142 - 85)\}$=$\{111, 110, 109, \dots, 57\}$. Thus, Figure 2(a) presents a super $(142, 0)$-EAT labeling of the subdivided star $T(3, 4, 7, 14, 27)$.

Now, we calculate the minimum edge-weight $a' = s + v + 1 = 31 + 56 + 1 = 88$ and the set of edge-labels $\{(88 - 31), (90 - 32), (92 - 33), \dots, (196 -$

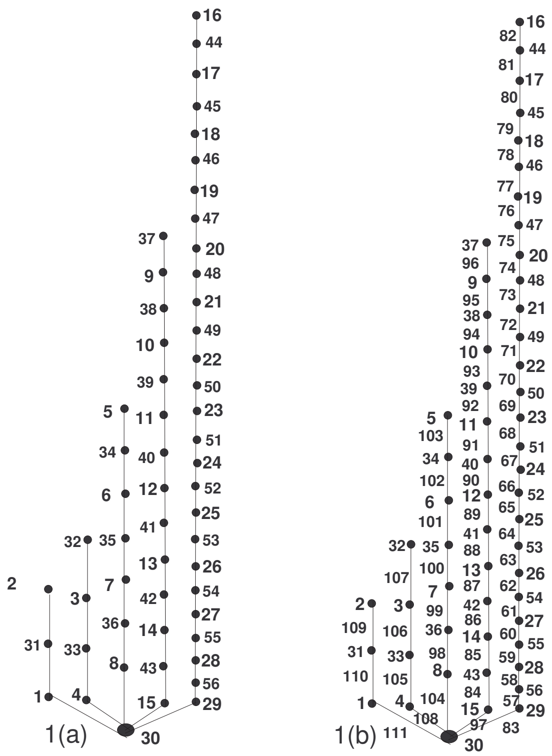

Figure 1. (a) (31,1)-EAV labeling of the subdivided star T(3,4,7,14,27).
(b) Super (142,0)-EAT labeling of the subdivided star T(3,4,7,14,27).

85)}=$\{57, 58, 59, \ldots, 111\}$. Consequently, Figure 2(a) gives a super $(88, 2)$-EAT labeling of the subdivided star $T(3, 4, 7, 14, 27)$.

Theorem 2.2. *For any odd $n \geq 3$, $G \cong T(n, n+1, 2n+1, 4n+2, 8n+3)$ admits a super $(a, 1)$-EAT labeling with $a = s + \frac{3v}{2}$, where $v = |V(G)|$ and $s = 8n + 7$.*

Proof. Let us consider the set of vertices and edges of the graph G defined as in the proof of Theorem 2.1. Now we define the vertex-labeling $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as in the same theorem. It follows that the set of edge-sums for all edges of G denoted by $A = \{a_i : 1 \leq i \leq e\} = \{8n + 6 + i : 1 \leq i \leq e\}$ forms an arithmetic sequence with common difference 1 and $B = \{b_j : 1 \leq j \leq e\} = \{v + j : 1 \leq j \leq e\}$ is a set of edge-labels. Define the set of edge-weights $C = \{\lambda(x) + \lambda(xy) + \lambda(y) : xy \in E(G)\}$=$\{a_{2i-1} + b_{e-i+1} : 1 \leq i \leq \frac{e+1}{2}\} \cup \left\{a_{2j} + b_{\frac{e-1}{2}-j+1} : 1 \leq j \leq \frac{e+1}{2} - 1\right\}$. It is easy to see that C constitutes an arithmetic sequence with $d = 1$ and $a = s + \frac{3v}{2} = 32n + 19$. Since all vertices receive the smallest labels, λ is a super $(a, 1)$-EAT labeling. ∎

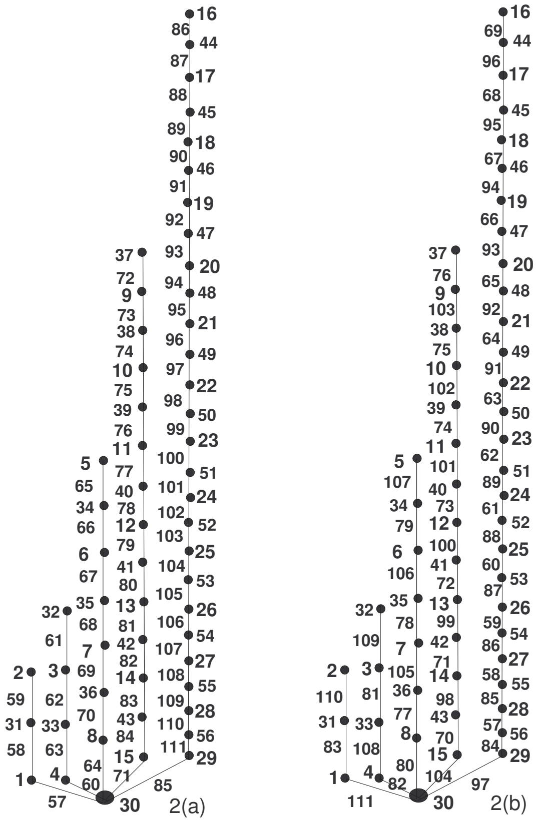

Figure 2. (a) Super (88,2)-EAT labeling of the subdivided star T(3,4,7,14,27).
(b) Super (115,1)-EAT labeling of the subdivided star T(3,4,7,14,27).

As a consequence of Theorem 2.2, to find a super $(a, 1)$-EAT labeling on $T(3, 4, 7, 14, 27)$, define $A = \{a_1, a_2, a_3 \ldots, a_e\} = \{31, 32, 33, \ldots, 85\}$ and $B = \{b_1, b_2, b_3, \ldots, b_e\} = \{57, 58, 59, \ldots, 111\}$. The set of edge-weights can be obtained by $C = \{a_{2i-1} + b_{e-i+1} : 1 \leq i \leq 28\} \cup \left\{a_{2j} + b_{\frac{e-1}{2}-j+1} : 1 \leq j \leq 27\right\} = \{31 + 111, 33 + 110, \ldots, 85+84\} \cup \{32+83, 34+82, \ldots, 84+57\} = \{142, 143, \ldots, 169\} \cup \{115, 116, \ldots, 141\} = \{115, 116, 117, \ldots, 169\}$. We note that the minimum edge-weight in the set C is 115. It also can be calculated by $a = s + \frac{3v}{2} = 31 + \frac{3(56)}{2} = 115$. Consequently, Figure 2(b) shows a super $(115, 1)$-EAT labeling of the subdivided star $T(3, 4, 7, 14, 27)$.

Theorem 2.3. *For any odd* $n \geq 3$, $G \cong T(n, n+1, 2n+1, 4n+2, 8n+3, 16n+5)$ *admits a super* $(a, 0)$-EAT *labeling with* $a = s + v + e$ *and a super* $(a', 2)$-EAT *labeling with* $a' = s + v + 1$, *where* $v = |V(G)|$ *and* $s = 16n + 10$.

Proof. Let us denote the vertices and edges of G as follows.

$$V(G) = \{c\} \cup \left\{x_i^{l_i} : 1 \le i \le 6, 1 \le l_i \le n_i\right\},$$
$$E(G) = \{cx_i^1 : 1 \le i \le 6\} \cup \left\{x_i^{l_i} x_i^{l_i+1} : 1 \le i \le 6, 1 \le l_i \le n_i - 1\right\}.$$

If $v = |V(G)|$ and $e = |E(G)|$, then $v = 32n + 13$, and $e = v - 1$.

Now, we define $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as follows: $\lambda(c) = 16n + 9$.

For $1 \le i \le 6$, $1 \le l_i \le n_i$ and l_i odd, we define:

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+1) - \frac{l_2-1}{2}, & \text{for } u = x_2^{l_2}, \\ (2n+2) - \frac{l_3-1}{2}, & \text{for } u = x_3^{l_3}, \\ (4n+3) - \frac{l_4-1}{2}, & \text{for } u = x_4^{l_4}, \\ (8n+5) - \frac{l_5-1}{2}, & \text{for } u = x_5^{l_5}, \\ (16n+8) - \frac{l_6-1}{2}, & \text{for } u = x_6^{l_6}, \end{cases}$$

and for l_i even we construct:

$$\lambda(u) = \begin{cases} (16n+9) + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (17n+10) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (18n+10) - \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (20n+11) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4}, \\ (24n+12) - \frac{l_5}{2}, & \text{for } u = x_5^{l_5}, \\ (32n+14) - \frac{l_6}{2}, & \text{for } u = x_6^{l_6}. \end{cases}$$

The set of all edge-sums generated by the above formulas is $\{\lambda(x) + \lambda(y): xy \in E(G)\}$=$\{16n + 9 + i : 1 \le i \le e\}$. It forms a sequence of consecutive integers starting from the minimum edge-sum $s = 16n + 10$. Thus, by Definition 1.1, λ is a $(16n + 10, 1)$-EAV labeling. As a consequence of Proposition 1.10, λ can be extended to a super $(a, 0)$-EAT labeling with magic constant $a = s + v + e = 80n + 35$. The set of edge-labels is $\{a - (16n + 9 + i) : 1 \le i \le e\}$. Similarly, by Proposition 1.10, λ can be extended to a super $(a', 2)$-EAT labeling with the minimum edge-weight $a' = s + v + 1 = 48n + 24$. The set of edge-labels can be obtained by $\{a' - (48n + 23 + i) : 1 \le i \le e\}$. ■

Theorem 2.4. *For any odd $n \ge 3$, $G \cong T(n, n+1, 2n+1, 4n+2, 8n+3, 16n+5, 32n+9)$ admits a super $(a, 0)$-EAT labeling with $a = s + v + e$ and a super $(a', 2)$-EAT labeling with $a' = s + v + 1$, where $v = |V(G)|$ and $s = 32n + 15$.*

Proof. Let us denote the vertices and edges of G as follows.

$$V(G) = \{c\} \cup \left\{x_i^{l_i} : 1 \le i \le 7, 1 \le l_i \le n_i\right\},$$
$$E(G) = \{cx_i^1 : 1 \le i \le 7\} \cup \left\{x_i^{l_i} x_i^{l_i+1} : 1 \le i \le 7, 1 \le l_i \le n_i - 1\right\}.$$

If $v = |V(G)|$ and $e = |E(G)|$ then $v = 64n + 22$, and $e = 64n + 21$.

Now, we define $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as follows: $\lambda(c) = 32n + 14$.

For $1 \le i \le 7$, $1 \le l_i \le n_i$ and l_i odd, we define:

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+1) - \frac{l_2-1}{2}, & \text{for } u = x_2^{l_2}, \\ (2n+2) - \frac{l_3-1}{2}, & \text{for } u = x_3^{l_3}, \\ (4n+3) - \frac{l_4-1}{2}, & \text{for } u = x_4^{l_4} \\ (8n+5) - \frac{l_5-1}{2}, & \text{for } u = x_5^{l_5}, \\ (16n+8) - \frac{l_6-1}{2}, & \text{for } u = x_6^{l_6}, \\ (32n+13) - \frac{l_7-1}{2}, & \text{for } u = x_7^{l_7}, \end{cases}$$

and for l_i even we construct:

$$\lambda(u) = \begin{cases} (32n+14) + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (33n+15) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (34n+15) - \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (36n+16) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4}, \\ (40n+17) - \frac{l_5}{2}, & \text{for } u = x_5^{l_5}, \\ (48n+19) - \frac{l_6}{2}, & \text{for } u = x_6^{l_6}, \\ (64n+23) - \frac{l_7}{2}, & \text{for } u = x_7^{l_7}. \end{cases}$$

The set of all edge-sums generated by the above formulas is $\{\lambda(x) + \lambda(y)\colon xy \in E(G)\}$=$\{32n + 14 + i : 1 \le i \le e\}$. It forms a sequence of consecutive integers starting from the minimum edge-sum $s = 32n + 15$. Thus, by Definition 1.1, λ is a $(32n + 15, 1)$-EAV labeling. As a consequence of Proposition 1.10, λ can be extended to a super $(a, 0)$-EAT labeling with magic constant $a = s + v + e = 160n + 58$. The set of edge-labels is $\{a - (16n + 9 + i) : 1 \le i \le e\}$. Similarly, by Proposition 1.10, λ can be extended to a super $(a', 2)$-EAT labeling with the minimum edge-weight $a' = s + v + 1 = 96n + 28$. The set of edge-labels can be obtained by $\{a' - (96n + 27 + i) : 1 \le i \le e\}$. ∎

Theorem 2.5. *For any even $n \ge 3$, $G \cong T(n, n+1, 2n+1, 4n+2, 8n+3, 16n+5, 32n+9)$ admits a super $(a, 1)$-EAT labeling with $a = s + \frac{3v}{2}$, where $v = |V(G)|$ and $s = 32n + 15$.*

Proof. Let us consider the set of vertices and edges of G defined as in Theorem 2.4. Now we define the vertex-labeling $\lambda : V(G) \to \{1, 2, \dots, v\}$ as in the same theorem. It follows that the set of edge-sums for all edges of G denoted by $A = \{a_i : 1 \le i \le e\} = \{32n + 14 + i : 1 \le i \le e\}$ forms an arithmetic sequence with common difference 1 and $B = \{b_j : 1 \le j \le e\} = \{v + j : 1 \le j \le e\}$ is a set of edge-labels. Define the set of edge-weights $C = \{\lambda(x) + \lambda(xy) + \lambda(y) : xy \in E(G)\}$=$\{a_{2i-1} + b_{e-i+1} : 1 \le i \le \frac{e+1}{2}\} \cup \left\{a_{2j} + b_{\frac{e-1}{2}-j+1} : 1 \le j \le \frac{e+1}{2} - 1\right\}$. It is easy to see that C constitutes an arithmetic sequence with $d = 1$ and

$a = s + \frac{3v}{2} = 128n + 48$. Since all vertices receive the smallest labels, λ is a super $(a, 1)$-EAT labeling. ■

Theorem 2.6. *For any $r \geq 5$ and odd $n \geq 3$, $G \cong T(n, n+1, 2n+1, 4n+2, n_5, \ldots, n_r)$ admits a super $(a, 0)$-EAT labeling with $a = s + v + e$ and a super $(a', 2)$-EAT labeling with $a' = s + v + 1$ where $v = |V(G)|$, $s = (4n+5) + \sum_{m=5}^{r}[2^{m-5}(4n+1)+1]$ and $n_m = 2^{m-4}(4n+1)+1$ for $5 \leq m \leq r$.*

Proof. Let us denote the vertices and edges of G as follows.

$V(G) = \{c\} \cup \left\{x_i^{l_i} : 1 \leq i \leq r, 1 \leq l_i \leq n_i\right\}$,

$E(G) = \{cx_i^1 : 1 \leq i \leq r\} \cup \left\{x_i^{l_i} x_i^{l_i+1} : 1 \leq i \leq r, 1 \leq l_i \leq n_i - 1\right\}$.

If $v = |V(G)|$ and $e = |E(G)|$, then $v = (8n+5) + \sum_{m=5}^{r}[2^{m-4}(4n+1)+1]$ and $e = v-1$. Throughout the labeling, suppose $\alpha = (4n+4) + \sum_{m=5}^{r}[2^{m-5}(4n+1)+1]$.

Define $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as follows: $\lambda(c) = \alpha$.

For $1 \leq i \leq 4$, $1 \leq l_i \leq n_i$ and l_i odd, we define:

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+1) - \frac{l_2-1}{2}, & \text{for } u = x_2^{l_2}, \\ (2n+2) - \frac{l_3-1}{2}, & \text{for } u = x_3^{l_3}, \\ (4n+3) - \frac{l_4-1}{2}, & \text{for } u = x_4^{l_4}, \end{cases}$$

and for l_i even, we construct:

$$\lambda(u) = \begin{cases} \alpha + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (\alpha+n+1) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (\alpha+2n+1) - \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (\alpha+4n+2) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4}. \end{cases}$$

For $5 \leq i \leq r$, $1 \leq l_i \leq n_i$ and l_i odd, we define:

$$\lambda(x_i^{l_i}) = (4n+3) + \sum_{m=5}^{i}[2^{m-5}(4n+1)+1] - \frac{l_i - 1}{2},$$

and for l_i even, we construct:

$$\lambda(x_i^{l_i}) = (\alpha+4n+2) + \sum_{m=5}^{i}[2^{m-5}(4n+1)] - \frac{l_i}{2}.$$

The set of all edge-sums generated by the above formulas is $\{\lambda(x)+\lambda(y) \colon xy \in E(G)\} = \{\alpha + i : 1 \leq i \leq i\}$. It forms a sequence of consecutive integers starting from the minimum edge-sum $s = \alpha + 1$. Thus, by Definition 1.1, λ is a $(\alpha+1, 1)$-EAV labeling. As a consequence of Proposition 1.10, λ can be extended to a super $(a, 0)$-EAT labeling with magic constant $a = s + v + e = 2v + (4n+4) +$

$\sum_{m=5}^{r}[2^{m-5}(4n+1)+1] = (20n+14)+\sum_{m=5}^{r}[2^{m-5}(20n+5)+3]$. The set of edge-labels is $\{a-(\alpha+i) : 1 \leq i \leq e\}$. Similarly, by Proposition 1.10, λ can be extended to a super $(a', 2)$-EAT labeling with the minimum edge-weight $a' = s + v + 1 = v + (4n+6) + \sum_{m=5}^{r}[2^{m-5}(4n+1)+1] = (12n+11)+\sum_{m=5}^{r}[2^{m-5}(12n+3)+2]$. The set of edge-labels can be obtained by $\{a' - (\alpha + i) : 1 \leq i \leq e\}$. ∎

Theorem 2.7. *For any $r \geq 5$ and odd $n \geq 3$, $G \cong T(n, n+1, 2n+1, 4n+2, n_5, \ldots, n_r)$ admits a super $(a,1)$-EAT total labeling with $a = s + \frac{3v}{2}$ if v is even, where $v = |V(G)|$, $s = (4n+5) + \sum_{m=5}^{r}[2^{m-5}(4n+1)+1]$ and $n_m = 2^{m-4}(4n+1)+1$ for $5 \leq m \leq r$.*

Proof. Let us consider the vertices and edges of G defined as in Theorem 2.6. Now, we define the labeling $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as in the same theorem. It follows that the set of edge-sums for all edges of G denoted by $A = \{a_i : 1 \leq i \leq e\} = \{\alpha + i : 1 \leq i \leq e\}$ forms an arithmetic sequence with common difference 1 and $B = \{b_j : 1 \leq j \leq e\} = \{v + j : 1 \leq j \leq e\}$ is a set of edge-labels, where $\alpha = (4n+4) + \sum_{m=5}^{r}[2^{m-5}(4n+1)+1]$. Define the set of edge-weights $C = \{\lambda(x) + \lambda(xy) + \lambda(y) : xy \in E(G)\}$=$\{a_{2i-1} + b_{e-i+1} : 1 \leq i \leq \frac{e+1}{2}\} \cup \left\{a_{2j} + b_{\frac{e-1}{2}-j+1} : 1 \leq j \leq \frac{e+1}{2} - 1\right\}$. It is easy to see that C constitutes an arithmetic sequence with $d = 1$ and $a = s + \frac{3v}{2} = 128n + 48 + \frac{1}{2}\sum_{m=5}^{r}[2^{m-2}(4n+1)+5]$. Since all vertices receive the smallest labels, λ is a super $(a,1)$-EAT labeling. ∎

3. Conclusion

In this paper, we have shown that a subclass of subdivided stars denoted by $T(n, n+1, 2n+1, 4n+2, n_5, \ldots, n_r)$ admits a super (a,d)-EAT labeling for $d \in \{0, 1, 2\}$, where $n \geq 3$ is odd, $n_m = 2^{m-4}(4n+1)+1$, $r \geq 5$ and $5 \leq m \leq r$. It is a generalized form of the three-path tree studied by Lu [16, 17] and Ngurah *et al.* [18]. The choice of $\{n_i : 2 \leq i \leq r\}$ in the present results is different from the results which are derived by Javaid *et al.* [9]. Salman *et al.* [19] proved the existence of a super $(a,0)$-EAT labeling on a particular subclass of the subdivided stars denoted by $T(n_1, n_2, n_3 \ldots, n_r)$, where $n_1 = n_2 = n_3 = \cdots = n_r = n$ and $n \in \{2, 3\}$. Moreover, the scheme of a super (a,d)-EAT labeling developed in this paper does not work on $T(n_1, n_2, n_3, n_4, n_5, n_6)$, when $n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = 4$. Thus, we propose the following open problem.

Open Problem 3.1. For the subdivided star $T(n_1, n_2, n_3, \ldots, n_r)$, where $n_1 = n_2 = n_3 = \cdots = n_r = n \geq 1$, determine if there is a super (a,d)-EAT labeling.

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