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# ON SUPER EDGE-ANTIMAGIC TOTAL LABELING OF SUBDIVIDED STARS ${ }^{1}$ 

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#### Abstract

In 1980, Enomoto et al. proposed the conjecture that every tree is a super ( $a, 0$ )-edge-antimagic total graph. In this paper, we give a partial support for the correctness of this conjecture by formulating some super $(a, d)$ -edge-antimagic total labelings on a subclass of subdivided stars denoted by $T\left(n, n+1,2 n+1,4 n+2, n_{5}, n_{6}, \ldots, n_{r}\right)$ for different values of the edgeantimagic labeling parameter $d$, where $n \geq 3$ is odd, $n_{m}=2^{m-4}(4 n+1)+1$, $r \geq 5$ and $5 \leq m \leq r$.


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## 1. Introduction

All graphs in this paper are finite, simple and undirected. For a graph $G, V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively. A $(v, e)$-graph $G$ is a graph such that $|V(G)|=v$ and $|E(G)|=e$. Moreover, the theoretic ideas of graphs can be seen in [22]. A labeling (or valuation) of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). In this paper, the domain will be the set of all vertices and edges and such a labeling is called a total labeling. Some labelings use the vertex set only or the edge set only and we shall call them vertex-labelings or edge-labelings, respectively.

[^0]There are many types of graph labelings, for example harmonius, cordial, graceful and antimagic. The most complete recent survey of graph labelings can be found in [6]. In this paper, we focus on an antimagic total labeling. More details on an antimagic total labeling can be found in [4]. The subject of edge-magic total labeling of graphs has its origin in the works of Kotzig and Rosa [13, 14] on what they called magic valuations of graphs.

Definition 1.1. An $(s, d)$-edge-antimagic vertex $((s, d)$-EAV) labeling of a graph $G$ is a bijective function $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ such that the set of edge-sums of all edges in $G,\{w(x y)=\lambda(x)+\lambda(y): x y \in E(G)\}$, forms an arithmetic progression $\{s, s+d, s+2 d, \ldots, s+(e-1) d\}$, where $s>0$ and $d \geq 0$ are two fixed integers.

Simanjuntak et al. [21] proved that the odd cycle $C_{2 n+1}$, the odd path $P_{2 n+1}$ and the even path $P_{2 n}$ have a $(n+2,1)$-EAV labeling, where $n \geq 1$. They also proved that the odd path $P_{2 n+1}$ has a $(n+3,1)$-EAV labeling and the path $P_{n}$ admits a $(3,2)$-EAV labeling for $n \geq 1$. Moreover, Bača, Miller, Simanjuntak, Lin and Bertault [2, 21] proved the following results.

- If a non-tree connected graph $G$ has an $(a, d)$-EAV labeling then $d=1$.
- The cycle $C_{n}$ has no $(a, d)$-EAV labeling for $d>1$ and $n \geq 3$.
- The complete graph $K_{n}$ has no $(a, d)$-EAV labeling, where $n>3$.
- The symmetric complete bipartite graph $K_{n, n}$ has no ( $a, d$ )-EAV labeling, where $n>1$.

Definition 1.2. An $(a, d)$-edge-antimagic total $((a, d)$-EAT) labeling of a graph $G$ is a bijective function $\lambda: V(G) \cup E(G) \rightarrow\{1,2, \ldots, v+e\}$ such that the set of edge-weights of all edges in $G,\{w(x y)=\lambda(x)+\lambda(x y)+\lambda(y): x y \in E(G)\}$, forms an arithmetic progression $\{a, a+d, a+2 d, \ldots, a+(e-1) d\}$, where $a>0$ and $d \geq 0$ are two fixed integers. If such a labeling exists, then $G$ is said to be an $(a, d)$-EAT graph.
Definition 1.3. An $(a, d)$-EAT labeling $\lambda$ is called a super $(a, d)$-edge-antimagic total (super $(a, d)$-EAT) labeling of $G$ if $\lambda(V(G))=\{1,2, \ldots, v\}$. Thus, a super $(a, d)$-EAT graph is a graph that admits a super $(a, d)$-EAT labeling.

In the above definition, if $d=0$, then a super ( $a, 0$ )-EAT labeling is called a super edge-magic total (SEMT) labeling and $a$ is called a magic constant. For $d \neq 0$, $a$ is called minimum edge-weight. The definition of an $(a, d)$-EAT labeling was introduced by Simanjuntak, Bertault and Miller in [21] as a natural extension of an edge-magic total labeling defined by Kotzig and Rosa. A super $(a, d)$ EAT labeling is a natural extension of the notion of a super $(a, 0)$-EAT labeling defined by Enomoto, Lladó, Nakamigawa and Ringel in [5]. They also proposed the conjecture that every tree is a super $(a, 0)$-EAT graph. In the favour of
this conjecture, many authors have derived different results on a super $(a, d)$ EAT labeling for many particular classes of trees, for example path-like trees [3], banana trees [7], $w$-trees [11], extended $w$-trees [10, 12], subdivided stars $[8,9,18,19,16,17]$, subdivided $w$-trees [8] and caterpillars [20]. Lee and Shah [15] verified this conjecture by a computer search for trees with at most 17 vertices. However, this conjecture is still open.

Definition 1.4. For $n_{i} \geq 1, r \geq 2$ and $1 \leq i \leq r$, let $T\left(n_{1}, n_{2}, \ldots, n_{r}\right)$ be a subdivided star obtained by inserting $n_{i}-1$ vertices to each of the $i$-th edge of the star $K_{1, r}$. Thus, the subdivided star $T \underbrace{(1,1, \ldots, 1)}_{r-\text { times }}$ is the star $K_{1, r}$.

A star is a particular type of trees and many authors have investigated antimagicness for subdivided stars under certain conditions. Lu $[16,17]$ called the subdivided star $T(m, n, k)$ a three-path tree and proved that it is a super ( $a, 0$ )-EAT if $n, m$ are odd and $k=n+1$ or $k=n+2$. Ngurah et al. [18] proved that $T(m, n, k)$ is also a super ( $a, 0$ )-EAT graph if $n, m$ are odd and $k=n+3$ or $k=n+4$. Salman et al. [19] proved the existence of a super ( $a, 0$ )-EAT labeling on a particular subclass of the subdivided stars denoted by $S_{r}^{1}$ and $S_{r}^{2}$, where $S_{r}^{1} \cong T \underbrace{(2,2, \ldots, 2)}_{r-\text { times }}$ and $S_{r}^{2} \cong T \underbrace{(3,3, \ldots, 3)}_{r-\text { times }}$. Javaid et al. [8] investigated some results related to a super ( $a, 0$ )-EAT labeling on the subdivision of the star $K_{1,4}$ and the $w$-tree $W T(n, k)$. Javaid et al. [9] proved that a particular subclass of the subdivided stars in its generalized form denoted by $T\left(n, n, n+2, n+2, n_{5}, \ldots, n_{r}\right)$ admits a super $(a, d)$-EAT labeling for different values of $d$. Some of the results are as follows.

Theorem 1.5 [9]. For any odd $n \geq 3, T(n, n, n+2, n+2,2 n+3)$ admits a super ( $a, d$ )-EAT labeling for $d \in\{0,2\}$.

Theorem 1.6 [9]. For any odd $n \geq 3, T(n, n, n+2, n+2,2 n+3)$ admits a super ( $a, 1$ )-EAT labeling.

Theorem 1.7 [9]. For any odd $n \geq 3, T(n, n, n+2, n+2,2 n+3,4 n+5)$ admits a super ( $a, d$ )-EAT labeling for $d \in\{0,2\}$.

Theorem 1.8 [9]. For any $r \geq 5$ and odd $n \geq 3, T\left(n, n, n+2, n+2, n_{5}, \ldots, n_{r}\right)$ admits a super ( $a, d$ )-EAT labeling, where $n_{m}=1+(n+1) 2^{m-4}, 5 \leq m \leq r$ and $d \in\{0,2\}$.

Theorem 1.9 [9]. For any $r \geq 5$ and odd $n \geq 3, T\left(n, n, n+2, n+2, n_{5}, \ldots, n_{r}\right)$ admits a super ( $a, 1$ )-EAT labeling if $\left|T\left(n, n, n+2, n+2, n_{5}, \ldots, n_{r}\right)\right|$ is even, where $n_{m}=1+(n+1) 2^{m-4}$ for $5 \leq m \leq r$.

In this paper, we construct another generalized subclass of subdivided stars denoted by $T\left(n, n+1,2 n+1,4 n+2, n_{5}, n_{6} \ldots, n_{r}\right)$, where $n_{m}=2^{m-4}(4 n+1)+1$, $5 \leq m \leq r$ and $r \geq 5$. Moreover, it is proved that this subclass also admits some super $(a, d)$-EAT labelings for different values of $d$. Let us consider the following proposition which we will use in the main results.
Proposition 1.10 [2]. If a $(v, e)$-graph $G$ has an $(s, d)$-EAV labeling, then
(i) $G$ has a super $(s+v+1, d+1)$-EAT labeling,
(ii) $G$ has a super $(s+v+e, d-1)$-EAT labeling.

### 1.1. Bounds for the magic constant $a$

Ngurah et al. [18] found lower and upper bounds of the magic constant $a$ for a particular family of subdivided stars which are stated as follows.
Lemma 1.11. If $T(m, n, k)$ is a super ( $a, 0)$-EAT graph, then $\frac{1}{2 l}\left(5 l^{2}+3 l+6\right) \leq$ $a \leq \frac{1}{2 l}\left(5 l^{2}+11 l-6\right)$, where $l=m+n+k$.
The lower and upper bounds of the magic constant a proved by Salman et al. [19] are as follows.
Lemma 1.12. If $T \underbrace{(n, n, \ldots, n)}_{n-\text { times }}$ is a super ( $a, 0$ )-EAT graph, then $\frac{1}{2 l}\left(5 l^{2}+(9-\right.$ $\left.2 n) l+n^{2}-n\right) \leq a \leq \frac{1}{2 l}\left(5 l^{2}+(2 n+5) l+n-n^{2}\right)$, where $l=n^{2}$.
Now we find lower and upper bounds of the magic constant $a$ for the most extended family of the subdivided stars denoted by $T\left(n_{1}, n_{2}, n_{3}, \ldots, n_{r}\right)$ with any $n_{i} \geq 1$ for $1 \leq i \leq r$.
Lemma 1.13. If $T\left(n_{1}, n_{2}, n_{3}, \ldots, n_{r}\right)$ is a super ( $a, 0$ )-EAT graph, then $\frac{1}{2 l}\left(5 l^{2}+\right.$ $\left.(9-2 r) l+\left(r^{2}-r\right)\right) \leq a \leq \frac{1}{2 l}\left(5 l^{2}+(5+2 r) l-\left(r^{2}-r\right)\right)$, where $l=\sum_{i=1}^{r^{2 l}} n_{i}$.
Proof. Suppose that $T\left(n_{1}, n_{2}, n_{3}, \ldots, n_{r}\right)$ admits a super ( $a, 0$ )-EAT labeling with magic constant $a$ and $l=\sum_{i=1}^{r} n_{i}$. Then " $a$ " cannot be smaller than the sum obtained by assigning the smallest label 1 to the vertex of the degree $r$, the labels from 2 to $l+1-r$ to the vertices of degree 2 and the labels from $l+2-r$ to $l+1$ to the next $r$ vertices of degree 1 as

$$
l a \geq r+2 \sum_{i=2}^{l-r+1} i+\sum_{i=l-r+2}^{l+1} i+\sum_{i=l+2}^{2 l+1} i
$$

Consider $\sum_{i=2}^{l-r+1} i=\frac{l-r}{2}(l-r+3), \sum_{i=l-r+2}^{l+1} i=\frac{1}{2}\left(2 l r-r^{2}+3 r\right)$ and $\sum_{i=l+2}^{2 l+1} i=$ $\frac{l}{2}(3 l+3)$. Consequently, we have $l a \geq \frac{1}{2}\left(5 l^{2}+r^{2}-2 l r+9 l-r\right)$ or

$$
\begin{equation*}
a \geq \frac{1}{2 l}\left(5 l^{2}+r^{2}-2 l r+9 l-r\right) \tag{1}
\end{equation*}
$$

Similarly, the upper bound of "la" is obtained by assigning the largest label $l+1$ to the vertex of the degree $r$, the labels from $r+1$ to $l$ to the vertices of degree 2 and the labels from 1 to $r$ to the next $r$ vertices of degree 1 as

$$
l a \leq r(l+1)+2 \sum_{i=r+1}^{l} i+\sum_{i=1}^{r} i+\sum_{i=l+2}^{2 l+1} i .
$$

Consider $\sum_{i=r+1}^{l} i=\frac{3 l}{2}(l+1), \sum_{i=1}^{r} i=\frac{r}{2}(r+1)$ and $\sum_{i=l+2}^{2 l+1} i=\frac{l-r}{2}(l+r+1)$. Consequently, we have $l a \leq \frac{1}{2 l}\left(5 l^{2}-r^{2}+2 l r+5 l+r\right)$ or

$$
\begin{equation*}
a \leq \frac{1}{2 l}\left(5 l^{2}-r^{2}+2 l r+5 l+r\right) \tag{2}
\end{equation*}
$$

Combining (1) and (2), we get

$$
\frac{1}{2 l}\left(5 l^{2}+(9-2 r) l+\left(r^{2}-r\right)\right) \leq a \leq \frac{1}{2 l}\left(5 l^{2}+(5+2 r) l-\left(r^{2}-r\right)\right)
$$

### 1.2. Strategy of construction for labeling schemes

Before presenting the main results, let us consider the overall strategy which is applied to find the results related to super $(a, d)$-EAT labelings on the particular subclasses of the subdivided stars for different values of the labeling parameter $d$. It is important to know about three terms edge-label, edge-sum and edgeweight. Let $x y$ be an edge with end vertices $x$ and $y$. Suppose that the assigned labels to the edge is $\lambda(x y)$ and to the vertices are $\lambda(x)$ and $\lambda(y)$. Thus, $\lambda(x y)$, $\lambda(x)+\lambda(y)$ and $\lambda(x)+\lambda(x y)+\lambda(y)$ are called edge-label, edge-sum and edge-weight, respectively.

In order to construct a super $(a, d)$-EAT labeling for $d=0,2$ on the graph $G$, the following steps have been performed:

### 1.2.1. Working steps for super ( $a, 0$ )-EAT labeling

- Define a bijection $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ in such a way that the set of edge-sums $\{\lambda(x)+\lambda(y): x y \in E(G)\}$ forms a sequence of consecutive integers with minimum edge-sum, say, $s$.
- It follows that the graph $G$ admits an $(s, 1)$-EAV labeling.
- After getting an $(s, 1)$-EAV labeling on the graph $G$, the goal is to extend it to a super ( $a, 0$ )-EAT labeling with the help of the magic constant $a$.
- The magic constant can be calculated as $a=s+v+e$.
- Using set of edge-sums and the value of magic constant, the set of edge-labels can be obtained as $\{a-(\lambda(x)+\lambda(y)): x y \in E(G)\}$.

Consequently, the graph $G$ admits a super ( $a, 0$ )-EAT labeling.

### 1.2.2. Working steps for super ( $\left.a^{\prime}, 2\right)$-EAT labeling

- Define a bijection $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ in such a way that the set of edge-sums $\{\lambda(x)+\lambda(y): x y \in E(G)\}$ forms a sequence of consecutive integers with minimum edge-sum, say, $s$.
- It follows that the graph $G$ admits an $(s, 1)$-EAV labeling.
- After getting an $(s, 1)$-EAV labeling on the graph $G$, the goal is to extend it to a super $\left(a^{\prime}, 2\right)$-EAT labeling with the help of the minimum edge-weight $a^{\prime}$.
- The minimum edge-weight is calculated as $a^{\prime}=s+v+1$.
- Define the set of edge-weights as $\left\{a^{\prime}-2+2 i: 1 \leq i \leq e\right\}$.
- Define the set of edge-sums as $H=\left\{h_{i}: 1 \leq i \leq e\right\}$.
- Using $a^{\prime}$ and the set $H$, the set of edge-labels can be obtained as $\left\{\left(a^{\prime}-2+\right.\right.$ $\left.2 i)-h_{i}: 1 \leq i \leq e\right\}$.
Consequently, the graph $G$ admits a super ( $a^{\prime}, 2$ )-EAT labeling.
In this paper, a super $(a, 1)$-EAT labeling is formulated if the order of the graph $G$ is even. Thus, for the construction of a super ( $a, 1$ )-EAT labeling scheme, we proceed as follows.


### 1.2.3. Working steps for a super ( $a, 1$ )-EAT labeling

- Define a bijection $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ in such a way that the set of edge-sums $\{\lambda(x)+\lambda(y): x y \in E(G)\}$ forms a sequence of consecutive integers with minimum edge-sum, say, $s$.
- Define the set of edge-sums as $A=\left\{a_{i}: 1 \leq i \leq e\right\}$.
- The set of edges-labels is $B=\left\{b_{j}: 1 \leq j \leq e\right\}=\left\{v_{j}+1: 1 \leq j \leq e\right\}$.
- The set of edge-weights can be obtained as $C=\{\lambda(x)+\lambda(x y)+\lambda(y): x y \in E(G)\}$ $=\left\{a_{2 i-1}+b_{e-i+1}: 1 \leq i \leq \frac{e+1}{2}\right\} \cup\left\{a_{2 j}+b_{\frac{e-1}{2}-j+1}: 1 \leq j \leq \frac{e+1}{2}-1\right\}$.
- Thus, the minimum edge-weight is $a=s+\frac{3 v}{2}$.

Consequently, the graph $G$ admits a super $(a, 1)$-EAT labeling.

## 2. Main Results

In this section, we present the main results related to a super $(a, d)$-EAT labeling on a subclass of the subdivided stars for different values of the labeling parameter $d$.

Theorem 2.1. For any odd $n \geq 3, G \cong T(n, n+1,2 n+1,4 n+2,8 n+3)$ admits a super $(a, 0)$-EAT labeling with $a=s+v+e$ and a super ( $\left.a^{\prime}, 2\right)$-EAT labeling with $a^{\prime}=s+v+1$, where $v=|V(G)|$ and $s=8 n+7$.

Proof. Let us denote the vertices and edges of $G$ as follows.

$$
\begin{aligned}
& V(G)=\{c\} \cup\left\{x_{i}^{l_{i}}: 1 \leq i \leq 5,1 \leq l_{i} \leq n_{i}\right\} \\
& E(G)=\left\{c x_{i}^{1}: 1 \leq i \leq 5\right\} \cup\left\{x_{i}^{l_{i}} x_{i}^{l_{i}+1}: 1 \leq i \leq 5,1 \leq l_{i} \leq n_{i}-1\right\}
\end{aligned}
$$

If $v=|V(G)|$ and $e=|E(G)|$, then $v=16 n+8$ and $e=v-1$.
Now, we define $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ as follows: $\lambda(c)=8 n+6$.
For $1 \leq i \leq 5,1 \leq l_{i} \leq n_{i}$ and $l_{i}$ odd, we define:

$$
\lambda(u)= \begin{cases}\frac{l_{1}+1}{2}, & \text { for } u=x_{1}^{l_{1}}, \\ (n+1)-\frac{l_{2}-1}{2}, & \text { for } u=x_{2}^{l_{2}}, \\ (2 n+2)-\frac{l_{3}-1}{2}, & \text { for } u=x_{3}^{l_{3}} \\ (4 n+3)-\frac{l_{4}-1}{2}, & \text { for } u=x_{4}^{l_{4}}, \\ (8 n+5)-\frac{l_{5}-1}{2}, & \text { for } u=x_{5}^{l_{5}}\end{cases}
$$

and for $l_{i}$ even, we construct:

$$
\lambda(u)= \begin{cases}(8 n+6)+\frac{l_{1}}{2}, & \text { for } u=x_{1}^{l_{1}} \\ (9 n+7)-\frac{l_{2}}{2}, & \text { for } u=x_{2}^{l_{2}} \\ (10 n+7)-\frac{l_{3}}{2}, & \text { for } u=x_{3}^{l_{3}} \\ (12 n+8)-\frac{l_{4}}{2}, & \text { for } u=x_{4}^{l_{4}} \\ (16 n+9)-\frac{l_{5}}{2}, & \text { for } u=x_{5}^{l_{5}}\end{cases}
$$

The set of all edge-sums generated by the above formulas is $\{\lambda(x)+\lambda(y): x y \in$ $E(G)\}=\{8 n+6+i: 1 \leq i \leq e\}$. It forms a sequence of consecutive integers starting from the minimum edge-sum $s=8 n+7$. Thus, by Definition $1.1, \lambda$ is a $(8 n+7,1)$-EAV labeling. As a consequence of Proposition 1.10, $\lambda$ can be extended to a super $(a, 0)$-EAT labeling with magic constant $a=s+v+e=40 n+22$. The set of edge-labels is $\{a-(8 n+6+i): 1 \leq i \leq e\}$. Similarly, by Proposition $1.10, \lambda$ can be extended to a super $\left(a^{\prime}, 2\right)$-EAT labeling with the minimum edgeweight $a^{\prime}=s+v+1=24 n+16$. The set of edge-labels can be obtained by $\left\{a^{\prime}-(8 n+6+i): 1 \leq i \leq e\right\}$.

As a consequence of the labeling which is formulated in Theorem 2.1, Figure 1 (a) gives the set of edge-sums $\{31,32,33, \ldots, 85\}$ as a sequence of consecutive integers starting from $s=31$. Thus, the subdivided star $T(3,4,7,14,27)$ admits a $(31,1)$-EAV labeling. The magic constant can be obtained by $c=v+e+s=$ $56+55+31=142$. The set of edge-labels is $\{(142-31),(142-32),(142-$ $33), \ldots,(142-85)\}=\{111,110,109, \ldots, 57\}$. Thus, Figure 2(a) presents a super $(142,0)$-EAT labeling of the subdivided star $T(3,4,7,14,27)$.

Now, we calculate the minimum edge-weight $a^{\prime}=s+v+1=31+56+$ $1=88$ and the set of edge-labels $\{(88-31),(90-32),(92-33), \ldots,(196-$


Figure 1. (a) (31,1)-EAV labeling of the subdivided star $\mathrm{T}(3,4,7,14,27)$.
(b) Super ( 142,0 )-EAT labeling of the subdivided star $\mathrm{T}(3,4,7,14,27)$.
$85)\}=\{57,58,59, \ldots, 111\}$. Consequently, Figure 2(a) gives a super ( 88,2 )-EAT labeling of the subdivided star $T(3,4,7,14,27)$.

Theorem 2.2. For any odd $n \geq 3, G \cong T(n, n+1,2 n+1,4 n+2,8 n+3)$ admits a super ( $a, 1$-EAT labeling with $a=s+\frac{3 v}{2}$, where $v=|V(G)|$ and $s=8 n+7$.

Proof. Let us consider the set of vertices and edges of the graph $G$ defined as in the proof of Theorem 2.1. Now we define the vertex-labeling $\lambda: V(G) \rightarrow$ $\{1,2, \ldots, v\}$ as in the same theorem. It follows that the set of edge-sums for all edges of $G$ denoted by $A=\left\{a_{i}: 1 \leq i \leq e\right\}=\{8 n+6+i: 1 \leq i \leq e\}$ forms an arithmetic sequence with common difference 1 and $B=\left\{b_{j}: 1 \leq\right.$ $j \leq e\}=\{v+j: 1 \leq j \leq e\}$ is a set of edge-labels. Define the set of edgeweights $C=\{\lambda(x)+\lambda(x y)+\lambda(y): x y \in E(G)\}=\left\{a_{2 i-1}+b_{e-i+1}: 1 \leq i \leq\right.$ $\left.\frac{e+1}{2}\right\} \cup\left\{a_{2 j}+b_{\frac{e-1}{2}-j+1}: 1 \leq j \leq \frac{e+1}{2}-1\right\}$. It is easy to see that $C$ constitutes an arithmetic sequence with $d=1$ and $a=s+\frac{3 v}{2}=32 n+19$. Since all vertices receive the smallest labels, $\lambda$ is a super ( $a, 1$ )-EAT labeling.


Figure 2. (a) Super (88,2)-EAT labeling of the subdivided star T(3,4,7,14,27).
(b) Super ( 115,1 )-EAT labeling of the subdivided star $\mathrm{T}(3,4,7,14,27)$.

As a consequence of Theorem 2.2, to find a super ( $a, 1$ )-EAT labeling on $T(3,4,7$, 14,27 ), define $A=\left\{a_{1}, a_{2}, a_{3} \ldots, a_{e}\right\}=\{31,32,33, \ldots, 85\}$ and $B=\left\{b_{1}, b_{2}, b_{3}, \ldots\right.$, $\left.b_{e}\right\}=\{57,58,59, \ldots, 111\}$. The set of edge-weights can be obtained by $C=$ $\left\{a_{2 i-1}+b_{e-i+1}: 1 \leq i \leq 28\right\} \cup\left\{a_{2 j}+b_{\frac{e-1}{2}-j+1}: 1 \leq j \leq 27\right\}=\{31+111,33+$ $110, \ldots, 85+84\} \cup\{32+83,34+82, \ldots, 84+57\}=\{142,143, \ldots, 169\} \cup\{115,116$, $\ldots, 141\}=\{115,116,117, \ldots, 169\}$. We note that the minimum edge-weight in the set $C$ is 115. It also can be calculated by $a=s+\frac{3 v}{2}=31+\frac{3(56)}{2}=115$. Consequently, Figure 2(b) shows a super (115, 1)-EAT labeling of the subdivided star $T(3,4,7,14,27)$.

Theorem 2.3. For any odd $n \geq 3, G \cong T(n, n+1,2 n+1,4 n+2,8 n+3,16 n+5)$ admits a super ( $a, 0$ )-EAT labeling with $a=s+v+e$ and a super ( $a^{\prime}, 2$ )-EAT labeling with $a^{\prime}=s+v+1$, where $v=|V(G)|$ and $s=16 n+10$.

Proof. Let us denote the vertices and edges of $G$ as follows.

$$
\begin{aligned}
V(G) & =\{c\} \cup\left\{x_{i}^{l_{i}}: 1 \leq i \leq 6,1 \leq l_{i} \leq n_{i}\right\} \\
E(G) & =\left\{c x_{i}^{1}: 1 \leq i \leq 6\right\} \cup\left\{x_{i}^{l_{i}} x_{i}^{l_{i}+1}: 1 \leq i \leq 6,1 \leq l_{i} \leq n_{i}-1\right\} .
\end{aligned}
$$

If $v=|V(G)|$ and $e=|E(G)|$, then $v=32 n+13$, and $e=v-1$.
Now, we define $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ as follows: $\lambda(c)=16 n+9$.
For $1 \leq i \leq 6,1 \leq l_{i} \leq n_{i}$ and $l_{i}$ odd, we define:

$$
\lambda(u)= \begin{cases}\frac{l_{1}+1}{2}, & \text { for } u=x_{1}^{l_{1}}, \\ (n+1)-\frac{l_{2}-1}{2}, & \text { for } u=x_{2}^{l_{2}}, \\ (2 n+2)-\frac{l_{3}-1}{2}, & \text { for } u=x_{3}^{l_{3}}, \\ (4 n+3)-\frac{l_{4}-1}{2}, & \text { for } u=x_{4}^{l_{4}}, \\ (8 n+5)-\frac{l_{5}-1}{2}, & \text { for } u=x_{5}^{l_{5}}, \\ (16 n+8)-\frac{l_{6}-1}{2}, & \text { for } u=x_{6}^{l_{6}},\end{cases}
$$

and for $l_{i}$ even we construct:

$$
\lambda(u)= \begin{cases}(16 n+9)+\frac{l_{1}}{2}, & \text { for } u=x_{1}^{l_{1}}, \\ (17 n+10)-\frac{l_{2}}{2}, & \text { for } u=x_{2}^{l_{2}}, \\ (18 n+10)-\frac{l_{3}}{2}, & \text { for } u=x_{3}^{l_{3}}, \\ (20 n+11)-\frac{l_{4}}{2}, & \text { for } u=x_{4}^{l_{4}}, \\ (24 n+12)-\frac{l_{5}}{2}, & \text { for } u=x_{5}^{l_{5}}, \\ (32 n+14)-\frac{l_{6}}{2}, & \text { for } u=x_{6}^{l_{6}} .\end{cases}
$$

The set of all edge-sums generated by the above formulas is $\{\lambda(x)+\lambda(y): x y \in$ $E(G)\}=\{16 n+9+i: 1 \leq i \leq e\}$. It forms a sequence of consecutive integers starting from the minimum edge-sum $s=16 n+10$. Thus, by Definition 1.1, $\lambda$ is a $(16 n+10,1)$-EAV labeling. As a consequence of Proposition $1.10, \lambda$ can be extended to a super ( $a, 0$ )-EAT labeling with magic constant $a=s+v+e=$ $80 n+35$. The set of edge-labels is $\{a-(16 n+9+i): 1 \leq i \leq e\}$. Similarly, by Proposition 1.10, $\lambda$ can be extended to a super ( $a^{\prime}, 2$ )-EAT labeling with the minimum edge-weight $a^{\prime}=s+v+1=48 n+24$. The set of edge-labels can be obtained by $\left\{a^{\prime}-(48 n+23+i): 1 \leq i \leq e\right\}$.

Theorem 2.4. For any odd $n \geq 3, G \cong T(n, n+1,2 n+1,4 n+2,8 n+3,16 n+$ $5,32 n+9)$ admits a super ( $a, 0$ )-EAT labeling with $a=s+v+e$ and a super $\left(a^{\prime}, 2\right)$-EAT labeling with $a^{\prime}=s+v+1$, where $v=|V(G)|$ and $s=32 n+15$.

Proof. Let us denote the vertices and edges of $G$ as follows.

$$
\begin{aligned}
& V(G)=\{c\} \cup\left\{x_{i}^{l_{i}}: 1 \leq i \leq 7,1 \leq l_{i} \leq n_{i}\right\} \\
& E(G)=\left\{c x_{i}^{1}: 1 \leq i \leq 7\right\} \cup\left\{x_{i}^{l_{i}} x_{i}^{l_{i}+1}: 1 \leq i \leq 7,1 \leq l_{i} \leq n_{i}-1\right\} .
\end{aligned}
$$

If $v=|V(G)|$ and $e=|E(G)|$ then $v=64 n+22$, and $e=64 n+21$.
Now, we define $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ as follows: $\lambda(c)=32 n+14$.

For $1 \leq i \leq 7,1 \leq l_{i} \leq n_{i}$ and $l_{i}$ odd, we define:

$$
\lambda(u)= \begin{cases}\frac{l_{1}+1}{2}, & \text { for } u=x_{1}^{l_{1}}, \\ (n+1)-\frac{l_{2}-1}{2}, & \text { for } u=x_{2}^{l_{2}}, \\ (2 n+2)-\frac{l_{3}-1}{2}, & \text { for } u=x_{3}^{l_{3}}, \\ (4 n+3)-\frac{l_{4}-1}{2}, & \text { for } u=x_{4}^{l_{4}} \\ (8 n+5)-\frac{l_{5}-1}{2}, & \text { for } u=x_{5}^{l_{5}} \\ (16 n+8)-\frac{l_{6}-1}{2}, & \text { for } u=x_{6}^{l_{6}}, \\ (32 n+13)-\frac{l_{7}-1}{2}, & \text { for } u=x_{7}^{l_{7}}\end{cases}
$$

and for $l_{i}$ even we construct:

$$
\lambda(u)= \begin{cases}(32 n+14)+\frac{l_{1}}{2}, & \text { for } u=x_{1}^{l_{1}}, \\ (33 n+15)-\frac{l_{2}}{2}, & \text { for } u=x_{2}^{l_{2}}, \\ (34 n+15)-\frac{l_{3}}{2}, & \text { for } u=x_{3}^{l_{3}}, \\ (36 n+16)-\frac{l_{4}}{2}, & \text { for } u=x_{4}^{l_{4}}, \\ (40 n+17)-\frac{l_{5}}{2}, & \text { for } u=x_{5}^{l_{5}}, \\ (48 n+19)-\frac{l_{6}}{2}, & \text { for } u=x_{6}^{l_{6}}, \\ (64 n+23)-\frac{l_{7}}{2}, & \text { for } u=x_{7}^{l_{7}},\end{cases}
$$

The set of all edge-sums generated by the above formulas is $\{\lambda(x)+\lambda(y): x y \in$ $E(G)\}=\{32 n+14+i: 1 \leq i \leq e\}$. It forms a sequence of consecutive integers starting from the minimum edge-sum $s=32 n+15$. Thus, by Definition 1.1, $\lambda$ is a $(32 n+15,1)$-EAV labeling. As a consequence of Proposition $1.10, \lambda$ can be extended to a super ( $a, 0$ )-EAT labeling with magic constant $a=s+v+e=$ $160 n+58$. The set of edge-labels is $\{a-(16 n+9+i): 1 \leq i \leq e\}$. Similarly, by Proposition 1.10, $\lambda$ can be extended to a super ( $a^{\prime}, 2$ )-EAT labeling with the minimum edge-weight $a^{\prime}=s+v+1=96 n+28$. The set of edge-labels can be obtained by $\left\{a^{\prime}-(96 n+27+i): 1 \leq i \leq e\right\}$.

Theorem 2.5. For any even $n \geq 3, G \cong T(n, n+1,2 n+1,4 n+2,8 n+3,16 n+$ $5,32 n+9)$ admits a super ( $a, 1$ )-EAT labeling with $a=s+\frac{3 v}{2}$, where $v=|V(G)|$ and $s=32 n+15$.

Proof. Let us consider the set of vertices and edges of $G$ defined as in Theorem 2.4. Now we define the vertex-labeling $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ as in the same theorem. It follows that the set of edge-sums for all edges of $G$ denoted by $A=\left\{a_{i}: 1 \leq i \leq e\right\}=\{32 n+14+i: 1 \leq i \leq e\}$ forms an arithmetic sequence with common difference 1 and $B=\left\{b_{j}: 1 \leq j \leq e\right\}=\{v+j: 1 \leq j \leq e\}$ is a set of edge-labels. Define the set of edge-weights $C=\{\lambda(x)+\lambda(x y)+\lambda(y): x y \in$ $E(G)\}=\left\{a_{2 i-1}+b_{e-i+1}: 1 \leq i \leq \frac{e+1}{2}\right\} \cup\left\{a_{2 j}+b_{\frac{e-1}{2}-j+1}: 1 \leq j \leq \frac{e+1}{2}-1\right\}$. It is easy to see that $C$ constitutes an arithmetic sequence with $d=1$ and
$a=s+\frac{3 v}{2}=128 n+48$. Since all vertices receive the smallest labels, $\lambda$ is a super ( $a, 1$ )-EAT labeling.

Theorem 2.6. For any $r \geq 5$ and odd $n \geq 3, G \cong T(n, n+1,2 n+1,4 n+$ $2, n_{5}, \ldots, n_{r}$ ) admits a super ( $a, 0$ )-EAT labeling with $a=s+v+e$ and a super $\left(a^{\prime}, 2\right)$-EAT labeling with $a^{\prime}=s+v+1$ where $v=|V(G)|, s=(4 n+5)+$ $\sum_{m=5}^{r}\left[2^{m-5}(4 n+1)+1\right]$ and $n_{m}=2^{m-4}(4 n+1)+1$ for $5 \leq m \leq r$.

Proof. Let us denote the vertices and edges of $G$ as follows.

$$
\begin{aligned}
& V(G)=\{c\} \cup\left\{x_{i}^{l_{i}}: 1 \leq i \leq r, 1 \leq l_{i} \leq n_{i}\right\} \\
& E(G)=\left\{c x_{i}^{1}: 1 \leq i \leq r\right\} \cup\left\{x_{i}^{l_{i}} x_{i}^{l_{i}+1}: 1 \leq i \leq r, 1 \leq l_{i} \leq n_{i}-1\right\}
\end{aligned}
$$

If $v=|V(G)|$ and $e=|E(G)|$, then $v=(8 n+5)+\sum_{m=5}^{r}\left[2^{m-4}(4 n+1)+1\right]$ and $e=v-1$. Throughout the labeling, suppose $\alpha=(4 n+4)+\sum_{m=5}^{r}\left[2^{m-5}(4 n+1)+1\right]$.

Define $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ as follows: $\lambda(c)=\alpha$.
For $1 \leq i \leq 4,1 \leq l_{i} \leq n_{i}$ and $l_{i}$ odd, we define:

$$
\lambda(u)= \begin{cases}\frac{l_{1}+1}{2}, & \text { for } u=x_{1}^{l_{1}} \\ (n+1)-\frac{l_{2}-1}{2}, & \text { for } u=x_{2}^{l_{2}} \\ (2 n+2)-\frac{l_{3}-1}{2}, & \text { for } u=x_{3}^{l_{3}} \\ (4 n+3)-\frac{l_{4}-1}{2}, & \text { for } u=x_{4}^{l_{4}}\end{cases}
$$

and for $l_{i}$ even, we construct:

$$
\lambda(u)= \begin{cases}\alpha+\frac{l_{1}}{2}, & \text { for } u=x_{1}^{l_{1}} \\ (\alpha+n+1)-\frac{l_{2}}{2}, & \text { for } u=x_{2}^{l_{2}} \\ (\alpha+2 n+1)-\frac{l_{3}}{2}, & \text { for } u=x_{3}^{l_{3}} \\ (\alpha+4 n+2)-\frac{l_{4}}{2}, & \text { for } u=x_{4}^{l_{4}}\end{cases}
$$

For $5 \leq i \leq r, 1 \leq l_{i} \leq n_{i}$ and $l_{i}$ odd, we define:

$$
\lambda\left(x_{i}^{l_{i}}\right)=(4 n+3)+\sum_{m=5}^{i}\left[2^{m-5}(4 n+1)+1\right]-\frac{l_{i}-1}{2}
$$

and for $l_{i}$ even, we construct:

$$
\lambda\left(x_{i}^{l_{i}}\right)=(\alpha+4 n+2)+\sum_{m=5}^{i}\left[2^{m-5}(4 n+1)\right]-\frac{l_{i}}{2}
$$

The set of all edge-sums generated by the above formulas is $\{\lambda(x)+\lambda(y): x y \in$ $E(G)\}=\{\alpha+i: 1 \leq i \leq i\}$. It forms a sequence of consecutive integers starting from the minimum edge-sum $s=\alpha+1$. Thus, by Definition $1.1, \lambda$ is a $(\alpha+1,1)$ EAV labeling. As a consequence of Proposition $1.10, \lambda$ can be extended to a super $(a, 0)$-EAT labeling with magic constant $a=s+v+e=2 v+(4 n+4)+$
$\sum_{m=5}^{r}\left[2^{m-5}(4 n+1)+1\right]=(20 n+14)+\sum_{m=5}^{r}\left[2^{m-5}(20 n+5)+3\right]$. The set of edgelabels is $\{a-(\alpha+i): 1 \leq i \leq e\}$. Similarly, by Proposition 1.10, $\lambda$ can be extended to a super $\left(a^{\prime}, 2\right)$-EAT labeling with the minimum edge-weight $a^{\prime}=s+v+1=$ $v+(4 n+6)+\sum_{m=5}^{r}\left[2^{m-5}(4 n+1)+1\right]=(12 n+11)+\sum_{m=5}^{r}\left[2^{m-5}(12 n+3)+2\right]$. The set of edge-labels can be obtained by $\left\{a^{\prime}-(\alpha+i): 1 \leq i \leq e\right\}$.

Theorem 2.7. For any $r \geq 5$ and odd $n \geq 3, G \cong T(n, n+1,2 n+1,4 n+$ $2, n_{5}, \ldots, n_{r}$ ) admits a super ( $a, 1$ )-EAT total labeling with $a=s+\frac{3 v}{2}$ if $v$ is even, where $v=|V(G)|$, $s=(4 n+5)+\sum_{m=5}^{r}\left[2^{m-5}(4 n+1)+1\right]$ and $n_{m}=$ $2^{m-4}(4 n+1)+1$ for $5 \leq m \leq r$.

Proof. Let us consider the vertices and edges of $G$ defined as in Theorem 2.6. Now, we define the labeling $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ as in the same theorem. It follows that the set of edge-sums for all edges of $G$ denoted by $A=\left\{a_{i}\right.$ : $1 \leq i \leq e\}=\{\alpha+i: 1 \leq i \leq e\}$ forms an arithmetic sequence with common difference 1 and $B=\left\{b_{j}: 1 \leq j \leq e\right\}=\{v+j: 1 \leq j \leq e\}$ is a set of edge-labels, where $\alpha=(4 n+4)+\sum_{m=5}^{r}\left[2^{m-5}(4 n+1)+1\right]$. Define the set of edge-weights $C=\{\lambda(x)+\lambda(x y)+\lambda(y): x y \in E(G)\}=\left\{a_{2 i-1}+b_{e-i+1}: 1 \leq i \leq\right.$ $\left.\frac{e+1}{2}\right\} \cup\left\{a_{2 j}+b_{\frac{e-1}{2}-j+1}: 1 \leq j \leq \frac{e+1}{2}-1\right\}$. It is easy to see that $C$ constitutes an arithmetic sequence with $d=1$ and $a=s+\frac{3 v}{2}=128 n+48+\frac{1}{2} \sum_{m=5}^{r}\left[2^{m-2}(4 n+\right.$ $1)+5]$. Since all vertices receive the smallest labels, $\lambda$ is a super $(a, 1)$-EAT labeling.

## 3. Conclusion

In this paper, we have shown that a subclass of subdivided stars denoted by $T\left(n, n+1,2 n+1,4 n+2, n_{5}, \ldots, n_{r}\right)$ admits a super ( $a, d$ )-EAT labeling for $d \in$ $\{0,1,2\}$, where $n \geq 3$ is odd, $n_{m}=2^{m-4}(4 n+1)+1, r \geq 5$ and $5 \leq m \leq r$. It is a generalized form of the three-path tree studied by Lu [16, 17] and Ngurah et al. [18]. The choice of $\left\{n_{i}: 2 \leq i \leq r\right\}$ in the present results is different from the results which are derived by Javaid et al. [9]. Salman et al. [19] proved the existence of a super $(a, 0)$-EAT labeling on a particular subclass of the subdivided stars denoted by $T\left(n_{1}, n_{2}, n_{3} \ldots, n_{r}\right)$, where $n_{1}=n_{2}=n_{3}=\cdots=n_{r}=n$ and $n \in\{2,3\}$. Moreover, the scheme of a super $(a, d)$-EAT labeling developed in this paper does not work on $T\left(n_{1}, n_{2}, n_{3}, n_{4}, n_{5}, n_{6}\right)$, when $n_{1}=n_{2}=n_{3}=n_{4}=$ $n_{5}=n_{6}=4$. Thus, we propose the following open problem.

Open Problem 3.1. For the subdivided star $T\left(n_{1}, n_{2}, n_{3}, \ldots, n_{r}\right)$, where $n_{1}=$ $n_{2}=n_{3}=\cdots=n_{r}=n \geq 1$, determine if there is a super ( $a, d$ )-EAT labeling.

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