Discussiones Mathematicae
Graph Theory 34 (2014) 691–705
doi:10.7151/dmgt.1764

ON SUPER EDGE-ANTIMAGIC TOTAL LABELING OF SUBDIVIDED STARS[1]

Muhammad Javaid

Department of Mathematics
National University of Computer and
Emerging Sciences, Lahore Campus, Pakistan

e-mail: mjavaidmath@gmail.com
javaidmath@gmail.com

Abstract

In 1980, Enomoto *et al.* proposed the conjecture that every tree is a super $(a, 0)$-edge-antimagic total graph. In this paper, we give a partial support for the correctness of this conjecture by formulating some super (a, d)-edge-antimagic total labelings on a subclass of subdivided stars denoted by $T(n, n+1, 2n+1, 4n+2, n_5, n_6, \ldots, n_r)$ for different values of the edge-antimagic labeling parameter d, where $n \geq 3$ is odd, $n_m = 2^{m-4}(4n+1)+1$, $r \geq 5$ and $5 \leq m \leq r$.

Keywords: super (a, d)-EAT labeling, subdivision of star.

2010 Mathematics Subject Classification: 05C78.

1. Introduction

All graphs in this paper are finite, simple and undirected. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively. A (v, e)-graph G is a graph such that $|V(G)| = v$ and $|E(G)| = e$. Moreover, the theoretic ideas of graphs can be seen in [22]. A *labeling* (or *valuation*) of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). In this paper, the domain will be the set of all vertices and edges and such a labeling is called a *total labeling*. Some labelings use the vertex set only or the edge set only and we shall call them *vertex-labelings* or *edge-labelings*, respectively.

[1]The research contents of this paper are partially supported by the Higher Education Commission (HEC) of Pakistan.

There are many types of graph labelings, for example harmonius, cordial, graceful and antimagic. The most complete recent survey of graph labelings can be found in [6]. In this paper, we focus on an antimagic total labeling. More details on an antimagic total labeling can be found in [4]. The subject of edge-magic total labeling of graphs has its origin in the works of Kotzig and Rosa [13, 14] on what they called magic valuations of graphs.

Definition 1.1. An (s, d)*-edge-antimagic vertex* ((s, d)-EAV) *labeling* of a graph G is a bijective function $\lambda : V(G) \to \{1, 2, \ldots, v\}$ such that the set of edge-sums of all edges in G, $\{w(xy) = \lambda(x) + \lambda(y) : xy \in E(G)\}$, forms an arithmetic progression $\{s, s + d, s + 2d, \ldots, s + (e - 1)d\}$, where $s > 0$ and $d \geq 0$ are two fixed integers.

Simanjuntak *et al.* [21] proved that the odd cycle C_{2n+1}, the odd path P_{2n+1} and the even path P_{2n} have a $(n+2, 1)$-EAV labeling, where $n \geq 1$. They also proved that the odd path P_{2n+1} has a $(n + 3, 1)$-EAV labeling and the path P_n admits a $(3, 2)$-EAV labeling for $n \geq 1$. Moreover, Bača, Miller, Simanjuntak, Lin and Bertault [2, 21] proved the following results.

- If a non-tree connected graph G has an (a, d)-EAV labeling then $d = 1$.
- The cycle C_n has no (a, d)-EAV labeling for $d > 1$ and $n \geq 3$.
- The complete graph K_n has no (a, d)-EAV labeling, where $n > 3$.
- The symmetric complete bipartite graph $K_{n,n}$ has no (a, d)-EAV labeling, where $n > 1$.

Definition 1.2. An (a, d)*-edge-antimagic total* ((a, d)-EAT) *labeling* of a graph G is a bijective function $\lambda : V(G) \cup E(G) \to \{1, 2, \ldots, v + e\}$ such that the set of edge-weights of all edges in G, $\{w(xy) = \lambda(x) + \lambda(xy) + \lambda(y) : xy \in E(G)\}$, forms an arithmetic progression $\{a, a + d, a + 2d, \ldots, a + (e - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. If such a labeling exists, then G is said to be an (a, d)-EAT *graph*.

Definition 1.3. An (a, d)-EAT labeling λ is called a *super* (a, d)*-edge-antimagic total* (*super* (a, d)-EAT) *labeling* of G if $\lambda(V(G)) = \{1, 2, \ldots, v\}$. Thus, a *super* (a, d)-EAT *graph* is a graph that admits a super (a, d)-EAT labeling.

In the above definition, if $d = 0$, then a super $(a, 0)$-EAT labeling is called a *super edge-magic total* (SEMT) *labeling* and a is called a *magic constant*. For $d \neq 0$, a is called *minimum edge-weight*. The definition of an (a, d)-EAT labeling was introduced by Simanjuntak, Bertault and Miller in [21] as a natural extension of an edge-magic total labeling defined by Kotzig and Rosa. A super (a, d)-EAT labeling is a natural extension of the notion of a super $(a, 0)$-EAT labeling defined by Enomoto, Lladó, Nakamigawa and Ringel in [5]. They also proposed the conjecture that every tree is a super $(a, 0)$-EAT graph. In the favour of this conjecture, many authors have derived different results on a super (a, d)-EAT labeling for many particular classes of trees, for example path-like trees [3], banana trees [7], w-trees [11], extended w-trees [10, 12], subdivided stars [8, 9, 18, 19, 16, 17], subdivided w-trees [8] and caterpillars [20]. Lee and Shah [15] verified this conjecture by a computer search for trees with at most 17 vertices. However, this conjecture is still open.

Definition 1.4. For $n_i \geq 1$, $r \geq 2$ and $1 \leq i \leq r$, let $T(n_1, n_2, \ldots, n_r)$ be a subdivided star obtained by inserting $n_i - 1$ vertices to each of the i-th edge of the star $K_{1,r}$. Thus, the subdivided star $T\underbrace{(1, 1, \ldots, 1)}_{r-times}$ is the star $K_{1,r}$.

A star is a particular type of trees and many authors have investigated antimagicness for subdivided stars under certain conditions. Lu [16, 17] called the subdivided star $T(m, n, k)$ a *three-path tree* and proved that it is a super $(a, 0)$-EAT if n, m are odd and $k = n + 1$ or $k = n + 2$. Ngurah *et al.* [18] proved that $T(m, n, k)$ is also a super $(a, 0)$-EAT graph if n, m are odd and $k = n + 3$ or $k = n + 4$. Salman *et al.* [19] proved the existence of a super $(a, 0)$-EAT labeling on a particular subclass of the subdivided stars denoted by S_r^1 and S_r^2, where $S_r^1 \cong T\underbrace{(2, 2, \ldots, 2)}_{r-times}$ and $S_r^2 \cong T\underbrace{(3, 3, \ldots, 3)}_{r-times}$. Javaid *et al.* [8] investigated some results related to a super $(a, 0)$-EAT labeling on the subdivision of the star $K_{1,4}$ and the w-tree $WT(n, k)$. Javaid *et al.* [9] proved that a particular subclass of the subdivided stars in its generalized form denoted by $T(n, n, n+2, n+2, n_5, \ldots, n_r)$ admits a super (a, d)-EAT labeling for different values of d. Some of the results are as follows.

Theorem 1.5 [9]**.** *For any odd* $n \geq 3$*,* $T(n, n, n+2, n+2, 2n+3)$ *admits a super* (a, d)-EAT *labeling for* $d \in \{0, 2\}$*.*

Theorem 1.6 [9]**.** *For any odd* $n \geq 3$*,* $T(n, n, n+2, n+2, 2n+3)$ *admits a super* $(a, 1)$-EAT *labeling.*

Theorem 1.7 [9]**.** *For any odd* $n \geq 3$*,* $T(n, n, n+2, n+2, 2n+3, 4n+5)$ *admits a super* (a, d)-EAT *labeling for* $d \in \{0, 2\}$*.*

Theorem 1.8 [9]**.** *For any* $r \geq 5$ *and odd* $n \geq 3$*,* $T(n, n, n+2, n+2, n_5, \ldots, n_r)$ *admits a super* (a, d)-EAT *labeling, where* $n_m = 1 + (n+1)2^{m-4}$*,* $5 \leq m \leq r$ *and* $d \in \{0, 2\}$*.*

Theorem 1.9 [9]**.** *For any* $r \geq 5$ *and odd* $n \geq 3$*,* $T(n, n, n+2, n+2, n_5, \ldots, n_r)$ *admits a super* $(a, 1)$-EAT *labeling if* $|T(n, n, n+2, n+2, n_5, \ldots, n_r)|$ *is even, where* $n_m = 1 + (n+1)2^{m-4}$ *for* $5 \leq m \leq r$*.*

In this paper, we construct another generalized subclass of subdivided stars denoted by $T(n, n+1, 2n+1, 4n+2, n_5, n_6 \ldots, n_r)$, where $n_m = 2^{m-4}(4n+1)+1$, $5 \le m \le r$ and $r \ge 5$. Moreover, it is proved that this subclass also admits some super (a,d)-EAT labelings for different values of d. Let us consider the following proposition which we will use in the main results.

Proposition 1.10 [2]. *If a (v,e)-graph G has an (s,d)-EAV labeling, then*
(i) *G has a super $(s+v+1, d+1)$-EAT labeling,*
(ii) *G has a super $(s+v+e, d-1)$-EAT labeling.*

1.1. Bounds for the magic constant a

Ngurah *et al.* [18] found lower and upper bounds of the magic constant a for a particular family of subdivided stars which are stated as follows.

Lemma 1.11. *If $T(m,n,k)$ is a super $(a,0)$-EAT graph, then $\frac{1}{2l}(5l^2+3l+6) \le a \le \frac{1}{2l}(5l^2+11l-6)$, where $l = m+n+k$.*

The lower and upper bounds of the magic constant a proved by Salman *et al.* [19] are as follows.

Lemma 1.12. *If $T\underbrace{(n, n, \ldots, n)}_{n-times}$ is a super $(a,0)$-EAT graph, then $\frac{1}{2l}\big(5l^2+(9-2n)l+n^2-n\big) \le a \le \frac{1}{2l}\big(5l^2+(2n+5)l+n-n^2\big)$, where $l = n^2$.*

Now we find lower and upper bounds of the magic constant a for the most extended family of the subdivided stars denoted by $T(n_1, n_2, n_3, \ldots, n_r)$ with any $n_i \ge 1$ for $1 \le i \le r$.

Lemma 1.13. *If $T(n_1, n_2, n_3, \ldots, n_r)$ is a super $(a,0)$-EAT graph, then $\frac{1}{2l}\big(5l^2+(9-2r)l+(r^2-r)\big) \le a \le \frac{1}{2l}\big(5l^2+(5+2r)l-(r^2-r)\big)$, where $l = \sum_{i=1}^{r} n_i$.*

Proof. Suppose that $T(n_1, n_2, n_3, \ldots, n_r)$ admits a super $(a,0)$-EAT labeling with magic constant a and $l = \sum_{i=1}^{r} n_i$. Then "la" cannot be smaller than the sum obtained by assigning the smallest label 1 to the vertex of the degree r, the labels from 2 to $l+1-r$ to the vertices of degree 2 and the labels from $l+2-r$ to $l+1$ to the next r vertices of degree 1 as

$$la \ge r + 2\sum_{i=2}^{l-r+1} i + \sum_{i=l-r+2}^{l+1} i + \sum_{i=l+2}^{2l+1} i.$$

Consider $\sum_{i=2}^{l-r+1} i = \frac{l-r}{2}(l-r+3)$, $\sum_{i=l-r+2}^{l+1} i = \frac{1}{2}(2lr-r^2+3r)$ and $\sum_{i=l+2}^{2l+1} i = \frac{l}{2}(3l+3)$. Consequently, we have $la \ge \frac{1}{2}(5l^2+r^2-2lr+9l-r)$ or

$$a \ge \frac{1}{2l}\left(5l^2+r^2-2lr+9l-r\right) \tag{1}$$

Similarly, the upper bound of "la" is obtained by assigning the largest label $l+1$ to the vertex of the degree r, the labels from $r+1$ to l to the vertices of degree 2 and the labels from 1 to r to the next r vertices of degree 1 as

$$la \leq r(l+1) + 2\sum_{i=r+1}^{l} i + \sum_{i=1}^{r} i + \sum_{i=l+2}^{2l+1} i.$$

Consider $\sum_{i=r+1}^{l} i = \frac{3l}{2}(l+1)$, $\sum_{i=1}^{r} i = \frac{r}{2}(r+1)$ and $\sum_{i=l+2}^{2l+1} i = \frac{l-r}{2}(l+r+1)$. Consequently, we have $la \leq \frac{1}{2l}(5l^2 - r^2 + 2lr + 5l + r)$ or

$$a \leq \frac{1}{2l}\left(5l^2 - r^2 + 2lr + 5l + r\right) \tag{2}$$

Combining (1) and (2), we get

$$\frac{1}{2l}\left(5l^2 + (9-2r)l + (r^2 - r)\right) \leq a \leq \frac{1}{2l}\left(5l^2 + (5+2r)l - (r^2 - r)\right).$$

■

1.2. Strategy of construction for labeling schemes

Before presenting the main results, let us consider the overall strategy which is applied to find the results related to super (a,d)-EAT labelings on the particular subclasses of the subdivided stars for different values of the labeling parameter d. It is important to know about three terms edge-label, edge-sum and edge-weight. Let xy be an edge with end vertices x and y. Suppose that the assigned labels to the edge is $\lambda(xy)$ and to the vertices are $\lambda(x)$ and $\lambda(y)$. Thus, $\lambda(xy)$, $\lambda(x)+\lambda(y)$ and $\lambda(x)+\lambda(xy)+\lambda(y)$ are called *edge-label, edge-sum* and *edge-weight*, respectively.

In order to construct a super (a,d)-EAT labeling for $d = 0, 2$ on the graph G, the following steps have been performed:

1.2.1. Working steps for super $(a, 0)$-EAT labeling

- Define a bijection $\lambda : V(G) \to \{1, 2, \ldots, v\}$ in such a way that the set of edge-sums $\{\lambda(x)+\lambda(y)\colon xy \in E(G)\}$ forms a sequence of consecutive integers with minimum edge-sum, say, s.
- It follows that the graph G admits an $(s,1)$-EAV labeling.
- After getting an $(s,1)$-EAV labeling on the graph G, the goal is to extend it to a super $(a,0)$-EAT labeling with the help of the magic constant a.
- The magic constant can be calculated as $a = s + v + e$.
- Using set of edge-sums and the value of magic constant, the set of edge-labels can be obtained as $\{a - (\lambda(x) + \lambda(y))\colon xy \in E(G)\}$.

Consequently, the graph G admits a super $(a, 0)$-EAT labeling.

1.2.2. Working steps for super $(a', 2)$-EAT labeling

- Define a bijection $\lambda : V(G) \to \{1, 2, \ldots, v\}$ in such a way that the set of edge-sums $\{\lambda(x) + \lambda(y)\colon\, xy \in E(G)\}$ forms a sequence of consecutive integers with minimum edge-sum, say, s.
- It follows that the graph G admits an $(s, 1)$-EAV labeling.
- After getting an $(s, 1)$-EAV labeling on the graph G, the goal is to extend it to a super $(a', 2)$-EAT labeling with the help of the minimum edge-weight a'.
- The minimum edge-weight is calculated as $a' = s + v + 1$.
- Define the set of edge-weights as $\{a' - 2 + 2i : 1 \le i \le e\}$.
- Define the set of edge-sums as $H = \{h_i : 1 \le i \le e\}$.
- Using a' and the set H, the set of edge-labels can be obtained as $\{(a' - 2 + 2i) - h_i : 1 \le i \le e\}$.

Consequently, the graph G admits a super $(a', 2)$-EAT labeling.

In this paper, a super $(a, 1)$-EAT labeling is formulated if the order of the graph G is even. Thus, for the construction of a super $(a, 1)$-EAT labeling scheme, we proceed as follows.

1.2.3. Working steps for a super $(a, 1)$-EAT labeling

- Define a bijection $\lambda : V(G) \to \{1, 2, \ldots, v\}$ in such a way that the set of edge-sums $\{\lambda(x) + \lambda(y)\colon\, xy \in E(G)\}$ forms a sequence of consecutive integers with minimum edge-sum, say, s.
- Define the set of edge-sums as $A = \{a_i : 1 \le i \le e\}$.
- The set of edges-labels is $B = \{b_j : 1 \le j \le e\} = \{v_j + 1 : 1 \le j \le e\}$.
- The set of edge-weights can be obtained as
  $C = \{\lambda(x) + \lambda(xy) + \lambda(y)\colon\, xy \in E(G)\}$
  $= \{a_{2i-1} + b_{e-i+1} :\ 1 \le i \le \frac{e+1}{2}\} \cup \{a_{2j} + b_{\frac{e-1}{2}-j+1} : 1 \le j \le \frac{e+1}{2} - 1\}$.
- Thus, the minimum edge-weight is $a = s + \frac{3v}{2}$.

Consequently, the graph G admits a super $(a, 1)$-EAT labeling.

2. Main Results

In this section, we present the main results related to a super (a, d)-EAT labeling on a subclass of the subdivided stars for different values of the labeling parameter d.

Theorem 2.1. *For any odd $n \ge 3$, $G \cong T(n, n+1, 2n+1, 4n+2, 8n+3)$ admits a super $(a, 0)$-EAT labeling with $a = s + v + e$ and a super $(a', 2)$-EAT labeling with $a' = s + v + 1$, where $v = |V(G)|$ and $s = 8n + 7$.*

Proof. Let us denote the vertices and edges of G as follows.

$V(G) = \{c\} \cup \left\{x_i^{l_i} : 1 \leq i \leq 5, 1 \leq l_i \leq n_i\right\}$,

$E(G) = \{cx_i^1 : 1 \leq i \leq 5\} \cup \left\{x_i^{l_i} x_i^{l_i+1} : 1 \leq i \leq 5, 1 \leq l_i \leq n_i - 1\right\}$.

If $v = |V(G)|$ and $e = |E(G)|$, then $v = 16n + 8$ and $e = v - 1$.

Now, we define $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as follows: $\lambda(c) = 8n + 6$.

For $1 \leq i \leq 5$, $1 \leq l_i \leq n_i$ and l_i odd, we define:

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+1) - \frac{l_2-1}{2}, & \text{for } u = x_2^{l_2}, \\ (2n+2) - \frac{l_3-1}{2}, & \text{for } u = x_3^{l_3}, \\ (4n+3) - \frac{l_4-1}{2}, & \text{for } u = x_4^{l_4}, \\ (8n+5) - \frac{l_5-1}{2}, & \text{for } u = x_5^{l_5}, \end{cases}$$

and for l_i even, we construct:

$$\lambda(u) = \begin{cases} (8n+6) + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (9n+7) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (10n+7) - \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (12n+8) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4}, \\ (16n+9) - \frac{l_5}{2}, & \text{for } u = x_5^{l_5}. \end{cases}$$

The set of all edge-sums generated by the above formulas is $\{\lambda(x) + \lambda(y): xy \in E(G)\}$=$\{8n + 6 + i : 1 \leq i \leq e\}$. It forms a sequence of consecutive integers starting from the minimum edge-sum $s = 8n + 7$. Thus, by Definition 1.1, λ is a $(8n+7, 1)$-EAV labeling. As a consequence of Proposition 1.10, λ can be extended to a super $(a, 0)$-EAT labeling with magic constant $a = s + v + e = 40n + 22$. The set of edge-labels is $\{a - (8n + 6 + i) : 1 \leq i \leq e\}$. Similarly, by Proposition 1.10, λ can be extended to a super $(a', 2)$-EAT labeling with the minimum edge-weight $a' = s + v + 1 = 24n + 16$. The set of edge-labels can be obtained by $\{a' - (8n + 6 + i) : 1 \leq i \leq e\}$. ∎

As a consequence of the labeling which is formulated in Theorem 2.1, Figure 1(a) gives the set of edge-sums $\{31, 32, 33, \ldots, 85\}$ as a sequence of consecutive integers starting from $s = 31$. Thus, the subdivided star $T(3, 4, 7, 14, 27)$ admits a $(31, 1)$-EAV labeling. The magic constant can be obtained by $c = v + e + s = 56 + 55 + 31 = 142$. The set of edge-labels is $\{(142 - 31), (142 - 32), (142 - 33), \ldots, (142 - 85)\}$=$\{111, 110, 109, \ldots, 57\}$. Thus, Figure 2(a) presents a super $(142, 0)$-EAT labeling of the subdivided star $T(3, 4, 7, 14, 27)$.

Now, we calculate the minimum edge-weight $a' = s + v + 1 = 31 + 56 + 1 = 88$ and the set of edge-labels $\{(88 - 31), (90 - 32), (92 - 33), \ldots, (196 -$

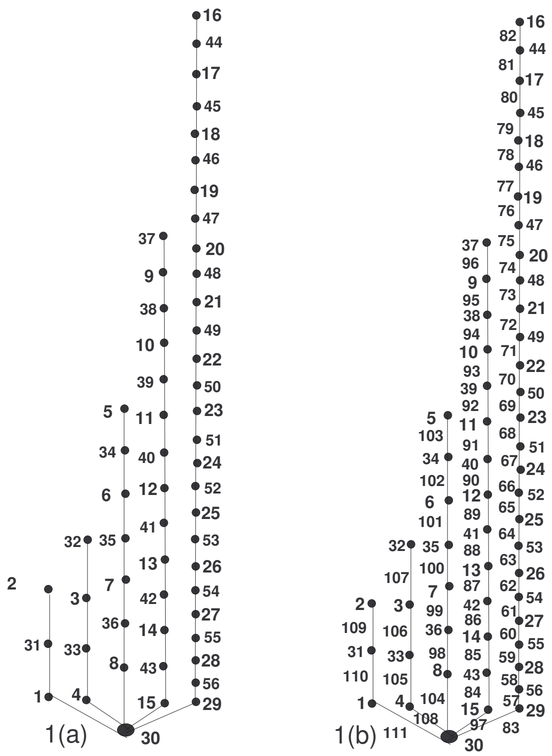

Figure 1. (a) (31,1)-EAV labeling of the subdivided star T(3,4,7,14,27).
(b) Super (142,0)-EAT labeling of the subdivided star T(3,4,7,14,27).

85)}=$\{57, 58, 59, \ldots, 111\}$. Consequently, Figure 2(a) gives a super $(88, 2)$-EAT labeling of the subdivided star $T(3, 4, 7, 14, 27)$.

Theorem 2.2. *For any odd* $n \geq 3$, $G \cong T(n, n+1, 2n+1, 4n+2, 8n+3)$ *admits a super* $(a, 1)$-EAT *labeling with* $a = s + \frac{3v}{2}$, *where* $v = |V(G)|$ *and* $s = 8n + 7$.

Proof. Let us consider the set of vertices and edges of the graph G defined as in the proof of Theorem 2.1. Now we define the vertex-labeling $\lambda : V(G) \rightarrow \{1, 2, \ldots, v\}$ as in the same theorem. It follows that the set of edge-sums for all edges of G denoted by $A = \{a_i : 1 \leq i \leq e\} = \{8n + 6 + i : 1 \leq i \leq e\}$ forms an arithmetic sequence with common difference 1 and $B = \{b_j : 1 \leq j \leq e\} = \{v + j : 1 \leq j \leq e\}$ is a set of edge-labels. Define the set of edge-weights $C = \{\lambda(x) + \lambda(xy) + \lambda(y) : xy \in E(G)\}$=$\{a_{2i-1} + b_{e-i+1} : 1 \leq i \leq \frac{e+1}{2}\} \cup \left\{a_{2j} + b_{\frac{e-1}{2}-j+1} : 1 \leq j \leq \frac{e+1}{2} - 1\right\}$. It is easy to see that C constitutes an arithmetic sequence with $d = 1$ and $a = s + \frac{3v}{2} = 32n + 19$. Since all vertices receive the smallest labels, λ is a super $(a, 1)$-EAT labeling. ∎

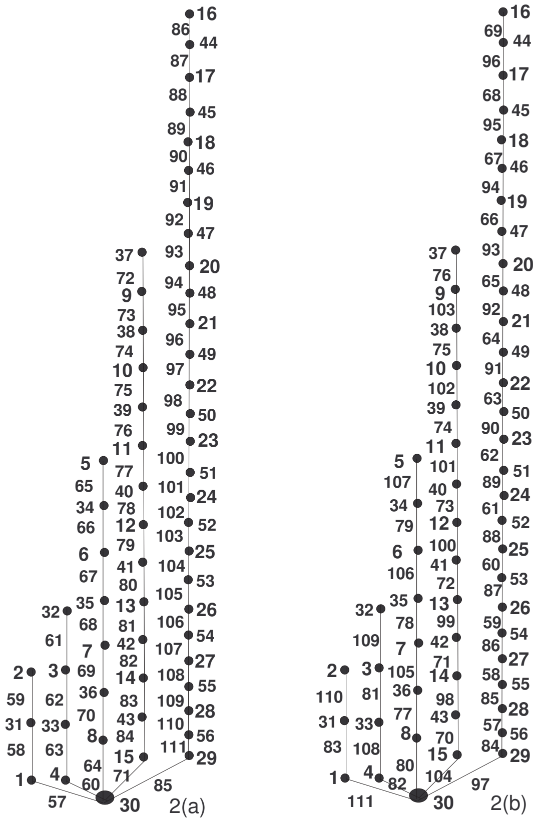

Figure 2. (a) Super (88,2)-EAT labeling of the subdivided star T(3,4,7,14,27).
(b) Super (115,1)-EAT labeling of the subdivided star T(3,4,7,14,27).

As a consequence of Theorem 2.2, to find a super $(a,1)$-EAT labeling on $T(3,4,7,14,27)$, define $A=\{a_1,a_2,a_3\ldots,a_e\}=\{31,32,33,\ldots,85\}$ and $B=\{b_1,b_2,b_3,\ldots,b_e\}=\{57,58,59,\ldots,111\}$. The set of edge-weights can be obtained by $C=\{a_{2i-1}+b_{e-i+1}:1\le i\le 28\}\cup\left\{a_{2j}+b_{\frac{e-1}{2}-j+1}:1\le j\le 27\right\}=\{31+111,33+110,\ldots,85+84\}\cup\{32+83,34+82,\ldots,84+57\}=\{142,143,\ldots,169\}\cup\{115,116,\ldots,141\}=\{115,116,117,\ldots,169\}$. We note that the minimum edge-weight in the set C is 115. It also can be calculated by $a=s+\frac{3v}{2}=31+\frac{3(56)}{2}=115$. Consequently, Figure 2(b) shows a super $(115,1)$-EAT labeling of the subdivided star $T(3,4,7,14,27)$.

Theorem 2.3. *For any odd $n\ge 3$, $G\cong T(n,n+1,2n+1,4n+2,8n+3,16n+5)$ admits a super $(a,0)$-EAT labeling with $a=s+v+e$ and a super $(a',2)$-EAT labeling with $a'=s+v+1$, where $v=|V(G)|$ and $s=16n+10$.*

Proof. Let us denote the vertices and edges of G as follows.

$$V(G) = \{c\} \cup \left\{x_i^{l_i} : 1 \leq i \leq 6, 1 \leq l_i \leq n_i\right\},$$
$$E(G) = \{cx_i^1 : 1 \leq i \leq 6\} \cup \left\{x_i^{l_i} x_i^{l_i+1} : 1 \leq i \leq 6, 1 \leq l_i \leq n_i - 1\right\}.$$

If $v = |V(G)|$ and $e = |E(G)|$, then $v = 32n + 13$, and $e = v - 1$.

Now, we define $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as follows: $\lambda(c) = 16n + 9$.

For $1 \leq i \leq 6$, $1 \leq l_i \leq n_i$ and l_i odd, we define:

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+1) - \frac{l_2-1}{2}, & \text{for } u = x_2^{l_2}, \\ (2n+2) - \frac{l_3-1}{2}, & \text{for } u = x_3^{l_3}, \\ (4n+3) - \frac{l_4-1}{2}, & \text{for } u = x_4^{l_4}, \\ (8n+5) - \frac{l_5-1}{2}, & \text{for } u = x_5^{l_5}, \\ (16n+8) - \frac{l_6-1}{2}, & \text{for } u = x_6^{l_6}, \end{cases}$$

and for l_i even we construct:

$$\lambda(u) = \begin{cases} (16n+9) + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (17n+10) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (18n+10) - \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (20n+11) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4}, \\ (24n+12) - \frac{l_5}{2}, & \text{for } u = x_5^{l_5}, \\ (32n+14) - \frac{l_6}{2}, & \text{for } u = x_6^{l_6}. \end{cases}$$

The set of all edge-sums generated by the above formulas is $\{\lambda(x) + \lambda(y): xy \in E(G)\}$=$\{16n + 9 + i : 1 \leq i \leq e\}$. It forms a sequence of consecutive integers starting from the minimum edge-sum $s = 16n + 10$. Thus, by Definition 1.1, λ is a $(16n + 10, 1)$-EAV labeling. As a consequence of Proposition 1.10, λ can be extended to a super $(a, 0)$-EAT labeling with magic constant $a = s + v + e = 80n + 35$. The set of edge-labels is $\{a - (16n + 9 + i) : 1 \leq i \leq e\}$. Similarly, by Proposition 1.10, λ can be extended to a super $(a', 2)$-EAT labeling with the minimum edge-weight $a' = s + v + 1 = 48n + 24$. The set of edge-labels can be obtained by $\{a' - (48n + 23 + i) : 1 \leq i \leq e\}$. ■

Theorem 2.4. *For any odd $n \geq 3$, $G \cong T(n, n+1, 2n+1, 4n+2, 8n+3, 16n+5, 32n+9)$ admits a super $(a, 0)$-EAT labeling with $a = s + v + e$ and a super $(a', 2)$-EAT labeling with $a' = s + v + 1$, where $v = |V(G)|$ and $s = 32n + 15$.*

Proof. Let us denote the vertices and edges of G as follows.

$$V(G) = \{c\} \cup \left\{x_i^{l_i} : 1 \leq i \leq 7, 1 \leq l_i \leq n_i\right\},$$
$$E(G) = \{cx_i^1 : 1 \leq i \leq 7\} \cup \left\{x_i^{l_i} x_i^{l_i+1} : 1 \leq i \leq 7, 1 \leq l_i \leq n_i - 1\right\}.$$

If $v = |V(G)|$ and $e = |E(G)|$ then $v = 64n + 22$, and $e = 64n + 21$.

Now, we define $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as follows: $\lambda(c) = 32n + 14$.

For $1 \le i \le 7$, $1 \le l_i \le n_i$ and l_i odd, we define:

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+1) - \frac{l_2-1}{2}, & \text{for } u = x_2^{l_2}, \\ (2n+2) - \frac{l_3-1}{2}, & \text{for } u = x_3^{l_3}, \\ (4n+3) - \frac{l_4-1}{2}, & \text{for } u = x_4^{l_4} \\ (8n+5) - \frac{l_5-1}{2}, & \text{for } u = x_5^{l_5}, \\ (16n+8) - \frac{l_6-1}{2}, & \text{for } u = x_6^{l_6}, \\ (32n+13) - \frac{l_7-1}{2}, & \text{for } u = x_7^{l_7}, \end{cases}$$

and for l_i even we construct:

$$\lambda(u) = \begin{cases} (32n+14) + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (33n+15) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (34n+15) - \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (36n+16) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4}, \\ (40n+17) - \frac{l_5}{2}, & \text{for } u = x_5^{l_5}, \\ (48n+19) - \frac{l_6}{2}, & \text{for } u = x_6^{l_6}, \\ (64n+23) - \frac{l_7}{2}, & \text{for } u = x_7^{l_7}. \end{cases}$$

The set of all edge-sums generated by the above formulas is $\{\lambda(x)+\lambda(y)\colon xy \in E(G)\}=\{32n+14+i : 1 \le i \le e\}$. It forms a sequence of consecutive integers starting from the minimum edge-sum $s = 32n+15$. Thus, by Definition 1.1, λ is a $(32n+15, 1)$-EAV labeling. As a consequence of Proposition 1.10, λ can be extended to a super $(a, 0)$-EAT labeling with magic constant $a = s+v+e = 160n+58$. The set of edge-labels is $\{a-(16n+9+i) : 1 \le i \le e\}$. Similarly, by Proposition 1.10, λ can be extended to a super $(a', 2)$-EAT labeling with the minimum edge-weight $a' = s+v+1 = 96n+28$. The set of edge-labels can be obtained by $\{a'-(96n+27+i) : 1 \le i \le e\}$. ■

Theorem 2.5. *For any even $n \ge 3$, $G \cong T(n, n+1, 2n+1, 4n+2, 8n+3, 16n+5, 32n+9)$ admits a super $(a, 1)$-EAT labeling with $a = s + \frac{3v}{2}$, where $v = |V(G)|$ and $s = 32n+15$.*

Proof. Let us consider the set of vertices and edges of G defined as in Theorem 2.4. Now we define the vertex-labeling $\lambda : V(G) \to \{1, 2, \dots, v\}$ as in the same theorem. It follows that the set of edge-sums for all edges of G denoted by $A = \{a_i : 1 \le i \le e\} = \{32n+14+i : 1 \le i \le e\}$ forms an arithmetic sequence with common difference 1 and $B = \{b_j : 1 \le j \le e\} = \{v+j : 1 \le j \le e\}$ is a set of edge-labels. Define the set of edge-weights $C = \{\lambda(x)+\lambda(xy)+\lambda(y) : xy \in E(G)\}=\{a_{2i-1}+b_{e-i+1} : 1 \le i \le \frac{e+1}{2}\} \cup \left\{a_{2j}+b_{\frac{e-1}{2}-j+1} : 1 \le j \le \frac{e+1}{2}-1\right\}$. It is easy to see that C constitutes an arithmetic sequence with $d = 1$ and

$a = s + \frac{3v}{2} = 128n + 48$. Since all vertices receive the smallest labels, λ is a super $(a, 1)$-EAT labeling. ∎

Theorem 2.6. *For any $r \geq 5$ and odd $n \geq 3$, $G \cong T(n, n+1, 2n+1, 4n+2, n_5, \ldots, n_r)$ admits a super $(a, 0)$-EAT labeling with $a = s + v + e$ and a super $(a', 2)$-EAT labeling with $a' = s + v + 1$ where $v = |V(G)|$, $s = (4n+5) + \sum_{m=5}^{r}[2^{m-5}(4n+1)+1]$ and $n_m = 2^{m-4}(4n+1)+1$ for $5 \leq m \leq r$.*

Proof. Let us denote the vertices and edges of G as follows.

$V(G) = \{c\} \cup \left\{x_i^{l_i} : 1 \leq i \leq r, 1 \leq l_i \leq n_i\right\}$,

$E(G) = \{cx_i^1 : 1 \leq i \leq r\} \cup \left\{x_i^{l_i} x_i^{l_i+1} : 1 \leq i \leq r, 1 \leq l_i \leq n_i - 1\right\}$.

If $v = |V(G)|$ and $e = |E(G)|$, then $v = (8n+5) + \sum_{m=5}^{r}[2^{m-4}(4n+1)+1]$ and $e = v - 1$. Throughout the labeling, suppose $\alpha = (4n+4) + \sum_{m=5}^{r}[2^{m-5}(4n+1)+1]$.

Define $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as follows: $\lambda(c) = \alpha$.

For $1 \leq i \leq 4$, $1 \leq l_i \leq n_i$ and l_i odd, we define:

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+1) - \frac{l_2-1}{2}, & \text{for } u = x_2^{l_2}, \\ (2n+2) - \frac{l_3-1}{2}, & \text{for } u = x_3^{l_3}, \\ (4n+3) - \frac{l_4-1}{2}, & \text{for } u = x_4^{l_4}, \end{cases}$$

and for l_i even, we construct:

$$\lambda(u) = \begin{cases} \alpha + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (\alpha+n+1) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (\alpha+2n+1) - \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (\alpha+4n+2) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4}. \end{cases}$$

For $5 \leq i \leq r$, $1 \leq l_i \leq n_i$ and l_i odd, we define:

$$\lambda(x_i^{l_i}) = (4n+3) + \sum_{m=5}^{i}[2^{m-5}(4n+1)+1] - \frac{l_i - 1}{2},$$

and for l_i even, we construct:

$$\lambda(x_i^{l_i}) = (\alpha+4n+2) + \sum_{m=5}^{i}[2^{m-5}(4n+1)] - \frac{l_i}{2}.$$

The set of all edge-sums generated by the above formulas is $\{\lambda(x)+\lambda(y)\colon xy \in E(G)\}$=$\{\alpha + i : 1 \leq i \leq i\}$. It forms a sequence of consecutive integers starting from the minimum edge-sum $s = \alpha + 1$. Thus, by Definition 1.1, λ is a $(\alpha+1, 1)$-EAV labeling. As a consequence of Proposition 1.10, λ can be extended to a super $(a, 0)$-EAT labeling with magic constant $a = s + v + e = 2v + (4n+4) +$

$\sum_{m=5}^{r}[2^{m-5}(4n+1)+1] = (20n+14)+\sum_{m=5}^{r}[2^{m-5}(20n+5)+3]$. The set of edge-labels is $\{a-(\alpha+i) : 1 \leq i \leq e\}$. Similarly, by Proposition 1.10, λ can be extended to a super $(a', 2)$-EAT labeling with the minimum edge-weight $a' = s+v+1 = v+(4n+6)+\sum_{m=5}^{r}[2^{m-5}(4n+1)+1] = (12n+11)+\sum_{m=5}^{r}[2^{m-5}(12n+3)+2]$. The set of edge-labels can be obtained by $\{a' - (\alpha+i) : 1 \leq i \leq e\}$. ∎

Theorem 2.7. *For any $r \geq 5$ and odd $n \geq 3$, $G \cong T(n, n+1, 2n+1, 4n+2, n_5, \ldots, n_r)$ admits a super $(a, 1)$-EAT total labeling with $a = s + \frac{3v}{2}$ if v is even, where $v = |V(G)|$, $s = (4n+5) + \sum_{m=5}^{r}[2^{m-5}(4n+1)+1]$ and $n_m = 2^{m-4}(4n+1)+1$ for $5 \leq m \leq r$.*

Proof. Let us consider the vertices and edges of G defined as in Theorem 2.6. Now, we define the labeling $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as in the same theorem. It follows that the set of edge-sums for all edges of G denoted by $A = \{a_i : 1 \leq i \leq e\} = \{\alpha + i : 1 \leq i \leq e\}$ forms an arithmetic sequence with common difference 1 and $B = \{b_j : 1 \leq j \leq e\} = \{v+j : 1 \leq j \leq e\}$ is a set of edge-labels, where $\alpha = (4n+4) + \sum_{m=5}^{r}[2^{m-5}(4n+1)+1]$. Define the set of edge-weights $C = \{\lambda(x) + \lambda(xy) + \lambda(y) : xy \in E(G)\}$=$\{a_{2i-1} + b_{e-i+1} : 1 \leq i \leq \frac{e+1}{2}\} \cup \left\{a_{2j} + b_{\frac{e-1}{2}-j+1} : 1 \leq j \leq \frac{e+1}{2} - 1\right\}$. It is easy to see that C constitutes an arithmetic sequence with $d = 1$ and $a = s + \frac{3v}{2} = 128n + 48 + \frac{1}{2}\sum_{m=5}^{r}[2^{m-2}(4n+1)+5]$. Since all vertices receive the smallest labels, λ is a super $(a, 1)$-EAT labeling. ∎

3. CONCLUSION

In this paper, we have shown that a subclass of subdivided stars denoted by $T(n, n+1, 2n+1, 4n+2, n_5, \ldots, n_r)$ admits a super (a, d)-EAT labeling for $d \in \{0, 1, 2\}$, where $n \geq 3$ is odd, $n_m = 2^{m-4}(4n+1)+1$, $r \geq 5$ and $5 \leq m \leq r$. It is a generalized form of the three-path tree studied by Lu [16, 17] and Ngurah *et al.* [18]. The choice of $\{n_i : 2 \leq i \leq r\}$ in the present results is different from the results which are derived by Javaid *et al.* [9]. Salman *et al.* [19] proved the existence of a super $(a, 0)$-EAT labeling on a particular subclass of the subdivided stars denoted by $T(n_1, n_2, n_3 \ldots, n_r)$, where $n_1 = n_2 = n_3 = \cdots = n_r = n$ and $n \in \{2, 3\}$. Moreover, the scheme of a super (a, d)-EAT labeling developed in this paper does not work on $T(n_1, n_2, n_3, n_4, n_5, n_6)$, when $n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = 4$. Thus, we propose the following open problem.

Open Problem 3.1. For the subdivided star $T(n_1, n_2, n_3, \ldots, n_r)$, where $n_1 = n_2 = n_3 = \cdots = n_r = n \geq 1$, determine if there is a super (a, d)-EAT labeling.

Acknowledgement

The author is deeply indebted to the anonymous referees for their valuable thoughts and comments which improved the original version of this paper.

References

[1] M. Bača, Y. Lin, M. Miller and M.Z. Youssef, *Edge-antimagic graphs*, Discrete Math. **307** (2007) 1232–1244.
doi:10.1016/j.disc.2005.10.038

[2] M. Bača, Y. Lin, M. Miller and R. Simanjuntak, *New constructions of magic and antimagic graph labelings*, Util. Math. **60** (2001) 229–239.

[3] M. Bača, Y. Lin and F.A. Muntaner-Batle, *Super edge-antimagic labelings of the path-like trees*, Util. Math. **73** (2007) 117–128.

[4] M. Bača and M. Miller, Super Edge-Antimagic Graphs (Brown Walker Press, Boca Raton, Florida USA, 2008).

[5] H. Enomoto, A.S. Lladó, T. Nakamigawa and G. Ringel, *Super edge-magic graphs*, SUT J. Math. **34** (1998) 105–109.

[6] J.A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin. (2011) #DS6.

[7] M. Hussain, E.T. Baskoro and Slamin, *On super edge-magic total labeling of banana trees*, Util. Math. **79** (2009) 243–251.

[8] M. Javaid, M. Hussain, K. Ali and H. Shaker, *On super edge-magic total labeling on subdivision of trees*, Util. Math. **89** (2012) 169–177.

[9] M. Javaid and A.A. Bhatti, *On super (a, d)-edge antimagic total labeling of subdivided stars*, Ars Combin. **105** (2012) 503–512.

[10] M. Javaid, A.A. Bhatti and M. Hussain, *On (a, d)-edge-antimagic total labeling of extended w-trees*, Util. Math. **87** (2012) 293–303.

[11] M. Javaid, M. Hussain, K. Ali and K.H. Dar, *Super edge-magic total labeling on w-trees*, Util. Math. **86** (2011) 183–191.

[12] M. Javaid, A.A. Bhatti, M. Hussain and K. Ali, *Super edge-magic total labeling on forest of extended w-trees*, Util. Math. **91** (2013) 155–162.

[13] A. Kotzig and A. Rosa, *Magic valuations of finite graphs*, Canad. Math. Bull. **13** (1970) 451–461.
doi:10.4153/CMB-1970-084-1

[14] A. Kotzig and A. Rosa, Magic Valuation of Complete Graphs (Centre de Recherches Mathematiques, Uni. de Montreal, 1972).

[15] S.M. Lee and Q.X. Shah, *All trees with at most* 17 *vertices are super edge-magic*, in: 16th MCCCC Conference, Carbondale SIU (2002).

[16] Y.-J. Lu, *A proof of three-path trees $P(m, n, t)$ being edge-magic*, College Mathematica **17(2)** (2001) 41–44.

[17] Y.-J. Lu, *A proof of three-path trees $P(m,n,t)$ being edge-magic* (II), College Mathematica **20(3)** (2004) 51–53.

[18] A.A.G. Ngurah, R. Simanjuntak and E.T. Baskoro, *On* (*super*) *edge-magic total labeling of subdivision of* $K_{1,3}$, SUT J. Math. **43** (2007) 127–136.

[19] A.N.M. Salman, A.A.G. Ngurah and N. Izzati, *On super edge-magic total labeling of a subdivision of a star* S_n, Util. Math. **81** (2010) 275–284.

[20] K.A. Sugeng, M. Miller, Slamin and M. Bača, (a,d)-*edge-antimagic total labelings of caterpillars*, Lect. Notes Comput. Sci. **3330** (2005) 169–180. doi:10.1007/978-3-540-30540-8_19

[21] R. Simanjuntak, F. Bertault and M. Miller, *Two new* (a,d)-*antimagic graph labelings*, in: Proc. 11th Australian Workshop on Combin. Algor. **11** (2000) 179–189.

[22] D.B. West, An Introduction to Graph Theory (Prentice Hall, 1996).

Received 11 June 2012
Revised 2 October 2013
Accepted 2 October 2013