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STRONG CHROMATIC INDEX OF PLANAR GRAPHS WITH LARGE GIRTH[1]

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Abstract

Let $\Delta \geq 4$ be an integer. In this note, we prove that every planar graph with maximum degree Δ and girth at least $10\Delta+46$ is strong $(2\Delta-1)$-edge-colorable, that is best possible (in terms of number of colors) as soon as G contains two adjacent vertices of degree Δ. This improves [6] when $\Delta \geq 6$.

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1. Introduction

A *strong k-edge-coloring* of a graph G is a mapping from $E(G)$ to $\{1, 2, \ldots, k\}$ such that every two adjacent edges or two edges adjacent to a same edge receive two distinct colors. In other words, the graph induced by each color class is an induced matching. This can also be seen as a vertex 2-distance coloring of the line graph of G. The *strong chromatic index* of G, denoted by $\chi'_s(G)$, is the smallest integer k such that G admits a strong k-edge-coloring. As already mentioned, we have $\chi'_s(G) = \chi(L(G)^2)$, where χ denotes the usual chromatic number and $L(G)^2$ the square of the line graph of G.

Strong edge-colorability was introduced by Fouquet and Jolivet [11, 12] and was used to solve the frequency assignment problem in some radio networks. Suppose that we have a set of transceivers communicating with each other over a shared medium. A transceiver x that wants to communicate with a transceiver y sends its message on a frequency α. However, every close transceiver of x receives the message dedicated to y on channel α. Suppose that transceivers x and y want to communicate with z, they cannot send a message to z on the same channel; otherwise z will not be able to understand the message (since the messages will interfere with each other). Also suppose that transceiver u wants to communicate with transceiver v, transceiver w wants to communicate with transceiver t, and v and w are close. Transceivers u and w cannot communicate their message on the same channel; otherwise v will receive two messages on the same channel: the message from u dedicated to it, and the message from w dedicated to t. Now in terms of graphs, if we consider the graph whose vertices are the transceivers, and there is an edge if the corresponding transceivers are close, then solving the frequency assignment problem is equivalent to find a strong edge coloring of the graph. For more details on applications and protocols see [4, 18, 20, 21].

An obvious upper bound on $\chi'_s(G)$ (given by a greedy coloring) is $2\Delta(\Delta - 1) + 1$ where Δ is the maximum degree of G. The following conjecture was posed by Erdős and Nešetřil [8, 9] and revised by Faudree, Schelp, Gyárfás and Tuza [10].

Conjecture 1 (Erdős and Nešetřil [8], [9], Faudree *et al.* [10])**.** *If G is a graph with maximum degree Δ, then*

$$\chi'_s(G) \leq \frac{5}{4}\Delta^2 \textit{ if } \Delta \textit{ is even and } \frac{1}{4}(5\Delta^2 - 2\Delta + 1) \textit{ otherwise.}$$

Moreover, they gave examples of graphs whose strong chromatic indices reach the upper bounds.

In the general case, the best known upper bound was given by Molloy and Reed [17] using the probabilistic method.

Theorem 2 (Molloy and Reed [17])**.** *For Δ large enough, every graph with maximum degree Δ has $\chi'_s(G) \leq 1.998\Delta^2$.*

For small maximum degrees, the cases $\Delta = 3$ and 4 were studied.

Theorem 3 (Andersen [1], Horák *et al.* [15])**.** *Every graph with maximum degree $\Delta \leq 3$ admits a strong 10-edge-coloring.*

This is best possible.

Theorem 4 (Cranston [7]). *Every graph with maximum degree $\Delta \leq 4$ admits a strong 22-edge-coloring.*

According to Conjecture 1, the best upper bound we may expect is 20.

The strong chromatic index was also studied for different families of graphs, as cycles, trees, d-dimensional cubes, chordal graphs, Kneser graphs, see [16]. For complexity issues, see [14, 16].

Faudree, Schelp, Gyárfás exhibited, for every integer $\Delta \geq 2$, a planar graph with maximum degree Δ and strong chromatic index $4\Delta - 4$. They established the following upper bound.

Theorem 5 (Faudree *et al.* [10]). *Planar graphs with maximum degree Δ are strong $(4\Delta + 4)$-edge-colorable.*

The proof of Theorem 5 is very nice and is as follows: first color the edges of the graph G properly with $\Delta + 1$ colors with Vizing's Theorem [23]. Then for each color i $(1 \leq i \leq \Delta + 1)$ consider the graph H_i where the vertices are the edges of G colored by i and there is an edge between two vertices of H_i if the corresponding edges are linked by an edge in G. Clearly, H_i is planar; so H_i is 4-vertex-colorable by the Four Color Theorem [2, 3] with the colors i^1, i^2, i^3, i^4. Map now these colors in G. We obtain a strong edge-coloring of G.

As a corollary of the proof of Theorem 5, one can observe that K_5-minor free graphs with maximum degree Δ are strong $(4\Delta + 4)$-edge-colorable. It suffices to notice that the graphs H_i are K_5-minor free (as they can be seen as the contraction of a subgraph of G) and so are 4-colorable.

Another corollary of this proof is that every planar graph G with girth at least 7 and maximum degree $\Delta \geq 7$ is strong 3Δ-edge-colorable: every planar graph G with maximum degree at least 7 is properly Δ-edge-colorable [22]; moreover if the girth of G is at least 7, then H_i is planar triangle-free and so is 3-vertex-colorable by Grötzsch's theorem [13].

Hence if G is planar with large girth and large maximum degree, then we have $\chi'_s(G) \leq 3\Delta$. The purpose of this paper is to prove that if the girth is large enough, then the upper bound can be strengthened to $2\Delta - 1$, which is best possible as soon as G contains two adjacent vertices of degree Δ. A first attempt was done by Borodin and Ivanova [6] who proved that every planar graph with maximum degree Δ is strong $(2\Delta - 1)$-colorable if its girth is at least $40 \left\lfloor \frac{\Delta}{2} \right\rfloor + 1$. Here we improved the girth condition as soon as $\Delta \geq 6$:

Theorem 6. *Let $\mathcal{F}_\Delta$ be the family of planar graphs with maximum degree at most Δ. Every graph of $\mathcal{F}_\Delta$ with girth at least $10\Delta + 46$ admits a strong $(2\Delta - 1)$-edge-coloring when $\Delta \geq 4$.*

Next section is devoted to the proof of Theorem 6.

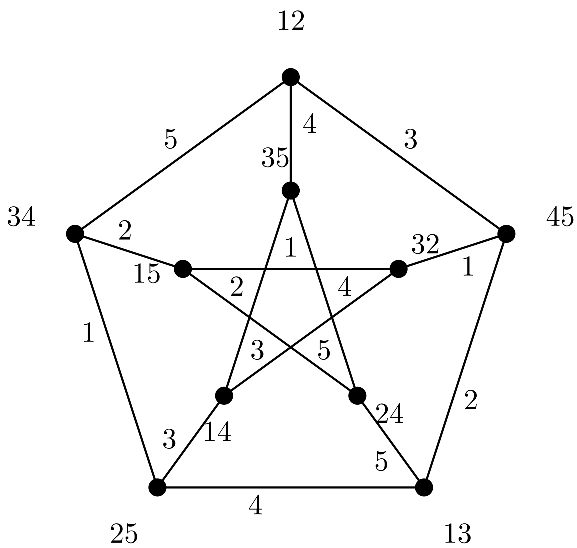

Figure 1. The odd graph O_3 and its edge labeling.

2. On Planar Graphs with Large Girth

A *walk* in a graph is a sequence of edges where two consecutive edges are adjacent. Throughout the paper, by *path* we mean a walk where every two consecutive edges are distinct. So a vertex or an edge can appear more than once in a path. By *cycle* we mean a closed path (the first and last edges of the sequence are adjacent).

The proof of Theorem 6 is based on the use of *odd graphs* and of their properties.

Let n be an integer; the odd graph O_n may be defined as follows:

- the vertices are the $(n-1)$-subsets of $\{1, 2, \dots, 2n-1\}$;
- two vertices are adjacent if and only if the corresponding subsets are disjoint.

The odd graph O_n is n-regular and distance transitive. Moreover, its odd-girth is $2n-1$ and its even-girth is 6 [5]. We will use the notation $S(x)$ to denote the subset assigned to the vertex x in O_n. Also we can label every edge xy by the label $\{1, \dots, 2n-1\} \setminus (S(x) \cup S(y))$. Remark that the obtained edge-labeling is a strong edge-coloring. As example, O_3 (the Petersen graph) is depicted in Figure 1. To prove Theorem 6, we establish that there is a path of length exactly $2(n-1)$ between every pair of vertices (not necessarily distinct) in the odd graph O_n ($n \geq 4$).

In the following we will consider the case $\Delta \geq 4$.

Let $G \in \mathcal{F}_\Delta$ be a counterexample to Theorem 6 with the minimum order. Clearly, G is connected.

(1) *G does not contain a vertex v adjacent to $d(v) - 1$ vertices of degree* 1.

By the way of contradiction, suppose G contains such a vertex v. Let u be a vertex of degree 1 adjacent to v. By the minimality of G, $G' = G - u$ admits a strong $(2\Delta - 1)$-edge-coloring. By a simple counting argument, it is easy to see that we can extend the coloring to uv, a contradiction.

Consider now $H = G - \{v : v \in G, d_G(v) = 1\}$.

(2) *The minimum degree of H is at least* 2 (*by* (1)). *Graph H is planar and has the same girth as G.*

The following observation is well-known [19].

(3) *Every planar graph with minimum degree at least* 2 *and girth at least* $5d + 1$ *contains a path consisting of d consecutive vertices of degree* 2.

Let $d = 2\Delta + 9$. It follows from the assumption on the girth, (2) and (3) that H contains a path $v_0v_1v_2 \cdots v_{d+1}$ in which every vertex v_i for $1 \leq i \leq d$ has degree 2. In G, the path $v_1 \cdots v_d$ is an induced path and every v_i $(1 \leq i \leq d)$ may be adjacent to some vertices of degree 1, by definition of H and (1).

Now, consider G' obtained from G by

- removing all the pendant vertices adjacent to $v_1 \cdots v_d$, and
- removing the vertices v_2 to v_{d-1}.

By the minimality of G, G' admits a strong $(2\Delta - 1)$-edge coloring ϕ. Our aim is to extend ϕ to G and get a contradiction.

Let $c_\phi(u)$ be the set of colors of the edges incident to u. We can assume that $|c_\phi(v_0)| = |c_\phi(v_{d+1})| = \Delta$ (by adding vertices of degree 1 adjacent to v_0 and v_{d+1} in G' as $2\Delta < d$ and so $|V(G')| < |V(G)|$). Let $x = \phi(v_0v_1)$ and $y = \phi(v_dv_{d+1})$. For a set C of colors, define $\overline{C} = \{1, \ldots, 2\Delta - 1\} \setminus C$.

Extending ϕ to G is equivalent to find a special path P in the odd graph O_Δ. This path P must have the following properties:

(P1) its length is $d + 1$; let $P = u_0u_1 \cdots u_{d+1}$;

(P2) u_0 is the vertex of O_Δ such that $S(u_0) = \overline{c_\phi(v_0)}$;

(P3) u_{d+1} is the vertex of O_Δ such that $S(u_{d+1}) = \overline{c_\phi(v_{d+1})}$;

(P4) the edge u_0u_1 is labeled with x;

(P5) the edge u_du_{d+1} is labeled with y.

Informally speaking, this path may be seen as a mapping of $v_0 \cdots v_{d+1}$ into O_Δ. If such a path exists, then one can extend ϕ to G by coloring the edges incident to v_i with colors of $\overline{S(u_i)}$; the edge v_iv_{i+1} is colored with the label of the edge u_iu_{i+1}.

The following part is dedicated to the proof of the existence of such a path.

(4) *Let xyz be a simple path of length 2 of O_n with $n \geq 3$. Then xyz is contained in a cycle of length 6.*

Proof. The claim follows directly from the fact that O_n is distance transitive and its even-girth is 6 [5]. However, let us exhibit such a cycle of length 6, as it is useful to establish property (5) below.

Let xyz be a path of length 2 of O_n. W.l.o.g. we can assume that $S(x) = X \cup b$, $S(y) = C \setminus (X \cup \{a, b\})$, $S(z) = X \cup \{a\}$ where $C = \{1, \ldots, 2n-1\}$, X is an arbitrary $(n-2)$-subset of C, and a, b are distinct elements of $C \setminus X$. Let us now exhibit a 6-cycle $xyzuvw$ going through xyz. Let $c \in C \setminus (X \cup \{a, b\})$. Vertex u (resp. v, w) is the vertex of O_n with the $(n-1)$-subset of $C \setminus (X \cup \{a, c\})$ (resp. $X \cup \{c\}$, $C \setminus (X \cup \{b, c\})$) (see Figure 2). □

The following property of odd graphs (which follows from (4)) is also useful for our proof.

(5) *Let x be a vertex of O_n with $n \geq 3$. Then x is contained in a cycle of length $2k$ for any integer $k \geq 3$.*

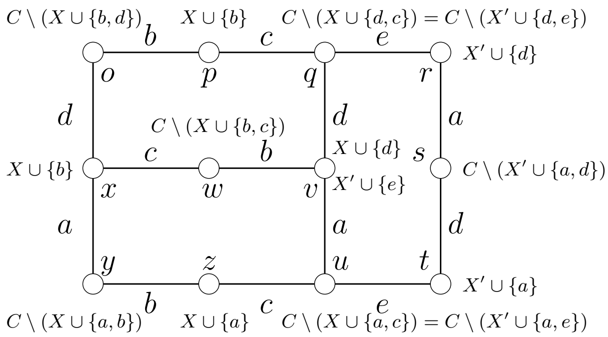

Figure 2. Vertex x is contained in a cycle of length $2k$ for any $k \geq 3$.

Proof. Let x be a vertex of O_n. By applying (4), one can observe that x is contained in the subgraph depicted in Figure 2, where C denotes the set $\{1, \ldots, 2n-1\}$, X and X' two $(n-2)$-subsets, and a, b, c, d, e five distinct elements.

Let C_1 (resp. C_2, C_3) be the cycle $xyzuvw$ (resp. $xyzuvqpo$, $xyzutsrqpo$) of length 6 (resp. 8, 10) containing x as depicted in Figure 2. Let $k = 3l + r$ with

$l \geq 1$ and $0 \leq r \leq 2$. We have $2k = 6(l-1) + (6+2r)$. Hence the cycle made of C_{r+1} and $(l-1)$ times C_1 is a cycle of length $2k$ containing x. □

We recall that a simple path is a path containing distinct vertices.

Claim 7. *Let u and v be two (not necessarily distinct) vertices of O_n with $n \geq 4$. There exists a simple path linking u and v of length exactly $2(n-1)$.*

Proof. Given two vertices (not necessarily distinct) u and v, we will exhibit a path, say $P = w_1 \cdots w_{2(n-1)+1}$, of length exactly $2(n-1)$ where $w_1 = u$, $w_{2(n-1)+1} = v$. We consider the following three cases with respect to the size of the intersection of $S(u)$ and $S(v)$.

Case: $|S(u) \cap S(v)| = k$ with $k = 0$ or $3 \leq k \leq n-1$. Let $S(u) \cap S(v) = \{x_1, \ldots, x_k\}$ and assume $S(u) = \{x_1, \ldots, x_k, y_{k+1}, \ldots, y_{n-1}\}$ and $S(v) = \{x_1, \ldots, x_k, t_{k+1}, \ldots, t_{n-1}\}$. Let $z_1, \ldots, z_{k+1}$ be the elements of $\{1, \ldots, 2n-1\} \setminus (S(u) \cup S(v))$.

We leave the vertex w_i by taking the edge labeled with $t_{k+(i+1)/2}$ when i is odd, and with $y_{k+i/2}$ otherwise. It follows that

$$S(w_i) = \begin{cases} \{z_1, \ldots, z_{k+1}, t_{k+2}, \ldots, t_{n-1}\}, & i = 2, \\ \{x_1, \ldots, x_k, t_{k+1}, \ldots, t_{k+(i-1)/2}, y_{k+(i+1)/2}, \ldots, y_{n-1}\}, & i \text{ is odd, } i \geq 3, \\ \{z_1, \ldots, z_{k+1}, y_{k+1}, y_{k-1+i/2}, t_{k+1+i/2}, \ldots, t_{n-1}\}, & i \text{ is even, } i \geq 4. \end{cases}$$

This path attains v after $2(n-1-k)$ steps; in other words, we have $w_{2(n-1-k)+1} = v$. If $k = 0$, then the result is obtained. Assume now $k \geq 3$. By the properties of O_n, vertex v is contained in a cycle of length $2k$ ($k \geq 3$), say C. We can make a loop around C. We obtain P.

Case: $|S(u) \cap S(v)| = 1$. Let $S(u) \cap S(v) = \{x_1\}$ and assume $S(u) = \{x_1, y_2, \ldots, y_{n-1}\}$ and $S(v) = \{x_1, t_2 \ldots, t_{n-1}\}$. Let z_1, z_2 be the elements of $\{1, \ldots, 2n-1\} \setminus (S(u) \cup S(v))$.

We leave w_1 by taking the edge labeled with z_2. Hence,

$$S(w_2) = \{z_1, t_2, \ldots, t_{n-1}\}.$$

Now we leave w_i $(3 \leq i \leq 2(n-1)-1)$ by the edge labeled with $y_{i/2+1}$ when i is even, and with $t_{(i-1)/2+1}$ otherwise. It follows that

$$S(w_3) = \{x_1, z_2, y_3, \ldots, y_{n-1}\} \text{ and } S(w_4) = \{z_1, y_2, t_3, \ldots, t_{n-1}\}$$

Moreover, when j is even and $j \geq 4$, we have

$$S(w_j) = \{z_1, y_2, \ldots, y_{j/2}, t_{j/2+1}, \ldots, t_{n-1}\}.$$

and, when j is odd and $j \geq 5$, we have

$$S(w_j) = \{x_1, z_2, t_2, \ldots, t_{(j-1)/2}, y_{(j+1)/2+1}, \cdots y_{n-1}\}.$$

We obtain

$$S\left(w_{2(n-1)}\right) = \{z_1, y_2, \ldots, y_{n-1}\}.$$

It remains to leave $w_{2(n-1)}$ by the edge labeled with z_2. Hence

$$S\left(w_{2(n-1)+1}\right) = \{x_1, t_2, \ldots, t_{n-1}\} = S(v),$$

as claimed.

Case: $|S(u) \cap S(v)| = 2$. Let $S(u) \cap S(v) = \{x_1, x_2\}$ and assume $S(u) = \{x_1, x_2, y_3, \ldots, y_{n-1}\}$ and $S(v) = \{x_1, x_2, t_3, \ldots, t_{n-1}\}$. Let z_1, z_2, z_3 be the elements of $\{1, \ldots, 2n-1\} \setminus (S(u) \cup S(v))$.

We leave w_1 by the edge labeled with z_1, we obtain

$$S(w_2) = \{z_2, z_3, t_3, \ldots, t_{n-1}\}.$$

Then we leave w_2 by the edge labeled with x_1. We have

$$S(w_3) = \{z_1, x_2, y_3, \ldots, y_{n-1}\}.$$

Now we leave w_i $(4 \leq i \leq 2(n-1)-2)$ with the edge labeled with $t_{(i+1)/2+1}$ when i is odd and with $y_{i/2+1}$ otherwise. Hence

$$S(w_i) = \begin{cases} \{x_1, z_2, z_3, t_4, \ldots, t_{n-1}\}, \ i = 4, \\ \{z_1, x_2, t_3, y_4, \ldots, y_{n-1}\}, \ i = 5, \\ \{z_1, x_2, t_3, \ldots, t_{(i+1)/2}, y_{(i+1)/2+1}, \ldots, y_{n-1}\}, \ i \text{ is odd and } i \geq 5, \\ \{x_1, z_2, z_3, y_3, \ldots, y_{i/2}, t_{i/2+2}, \ldots, t_{n-1}\}, \ \ i \text{ is even and } i \geq 6. \end{cases}$$

We obtain

$$S\left(w_{2(n-1)-1}\right) = \{z_1, x_2, t_3, \ldots, t_{n-1}\}.$$

We leave $w_{2(n-1)-1}$ by the edge labeled by x_1. We have

$$S\left(w_{2(n-1)}\right) = \{z_2, z_3, y_3, \ldots, y_{n-1}\}.$$

Finally we leave $w_{2(n-1)}$ by the edge labeled with z_1. We obtain

$$S\left(w_{2(n-1)+1}\right) = \{x_1, x_2, t_3, \ldots, t_{n-1}\}$$

as claimed. This completes the proof of the claim. □

We are now able to exhibit the path P linking u_0 and u_{d+1}. By Claim 7, let $P_s = u_0 s_1 \cdots s_{2(\Delta-1)-1} u_{d+1}$ be a path linking u_0 and u_{d+1} of length $2(\Delta-1)$ in O_Δ. Let u_1 be the neighbor of u_0 so that the edge u_0u_1 is labeled with x. Let u_d be the neighbor of u_{d+1} so that the edge $u_d u_{d+1}$ is labeled with y. As $\Delta \geq 3$, let t be a neighbor of u_0 distinct from u_1 and s_1, and let w be a neighbor of u_{d+1} distinct from u_d and $s_{2(\Delta-1)-1}$. Finally, let C_1 be a 6-cycle containing tu_0u_1 and let C_2 be a 6-cycle containing $wu_{d+1}u_d$.

1. We first start from u_0 making a loop around C_1 going through first u_1. Hence (P2) and (P4) are satisfied.

2. We then leave u_0 to u_{d+1} going through P_s.

3. Finally, we make a loop around C_2 going through first w. Hence (P3) and (P5) are satisfied.

Finally, observe that the length of P is 6 (loop on C_1) plus the length of P_s plus 6 (loop on C_2) that is equal to $2(\Delta - 1) + 12 = 2\Delta + 10 = d + 1$, as required by (P1).

3. Concluding Remark

The proof of Theorem 6 is based on the existence of a path P_s of length exactly $2(n-1)$ in O_n $(n \geq 4)$ between every pair of vertices. One possible way to improve the lower bound on the girth in Theorem 6 would be to decrease the length of P_s. However, the length of P_s is best possible: it does not exist an integer $l < 2(n-1)$ such that every pair of vertices is linked by a path of length exactly l.

Suppose by the way of contradiction that such an l exists and consider the following two cases depending on the parity of l.

Assume l is odd and consider the path P_s (of length l) linking a vertex x with itself. It forms an odd cycle of length strictly less than $2n-1$, contradicting the value of the odd-girth of O_n.

Assume l is even and consider the path P_s (of length l) linking two adjacent vertices x and y. Again, it forms an odd cycle of length strictly less than $2n-1$, contradicting the value of the odd-girth of O_n.

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