Discussiones Mathematicae
Graph Theory 34 (2014) 603–612
doi:10.7151/dmgt.1760

DOWNHILL DOMINATION IN GRAPHS

Teresa W. Haynes

Department of Mathematics and Statistics
East Tennessee State University
Johnson City, TN 37614, USA

e-mail: haynes@etsu.edu

Stephen T. Hedetniemi[1]

School of Computing
Clemson University
Clemson, SC 29634, USA

e-mail: hedet@clemson.edu

and

Jessie D. Jamieson[2]

William B. Jamieson[2]

Department of Mathematics
University of Nebraska-Lincoln
Lincoln, NE 68588, USA

e-mail: jdjamieson@huskers.unl.edu
wjamieson@huskers.unl.edu

Abstract

A path $\pi = (v_1, v_2, \ldots, v_{k+1})$ in a graph $G = (V, E)$ is a *downhill path* if for every i, $1 \leq i \leq k$, $\deg(v_i) \geq \deg(v_{i+1})$, where $\deg(v_i)$ denotes the degree of vertex $v_i \in V$. The *downhill domination number* equals the minimum cardinality of a set $S \subseteq V$ having the property that every vertex $v \in V$ lies on a downhill path originating from some vertex in S. We investigate downhill domination numbers of graphs and give upper bounds. In particular, we show that the downhill domination number of a graph is at most half its order, and that the downhill domination number of a tree is at most one third its order. We characterize the graphs obtaining each of these bounds.

[1]Professor Emeritus.
[2]Work completed while attending East Tennessee State University.

Keywords: downhill path, downhill domination number.

2010 Mathematics Subject Classification: 05C69, 05C38.

1. Introduction

In a graph $G = (V, E)$, the *degree* of a vertex is given by $\deg(v) = |\{u : uv \in E\}|$. A graph G is *r-regular* if $\deg(v) = r$ for every vertex $v \in V$. A *path of length* k in G is a sequence of distinct vertices $v_1, v_2, \ldots, v_{k+1}$, such that for every i, $1 \le i \le k$, $v_i v_{i+1} \in E$. Path parameters with degree constraints were defined in [1]. In particular, a path $v_1, v_2, \ldots, v_{k+1}$ is a *downhill path* if for every i, $1 \le i \le k$, $\deg(v_i) \ge \deg(v_{i+1})$. For $j \ge i$, we say that v_j is *on a downhill path* from v_i, or just that v_j is *downhill* from v_i.

If a vertex v is on a downhill path from a vertex u, then we say that u *downhill dominates* v, or that v is *downhill dominated* by u. As introduced in [1], a *downhill dominating set*, abbreviated DDS, is a set $S \subseteq V$ having the property that every vertex $v \in V$ lies on a downhill path originating from some vertex in S. In other words, the vertices of S downhill dominate the vertices of $V \setminus S$. The *downhill domination number* $\gamma_{dn}(G)$ equals the minimum cardinality of a DDS of G. A DDS having minimum cardinality is called a *γ_{dn}-set* of G. For example, a connected, r-regular graph G has $\gamma_{dn}(G) = 1$ since every vertex is downhill from every other vertex in G. Note that a DDS of a graph G does not necessarily dominate G in the standard sense of domination, and a dominating set of a graph G is not necessarily a DDS of G. In fact, the domination number $\gamma(G)$ and the downhill domination number $\gamma_{dn}(G)$ are incomparable in general. For instance, let G be the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ for $n \ge 6$. If n is even, then $\gamma(G) = 2 > 1 = \gamma_{dn}(G)$. On the other hand, for odd n, $\gamma(G) = 2 < \lfloor \frac{n}{2} \rfloor = \gamma_{dn}(G)$. For more details on domination, see [3].

As an application of downhill dominating sets, we propose a graph model where a vertex has more "power" than those vertices of lesser degree, and a vertex can dominate vertices along a path from it as long as it does not encounter a more "powerful" vertex. Thus, if each vertex represents a military site, for example, a minimum downhill dominating set could represent the minimum number of military bases powerful enough to protect all the sites. It was noted in [1] that although the definition of a downhill path is given in terms of the degrees of the vertices on the path, a similar definition can be given in terms of any function that assigns weights to the vertices of a graph, as is done in surveying when assigning elevations to the points of a topographic map, or in thermal imaging, in which the values assigned to the points in an image are a measure of their heat content.

In [1], we also explored relationships between $\gamma_{dn}(G)$ and other invariants. We obtained an upper bound on $\gamma_{dn}(G)$ in terms of the vertex independence number $\beta_0(G)$ by showing that any minimal DDS is an independent set. Moreover, it was shown in [1] that for any pair of positive integers, a and b, where $a \leq b$, there exists a graph G having $\gamma_{dn}(G) = a$ and $\beta_0(G) = b$. Also, we determined a Vizing-like result for the downhill domination number of a Cartesian product $G \square H$, that is, we showed that $\gamma_{dn}(G \square H) = \gamma_{dn}(G)\gamma_{dn}(H)$.

A well-known result of Ore [5] gives that for any graph G without isolated vertices, $\gamma(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$. Although $\gamma(G)$ and $\gamma_{dn}(G)$ are incomparable, in this paper we show that the same upper bound holds for the downhill domination number. We begin with terminology and preliminary results in Section 2 that will be used in the subsequent sections. In Section 3, we prove an upper bound on the downhill domination number of graphs and characterize the extremal graphs. In Section 4, we improve the bound for trees, define a family $\mathcal{T}$ of trees attaining the bound, and characterize the extremal trees. Specifically, we prove the following main results.

Theorem 1. *If G is a connected graph of order $n \geq 2$, then $\gamma_{dn}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$, with equality if and only if G is one of the complete graphs K_2 or K_3, or the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ of odd order.*

Theorem 2. *If T is a tree of order $n \geq 4$, then $\gamma_{dn}(T) \leq \left\lfloor \frac{n-1}{3} \right\rfloor$, with equality if and only if T is the path of order 4 or $T \in \mathcal{T}$.*

2. Preliminary Results

For a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \,|\, uv \in E\}$ of vertices adjacent to v. The *closed neighborhood* of a vertex $v \in V$ is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ of vertices is $N(S) = \bigcup_{v \in S} N(v)$, while the *closed neighborhood of a set* S is the set $N[S] = \bigcup_{v \in S} N[v]$. An *$S$-external private neighbor* of a vertex $v \in S$ is a vertex $u \in V \setminus S$ which is adjacent to v but to no other vertex of S. The set of all S-external private neighbors of $v \in S$ is called the *S-external private neighbor set* of v and is denoted by $\mathrm{epn}(v, S)$.

We will use of the following result from [1].

Lemma 3 [1]. *Any minimal downhill dominating set of a graph G is an independent set of G.*

In the proofs of our main results, we also make use of degree relationships between a vertex and its neighbors. Hedetniemi, Hedetniemi, Hedetniemi, and Lewis [4] identified the seven possible degree relationships as follows.

Definition 4. A vertex $u \in V(G)$ in a graph G is called

very strong if $\deg(u) \geq 2$ and for every vertex $v \in N(u)$, $\deg(u) > \deg(v)$;

strong if $\deg(u) \geq 2$ and for every vertex $v \in N(u)$, $\deg(u) \geq \deg(v)$, at least one neighbor $v \in N(u)$ has $\deg(u) > \deg(v)$, and at least one neighbor $w \in N(u)$ has $\deg(u) = \deg(w)$;

regular if $\deg(u) \geq 0$ and for every vertex $v \in N(u)$, $\deg(u) = \deg(v)$;

very typical if $\deg(u) \geq 2$, $\deg(u) \neq \deg(v)$ for all $v \in N(u)$, and at least one neighbor $v \in N(u)$ has $\deg(u) > \deg(v)$ and at least one neighbor $w \in N(u)$ has $\deg(u) < \deg(w)$;

typical if $\deg(u) \geq 3$ and there are three distinct vertices $v, w, x \in N(u)$ such that $\deg(v) < \deg(u) = \deg(x) < \deg(w)$;

weak if $\deg(u) \geq 2$ and for every vertex $v \in N(u)$, $\deg(u) \leq \deg(v)$, at least one neighbor $v \in N(u)$ has $\deg(u) < \deg(v)$, and at least one neighbor $w \in N(u)$ has $\deg(u) = \deg(w)$;

very weak if $\deg(u) \geq 1$ and for every vertex $v \in N(u)$, $\deg(u) < \deg(v)$.

For a graph G, let $VS(G)$ be the set of very strong vertices in G, $S(G)$ be the set of strong vertices in G, and $R(G)$ be the set of regular vertices of G. Our next observations follow directly from the above definitions and the minimality of a γ_{dn}-set.

Observation 5. *If D is a γ_{dn}-set of a graph G, then $D \subseteq R(G) \cup S(G) \cup VS(G)$.*

Observation 6. *If D is a γ_{dn}-set of a graph G, then $VS(G) \subseteq D$.*

In order to prove our next result, we need another definition.

Definition 7. Let G be a graph. For a vertex $v \in V(G)$, the *regular path neighborhood of* v, denoted $RPN(v)$, is the set all $u \in V(G)$ such that there is a v-u path $\Pi = (v = x_1, x_2, \ldots, x_k = u)$ for which $\deg(x_i) = \deg(v)$ for $1 \leq i \leq k$.

Lemma 8. *Let G be a connected graph. There exists a γ_{dn}-set of G that contains no regular vertices if and only if G is not regular.*

Proof. If G has a γ_{dn}-set which contains no regular vertices, then G is not regular.

Assume that G is not regular. Among all γ_{dn}-sets of G, select D to minimize $|D \cap R(G)|$, that is, D contains the minimum number of regular vertices. If $D \cap R(G) = \emptyset$, then the result holds. Thus, assume that there is a regular vertex $v \in D$. By Lemma 3, D is independent.

We first show that $RPN(v) \subseteq R(G) \cup S(G)$. Since $v \in R(G)$, v has a neighbor of the same degree. Hence, $RPN(v) \backslash \{v\} \neq \emptyset$. Let $u \in RPN(v)$. Since $\deg(u) = \deg(v)$, it follows from Definition 4 that u is either weak, typical, regular, or strong. Thus assume that u is weak or typical. Then there exists a vertex $y \in N(u)$ such that $\deg(y) > \deg(u)$. Hence, $y \notin RPN(v)$. Further since $\deg(u) = \deg(v)$ and there is a path between u and v consisting of vertices having degree $\deg(v)$, it follows that v does not downhill dominate y. Hence, there is some vertex $w \in D\backslash\{v\}$ such that y is downhill from w or $y = w$. But then w downhill dominates v and all the vertices downhill dominated by v, and so $D\setminus\{v\}$ is a DDS of G with cardinality less than $\gamma_{dn}(G)$, a contradiction. Hence, we may assume that every vertex in $RPN(v)$ is regular or strong.

We note that if $RPN(v) \subseteq R(G)$, then since G is connected, G must be regular, a contradiction. Therefore, there exists a strong vertex, say x, in $RPN(v)$. Further, from the definition of $RPN(v)$, v is downhill from x, implying that every vertex downhill from v is downhill from x. Hence, $(D \setminus \{v\}) \cup \{x\}$ is a γ_{dn}-set of G having fewer regular vertices than D, contradicting our choice of D. ∎

3. Proof of Theorem 1

We shall use the well-known theorem by Hall [2].

Hall's Theorem. *Let G be a bipartite graph with partite sets U and W. Then U can be matched to a subset of W if and only if for all $S \subseteq U$, $|N(S)| \geq |S|$.*

In order to prove Theorem 1, we need to establish some properties of $S(G)$ and $VS(G)$. We present these properties as separate results as they are interesting in their own right.

Proposition 9. *Let G be a connected graph of order $n \geq 3$. If $VS(G) \neq \emptyset$, then $VS(G)$ can be matched to $N(VS(G))$.*

Proof. Let G be a connected graph of order $n \geq 3$ and $VS(G) \neq \emptyset$. Now let $X_e \subseteq E(G)$ be the set of edges having at least one endvertex in $VS(G)$. By Lemma 3 and Observation 6, $VS(G)$ is an independent set. Thus, the edge induced subgraph $G[X_e]$ is a bipartite graph with partite sets $VS(G)$ and $N(VS(G))$. We wish to show that there exists a matching from $VS(G)$ to $N(VS(G))$ in the edge induced subgraph $G[X_e]$. By Hall's Theorem, it suffices to show that for all $X \subseteq VS(G)$, $|N(X)| \geq |X|$.

To establish this, we proceed by induction on $|X|$ for a subset $X \subseteq VS(G)$. Since X is an independent set and G has no isolated vertices, every vertex in X has a neighbor in $N(VS(G))$. Hence, the result holds for $|X| = 1$. For $|X| = 2$, suppose to the contrary that $|N(X)| < |X|$. Again since G has no isolated vertices, we have that $|N(X)| \geq 1$, so $|N(X)| = 1$. But then the two vertices of X each have degree one, while their common neighbor in $N(X)$ has degree at least two, contradicting that the vertices of X are very strong. Thus, $|N(X)| \geq |X| = 2$, and so the result holds for $1 \leq |X| \leq 2$.

Assume that $|N(X)| \geq |X|$ holds for any $X \subseteq VS(G)$ such that $|X| \leq k$ for some $k \geq 2$. Let $|X| = k + 1$, and suppose to the contrary that $|N(X)| < |X|$. Let $X' = X \backslash \{v\}$ for some $v \in X$. Since $X' \subseteq VS(G)$ and $|X'| = k$, by our inductive hypothesis, $|N(X')| \geq |X'|$. Thus, we obtain the following relations

$$|N(X')| \geq |X'| \tag{1}$$

$$|N(X)| < |X| \tag{2}$$

$$|X| = |X'| + 1. \tag{3}$$

From (2) and (3), we have that $|N(X)| \leq |X| - 1 = |X'|$. Thus, by (1), $|N(X)| \leq |X'| \leq |N(X')|$. Since $N(X') \subseteq N(X)$, we have $|N(X)| \geq |N(X')|$. Thus, $|N(X)| = |N(X')|$, implying that $|N(X')| = |X'|$. Moreover, by our inductive hypothesis, $|N(X'')| \geq |X''|$ for all $X'' \subseteq X'$. Thus, by Hall's Theorem, there is a matching in $G[X_e]$ between the vertices of X' and the vertices of $N(X')$. Label the vertices of $X' = \{x_1, x_2, \ldots, x_k\}$ and $N(X') = \{y_1, y_2, \ldots, y_k\}$ such that $M = \{x_i y_i \,|\, 1 \leq i \leq k, x_i \in X' \text{ and } y_i \in N(X')\}$ is a perfect matching.

Now there are exactly $\sum_{i=1}^{k} \deg_G(x_i)$ edges incident to vertices in X' and vertices in $N(X')$, implying that $\sum_{i=1}^{k} \deg_G(y_i) \geq \sum_{i=1}^{k} \deg_G(x_i)$. But since $x_i \in VS(G)$, $\deg(y_i) < \deg(x_i)$ for all $1 \leq i \leq k$, and so $\sum_{i=1}^{k} \deg(y_i) < \sum_{i=1}^{k} \deg(x_i)$, a contradiction.

Thus, we conclude that $|N(X)| \geq |X|$ for every set $X \subseteq VS(G)$ such that $|X| = k + 1$. By the Principle of Mathematical Induction, $|N(X)| \geq |X|$ where $|X| \geq 1$. Therefore, by Hall's Theorem, the set $VS(G)$ can be matched to $N(VS(G))$ in the subgraph $G[X_e]$, and so $VS(G)$ can be matched to $N(VS(G))$ in G. ■

Proposition 10. *Let G be a connected graph of order $n \geq 3$. If $VS(G) \neq \emptyset$, then $|VS(G)| < |N(VS(G))|$.*

Proof. Let G be a connected graph of order $n \geq 3$ and $X = VS(G) \neq \emptyset$. By Proposition 9, X can be matched to the set $N(X)$. Thus, $|X| \leq |N(X)|$. Suppose that $|X| = |N(X)| = k$ and that $X = \{x_1, x_2, \ldots, x_k\}$ and $N(VS(G)) = \{y_1, y_2, \ldots, y_k\}$, where x_i is matched to y_i for $1 \leq i \leq k$. Since X is an independent set, $\sum_{i=1}^{k} \deg(x_i) \leq \sum_{i=1} \deg(y_i)$. However, since x_i is very strong, $\deg(x_i) > \deg(y_i)$ for $1 \leq i \leq k$. Thus, $\sum_{i=1}^{k} \deg(x_i) > \sum_{i=1} \deg(y_i)$, a contradiction. Hence, $|X| < |N(X)|$. ■

Proposition 11. *Let G be a connected graph of order $n \geq 2$. If $v \in D \cap S(G)$, then there exists a vertex $x \in \operatorname{epn}(v, D)$ such that $\deg(x) = \deg(v)$.*

Proof. Let G be a connected graph of order $n \geq 2$ and $v \in D \cap S(G)$. Since v is a strong vertex, there exists $x \in N(v)$ such that $\deg(x) = \deg(v)$. By Lemma 3, D is an independent set, so $x \in V \setminus D$. Suppose to the contrary that $x \notin \operatorname{epn}(v, D)$, that is, x has another neighbor, say y, in D. If $\deg(y) \geq \deg(x)$, then x, v and the vertices downhill from v are downhill from y. Thus, $D \setminus \{v\}$ is a DDS of G with cardinality less than $\gamma_{dn}(G)$, a contradiction. Assume then that $\deg(y) < \deg(x)$. But then y is downhill from v, and so $D \setminus \{y\}$ is a DDS with cardinality less than $\gamma_{dn}(G)$. Hence, we conclude that $x \in \text{epn}(v, D)$, and every strong vertex in D has at least one neighbor of the same degree in its private neighborhood. ■

We are now ready to prove Theorem 1.

Theorem 1. *If G is a connected graph of order $n \geq 2$, then $\gamma_{dn}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$, with equality if and only if G is one of the complete graphs K_2 or K_3, or the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ of odd order.*

Proof. We first prove the upper bound. Let G be a connected graph of order $n \geq 2$. As noted in the introduction, if G is a regular graph, then $\gamma_{dn}(G) = 1$, and the result holds. Suppose now that G is not a regular graph. By Observation 5 and Lemma 8, we may choose a γ_{dn}-set D of G such that $D \subseteq S(G) \cup VS(G)$. By Observation 6, $VS(G) \subseteq D$. To prove the upper bound, it suffices to show that each vertex in D can be uniquely paired with a vertex in $V(G) \setminus D$. By Proposition 11, each vertex of $D \cap S(G)$ has a private neighbor that is of the same degree. Let S' be the set of these private neighbors. Thus, $S(G)$ can be matched to S'. By Proposition 9, there exists a matching from $VS(G)$ to $N(VS(G))$. Further, $S' \cap N(VS(G)) = \emptyset$ and $S' \cup N(VS(G)) \subseteq V \setminus D$. It follows that $\gamma_{dn}(G) \leq |D| = |D \cap S(G)| + |D \cap VS(G)| \leq |S'| + |N(VS(G))| \leq |V \setminus D| = n - \gamma_{dn}(G)$. Hence, $\gamma_{dn}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Next we prove the characterization. Clearly, if $G \in \{K_2, K_3\}$, then $\gamma_{dn}(G) = 1 = \left\lfloor \frac{n}{2} \right\rfloor$, and if $G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ with odd order n, then $\gamma_{dn}(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

Now suppose that G is a connected graph of order $n \geq 2$ with $\gamma_{dn}(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Since for any r-regular graph, $\gamma_{dn}(G) = 1$, if G is regular, then $n = \{2, 3\}$, implying that $G \in \{K_2, K_3\}$. Note also that $\gamma_{dn}(P_3) = 1$ and $P_3 = K_{1,2}$.

Henceforth, we may assume that G is not a regular graph and that $n \geq 4$. Again, we may assume that G has a γ_{dn}-set D, such that $D \subseteq S(G) \cup VS(G)$, D is an independent set, and $VS(G) \subseteq D$.

Let $X = VS(G) = \{x_1, x_2, \ldots, x_j\}$ and $Y = D \cap S(G) = \{y_1, y_2, \ldots, y_k\}$ for some integers j and k. Then $|D| = |X| + |Y| = j + k$.

If $k \geq 1$, then by Proposition 11, every vertex $y_i \in Y$ has a private neighbor $y'_i \in V \setminus D$ such that $\deg(y_i) = \deg(y'_i)$. Let $Y' = \{y'_i \mid 1 \leq i \leq k\}$. Then $|Y'| = |Y|$ and $Y' \cap N(X) = \emptyset$.

If $X = \emptyset$, then $D = Y$, and $|V \setminus D| \leq |D| + 1 = |Y| + 1 = |Y'| + 1$. Since each $y_i \in Y$ is a strong vertex, $\deg(y_i) \geq 2$. Moreover, since each y_i has exactly one external private neighbor in Y' and D is an independent set, it follows that each y_i has at least one neighbor in $V \setminus (D \cup Y')$, that is, $V \setminus (D \cup Y') \neq \emptyset$.

Hence, $|V \setminus D| \geq |Y'| + 1 = |Y| + 1 = |D| + 1$, and so, $|V \setminus D| = |D| + 1$. Then $V \setminus D = Y' \cup \{w\}$ for some vertex w. Since y_i is a strong vertex and $\deg(y_i) = \deg(y_i') \geq 2$, we have $N(y_i) = \{w, y_i'\}$ for $1 \leq i \leq k$. But then $\deg(w) \geq |Y| = |D| = \lfloor \frac{n}{2} \rfloor \geq 2 \geq \deg(y_i) = 2$, implying that no neighbor of y_i has degree less than $\deg(y_i)$, a contradiction since y_i is a strong vertex. Hence, we may assume that $X \neq \emptyset$, that is, $j \geq 1$.

Thus, we have

$$
\begin{aligned}
|Y'| + |N(X)| &\leq |V \setminus D| = \left\lceil \frac{n}{2} \right\rceil \\
|Y| + |N(X)| &\leq \left\lceil \frac{n}{2} \right\rceil \\
|Y| + |N(X)| &\leq |D| + 1 = |X| + |Y| + 1 \\
|N(X)| &\leq |X| + 1
\end{aligned} \tag{4}
$$

Since $X \neq \emptyset$, by Proposition 10, we have $|N(X)| > |X|$, so $|N(X)| = |X| + 1$. By Proposition 9, every vertex in X can be matched with a vertex in $N(X)$. Let $X' = N(X) = \{x_1', x_2', \ldots, x_j'\} \cup \{x\}$, such that $\{x_i x_i' \,|\, 1 \leq i \leq j\}$ is a matching from X to $N(X)$. Since D is an independent set and each vertex in Y' has exactly one neighbor in D, the number of edges incident to vertices of D is $m' = \sum_{i=1}^{k} \deg(y_i) + \sum_{i=1}^{j} \deg(x_i) \leq |Y'| + \sum_{i=1}^{j} \deg(x_i') + \deg(x)$. However, since x_i is very strong, $\deg(x_i) > \deg(x_i')$ for $1 \leq i \leq j$. Thus, $\sum_{i=1}^{j} \deg(x_i) \geq \sum_{i=1}^{j} \deg(x_i') + j$. And since y_i is strong, we have that $\deg(y_i) \geq 2$ for $1 \leq i \leq k$. Hence, $2k + \sum_{i=1}^{j} \deg(x_i') + j \leq \sum_{i=1}^{k} \deg(y_i) + \sum_{i=1}^{j} \deg(x_i) \leq |Y'| + \sum_{i=1}^{j} \deg(x_i') + \deg(x) = k + \sum_{i=1}^{j} \deg(x_i') + \deg(x)$. Thus, $\deg(x) \geq j + k = |X| + |Y| = |D| = \lfloor \frac{n}{2} \rfloor$. Since m' counts only the edges incident to a vertex in D and to a vertex in $V \setminus D$, it follows that x is adjacent to every vertex in D. Since $X \neq \emptyset$ and every vertex $x_i \in X$ is very strong, it follows that $\deg(x_i) > \deg(x) = \lfloor \frac{n}{2} \rfloor$ for $1 \leq i \leq j$. Since D is independent, we conclude that $N(x_i) = V \setminus D$ for each $x_i \in X$. But $V \setminus D = Y' \cup N(X)$ and $Y' \cap N(X) = \emptyset$, implying that $Y = \emptyset$. It follows that $|D| = |X| = j$ and $|V \setminus D| = |N(X)| = |X| + 1 = j + 1$. Moreover, since $\deg(x_i') < \deg(x_i)$ for $1 \leq i \leq j$, we have that $\deg(x_i') < |V \setminus D| = j + 1$. On the other hand, every vertex in D is adjacent to every vertex in X', and so $\deg(x_i') \geq |D| = j$, implying that $\deg(x_i') = j$ and $N(x_i') = D$. Thus, $V \setminus D$ is an independent set, and G is the complete bipartite graph $K_{j,j+1}$, as desired. ■

4. PROOF OF THEOREM 2

As we have seen, $\lfloor n/2 \rfloor$ is a sharp upper bound on $\gamma_{dn}(G)$ for connected graphs G. Restricting our attention to trees, we can improve this bound. For the purpose of characterizing the trees attaining this bound, we introduce a family $\mathcal{T}$ of trees

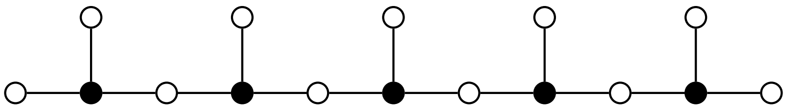

Figure 1. The caterpillar with code (2,0,1,0,1,0,1,0,2).

$T = T_k$ that can be obtained as follows. Let T_1 be a claw $K_{1,3}$. If $k \geq 2$, then for $1 \leq i \leq k-1$, T_{i+1} can be obtained recursively from T_i by attaching a path P_3 with an edge from the center of the added P_3 to a leaf in T_i. In other words, T is a tree with a set S of vertices of degree 3, $|S| = (n-1)/3$, S is an independent set, $V \setminus S$ is an independent set, and each vertex in $V \setminus S$ has degree 1 or 2.

For example, we consider a family of caterpillars. A *caterpillar* is a tree for which the removal of its leaves results in path, called its *spine*. The code of the caterpillar having spine $P_k = (v_1, v_2, \ldots, v_k)$ is the ordered k-tuple $(l_1, l_2, \ldots, l_k)$, where l_i is the number of leaves adjacent to v_i. The set of caterpillars with code $(2, 0, 1, 0, 1, 0, \ldots, 1, 0, 1, 0, 2)$ is a subset of $\mathcal{T}$. See Figure 1 for a caterpillar in this subfamily.

Theorem 2. *If T is a tree of order $n \geq 4$, then $\gamma_{dn}(T) \leq \left\lfloor \frac{n-1}{3} \right\rfloor$, with equality if and only if T is the path of order 4 or $T \in \mathcal{T}$.*

Proof. Let T be a tree of order $n \geq 4$. We first prove the upper bound. Note that if $\Delta(T) = 2$, then T is a path, and since $n \geq 4$, we have that $\gamma_{dn}(P_n) = 1 \leq \left\lfloor \frac{n-1}{3} \right\rfloor$. Hence, we may assume that $\Delta(T) \geq 3$.

Assume, for the purpose of a contradiction, that $\gamma_{dn}(T) > \left\lfloor \frac{n-1}{3} \right\rfloor$. By Lemma 8, T has a γ_{dn}-set D such that $D \subseteq S(T) \cup VS(T)$. To reach a contradiction, we show that T has size $m > n-1$, that is, T has a cycle. Since by Lemma 3, D is an independent set, every edge incident to a vertex in D is incident to a vertex in $V \setminus D$. Thus, it suffices to show that each vertex in D has degree at least 3 because this implies that $m \geq 3(\left\lfloor \frac{n-1}{3} \right\rfloor + 1) > n-1$.

Assume to the contrary that there exists a vertex $u \in D$ with $\deg(u) \leq 2$. Then u is either strong or very strong, so $\deg(u) = 2$. Since T is connected and $n \geq 4$, it follows that $u \in S(T)$, and u is adjacent to a leaf and to a vertex, say w, of degree two. If w is downhill from a vertex in D, then so is u and its leaf neighbor, implying that $D \setminus \{u\}$ is a DDS with cardinality less than $\gamma_{dn}(T)$, a contradiction. Thus, w is not downhill from any vertex in $D \setminus \{u\}$. Let $v = w_1, w_2, \ldots, w_k = w$ be a v-w path for some $v \in D \setminus \{u\}$. Since the v-w path is not a downhill path, there exists a w_i such that $\deg(w_{i+1}) > \deg(w_i)$. Let i be the largest index such that $\deg(w_{i+1}) > \deg(w_i)$ on the v-w path. Since $\deg(w) = 2 < \deg(w_{i+1})$, we have that $w \neq w_{i+1}$ and w is downhill from w_{i+1}. Therefore, $w_{i+1} \notin D$, and so $w_{i+1} \in V \setminus D$. Thus, there exists a downhill path from some vertex $v' \in D$ to w_{i+1}. But then w is downhill from v', a contradiction. Hence, we may conclude that every vertex in D has degree at least 3, proving the upper bound.

Clearly, the bound is sharp for the path P_4 and the claw $K_{1,3}$. Let T be a tree in $\mathcal{T}$ with $n \geq 5$ vertices. By the construction of T, the set of $\frac{n-1}{3}$ vertices of degree 3 in T are very strong vertices of T. By Observation 6, we have that every very strong vertex is in every γ_{dn}-set of T, so $\gamma_{dn}(T) \geq \frac{n-1}{3}$. Hence, $\gamma_{dn}(T) = \frac{n-1}{3}$.

Next, let T be a tree with order $n \geq 4$ and $\gamma_{dn}(T) = \frac{n-1}{3}$. Then $n-1$ is divisible by 3. If $n = 4$, then $T \in \{P_4, K_{1,3}\}$, the result holds. Thus, assume that $n \geq 7$. We show that $T = T_k \in \mathcal{T}$.

Let D be a γ_{dn}-set of T. By our previous argument, every vertex in D has degree at least three. Since $|D| = \frac{n-1}{3}$ and D is independent, $3(\frac{n-1}{3}) \leq m = n-1$. It follows every vertex of D has degree 3, and the edges of T are precisely the edges incident to a vertex of D and a vertex of $V \setminus D$. In other words, both D and $V \setminus D$ are independent sets. Note that a pair of vertices in D have at most one common neighbor in $V \setminus D$, else a cycle is formed. To show that $T \in \mathcal{T}$, it suffices to show that every vertex of $V \setminus D$ has degree 1 or 2.

Assume to the contrary, that $u \in V \setminus D$ and $\deg(u) \geq 3$. Without loss of generality, let v_1, v_2, and v_3 be neighbors of u. Necessarily, $v_i \in D$, for $1 \leq i \leq 3$. But then $(D \setminus \{v_1, v_2, v_3\}) \cup \{u\}$ is a DDS of T having cardinality less than $\gamma_{dn}(T)$, a contradiction. Hence, $T \in \mathcal{T}$. ■

It should be noted that a graph G having a downhill domination number less than $\left\lfloor \frac{n-1}{3} \right\rfloor$ does not imply that G is acyclic. Consider the complete graph K_n, for example.

References

[1] T.W. Haynes, S.T. Hedetniemi, J. Jamieson and W. Jamieson, *Downhill and uphill domination in graphs*, submitted for publication (2013).

[2] P. Hall, *On representation of subsets*, J. London Math. Soc. **10** (1935) 26–30.

[3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, 1998).

[4] J.D. Hedetniemi, S.M. Hedetniemi, S.T. Hedetniemi and T. Lewis, *Analyzing graphs by degrees*, AKCE Int. J. Graphs Comb., to appear.

[5] O. Ore, Theory of Graphs (Amer. Math. Soc. Colloq. Publ. 38, 1962).

Received 8 April 2013
Revised 9 September 2013
Accepted 9 September 2013