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DEGREE SEQUENCES OF MONOCORE GRAPHS

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Abstract

A k-monocore graph is a graph which has its minimum degree and degeneracy both equal to k. Integer sequences that can be the degree sequence of some k-monocore graph are characterized as follows. A nonincreasing sequence of integers $d_1, \ldots, d_n$ is the degree sequence of some k-monocore graph G, $0 \leq k \leq n-1$, if and only if $k \leq d_i \leq min\{n-1, k+n-i\}$ and $\sum d_i = 2m$, where m satisfies $\lceil \frac{k \cdot n}{2} \rceil \leq m \leq k \cdot n - \binom{k+1}{2}$.

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1. Introduction

One of the basic properties of graphs is the existence of subgraphs with specified degree conditions. (See [10] and [19] for basic terminology.)

Definition. The *k-core* of a graph G, $C_k(G)$, is the maximal induced subgraph $H \subseteq G$ such that the minimum degree $\delta(H) \geq k$, if it exists.

Cores were introduced by Seidman [17] and have been studied extensively in [5]. They have mostly been studied in the context of random graph theory (e.g. [15]).

Cores have applications outside of mathematics. Seidman briefly explores social networks in his paper. Cores have applications in computer science to network visualization [2, 13]. They also have applications in bioinformatics [1, 3, 20].

It is easy to show that the k-core is well-defined and that the cores of a graph are nested. There is a simple algorithm for determining the k-core of a graph, which we shall call the k-core algorithm.

Algorithm 1 (k-core Algorithm)**.** Iteratively delete vertices of degree less than k until none remain.

It is straightforward to show that this will produce the k-core if it exists. This algorithm runs in $O(m)$ time [4]. This suggests a way to order the vertices of a graph by successively deleting or adding vertices of small degree.

Definition. A *vertex deletion sequence of a graph* G is a sequence that contains each of its vertices exactly once and is formed by successively deleting a vertex of smallest degree. A *construction sequence of a graph* is the reversal of a corresponding deletion sequence.

We can also consider the maximum value in a deletion sequence.

Definition [14]**.** A graph is k-*degenerate* if its vertices can be successively deleted so that when deleted, each has degree at most k. The *degeneracy* $D(G)$ of a graph G is the smallest k such that it is k-degenerate.

The maximum core number of a graph is the maximum k such that G has a k-core. As a corollary of the k-core algorithm, we have the following min-max relationship.

Corollary 2. *For any graph, its maximum core number is equal to its degeneracy.*

It is immediate that the degeneracy of a graph is bounded by its minimum and maximum degrees, $\delta(G) \leq D(G) \leq \triangle(G)$. It is easy to characterize the extremal graphs for the upper bound. Instead, we consider the extremal graphs for the lower bound $\delta(G) \leq D(G)$.

Definition. A graph G is k-*monocore* if $D(G) = \delta(G)$.

Thus a graph is k-monocore if and only if $C_i(G) = \begin{cases} G & i \leq k, \\ \emptyset & i > k. \end{cases}$ Many important classes of graphs are monocore. These include regular graphs, trees, forests without trivial components, complete multipartite graphs, wheels, maximal outerplanar graphs, [5] and minimally k-connected graphs ([8] p. 21–24). The 0-monocore graphs are exactly the empty graphs, the 1-monocore graphs are exactly the nontrivial trees, and the 2-monocore graphs are characterized in [5]. Monocore graphs have applications to several problems in graph coloring [7].

A maximal k-degenerate graph cannot have any more edges added and remain k-degenerate. The size of a maximal k-degenerate graph was determined by Lick and White [14].

Theorem 3. *The size of a maximal k-degenerate graph with order n is $k \cdot n - \binom{k+1}{2}$.*

Other properties of maximal k-degenerate graphs were studied in [16, 18, 12] and [6]. The following lemmas are used in characterizing the degree sequences of maximal k-degenerate graphs, and are also essential for proving the main theorem.

Lemma 4 [6]**.** *Let G be maximal k-degenerate with order n and nonincreasing degree sequence $d_1, \ldots, d_n$. Then $d_i \leq k + n - i$.*

Proof. Assume to the contrary that $d_i > k + n - i$ for some i. Let H be the graph formed by deleting the $n - i$ vertices of smallest degree. Then $\delta(H) > k$, so G has a $k+1$-core. ■

Lemma 5 [6]**.** *Let $d_1, \ldots, d_n$ be nonincreasing sequence of integers with $\sum d_i = 2\left[k \cdot n - \binom{k+1}{2}\right]$ such that $k \leq d_i \leq min\{n-1, k+n-i\}$. Then at most $k+1$ terms of the sequence achieve the upper bound.*

Proof. Visualize the problem as stacking boxes in adjacent columns so that the height of the i^{th} column is d_i. If all the terms other than d_n that achieve the upper bound are at the beginning of the sequence, then there are at most k, since $\sum d_i = 2k \cdot n - k(k+1) = k(n-1) + (n-k)k$. Filling the row at height $k+1$ would require $n-k-1$ more boxes, which would have to be moved from at least two of the columns. Similarly, filling more rows requires disrupting at least as many columns. Thus there are at most $k+1$ terms that achieve the upper bound when all the columns that achieve the upper bound are at the beginning or end of the sequence. Suppose there is a sequence that is a counterexample, and let it maximize the number of columns at the beginning or end that achieve the maximum. There must be a column somewhere in the middle that achieves the upper bound. Then some boxes can be moved to a column or row next to the the run of those at the beginning or end that to achieve the upper bound, producing a contradiction. ■

The following theorem characterizes degree sequences of maximal k-degenerate graphs. A different characterization was offered in [9].

Theorem 6 [6]**.** *A nonincreasing sequence of integers $d_1, \ldots, d_n$ is the degree sequence of a maximal k-degenerate graph G if and only if $k \leq d_i \leq \min\{n-1, k+n-i\}$ and $\sum d_i = 2\left[k \cdot n - \binom{k+1}{2}\right]$ for $0 \leq k \leq n-1$.*

2. The Main Theorem

We will first examine the maximal k-monocore graphs. In fact, these are just maximal k-degenerate graphs. We have already seen that maximal k-degenerate graphs are k-monocore. A partial converse to this result is true.

Lemma 7. *Every k-monocore graph is contained in a maximal k-degenerate graph.*

Proof. Let G be k-monocore. Determine a deletion sequence for G, and reverse it to obtain a construction sequence. Now construct graph G' by adding not only the edges of G, but enough extra edges so that $\min\{k, i-1\}$ edges are added when the i^{th} vertex is added, which is always possible since there are $i-1$ vertices available. The resulting graph is maximal k-degenerate. ∎

Adding an edge to a maximal k-degenerate graph creates a $k+1$-core, so maximal k-monocore graphs are maximal k-degenerate.

We can use this lemma to determine sharp bounds on the size of a k-monocore graph.

Proposition 8. *The size m of a k-monocore graph G of order n satisfies*

$$\left\lceil \frac{k \cdot n}{2} \right\rceil \leq m \leq k \cdot n - \binom{k+1}{2}.$$

Proof. The sum of the degrees of G is at least $k \cdot n$, so $m \geq \left\lceil \frac{k \cdot n}{2} \right\rceil$. The upper bound follows from the previous lemma. ∎

Both bounds are sharp. The graphs achieving the upper bound are maximal k-degenerate graphs. For n or k even, the graphs achieving the lower bound are just regular graphs, and for n and k both odd, they are graphs with exactly one vertex of degree $k+1$, and all others of degree k.

Some observations about the degree sequences of k-monocore graphs are immediate.

Lemma 9. *If a nonincreasing sequence of integers $d_1, \ldots, d_n$ is the degree sequence of a k-monocore graph G, $0 \leq k \leq n-1$, then $k \leq d_i \leq \min\{n-1, k+n-i\}$ and $\sum d_i = 2m$, where $\left\lceil \frac{k \cdot n}{2} \right\rceil \leq m \leq k \cdot n - \binom{k+1}{2}$.*

Proof. For the first inequalities, the lower bound is obvious, and the upper bound follows from the corresponding result for maximal k-degenerate graphs. The latter equation follows from the First Theorem of Graph Theory and the previous result. ∎

Let A be the set of sequences satisfying the conclusion of Lemma 9. Our main theorem states that the converse to this result holds, that is, every sequence in A is the degree sequence of some k-monocore graph. We show that the converse is equivalent to a simpler statement that limits how many sequences we need to consider.

Lemma 10. *Let B be the set of nonincreasing sequences of integers $d_1, \ldots, d_n$ satisfying $d_1 \le n-1$, $d_k = d_n = k$, and $\sum d_i = 2m$. The converse to Lemma 9 holds if and only if every sequence in B is the degree sequence of some k-monocore graph.*

Proof. ($\Rightarrow$) Since $B \subseteq A$, if every sequence in A is the degree sequence of some k-monocore graph, then so is every sequence in B.

($\Leftarrow$) Assume that every sequence in B is the degree sequence of some k-monocore graph. We use induction on the order r. If $r = k+1$, the only possible sequence is $k+1$ k's, so $G = K_{k+1}$, which is k-monocore. Hence we assume that every sequence in A with order $r \ge k+1$ is the degree sequence of some k-monocore graph. Let $D : d_1, \ldots, d_{r+1}$ be a sequence in A. If D has fewer than k integers larger than k, then $D \in B$, so it is the degree sequence of some k-monocore graph by assumption. (In fact, any graph G satisfying D is a k-core since $d_n = \delta(G) = k$ and G cannot have a $k+1$-core, so G is k-monocore.)

Hence we assume additionally that D has at least k integers larger than k, so $D \notin B$. Let $D' : d'_1, \ldots, d'_r$ be the sequence formed by deleting $d_{r+1} = k$ and decreasing k other numbers greater than k by one, including any that achieve the upper bound. (There are at most k by Lemma 5.) Then $D' \in A$ and has length r, so it is the degree sequence for some k-monocore graph H. Add vertex v_{r+1} to H, making it adjacent to the k vertices with degrees that were decreased to form D'. Then the resulting graph G has degree sequence D and is k-monocore. ■

In light of this lemma, we need only consider sequences that end with many k's to prove the converse to Lemma 9. We use two operations to limit the number of k's at the end of the sequence that we must consider. They require that a large enough independent set of edges exists in a k-core.

Proposition 11. *The edge independence number $\alpha'(G)$ of a k-core G satisfies $\alpha'(G) \ge \left\lceil \frac{k}{2} \right\rceil$ and the graphs for which this is an equality are exactly empty graphs, stars, complete graphs, and $K_{2i+1} - tK_2$, $1 \le t \le i$.*

Proof. We begin with the special cases $k = 0, \ldots, 3$. Certainly the result holds for empty graphs, which are exactly the 0-monocore graphs. Obviously any nontrivial tree contains an edge and only stars have diameter at most two, so the result holds for $k = 1$. Any 2-core contains a cycle and only C_3 has edge independence number one. If a 2-core contains more than one triangle, then each contains an edge not on the other, so the result holds for $k = 2$.

Deleting two vertices from a 3-core leaves a 1-core, so the bound holds. Equality certainly holds for the indicated graphs, so assume it holds for a 3-core with $n \ge 6$. Deleting two vertices must leave a star, and each of its leaves must be adjacent to both of the vertices deleted. But then the graph contains $K_{3,3}$, producing a contradiction.

We use induction on $r \geq 2$. Assume the bound and extremal graphs hold for all k with $2 \leq k \leq r$ and let G be an $r+2$-core. Let $e = uv$ be an edge of G. Then $G - u - v$ is an r-core, so $\alpha'(G - u - v) \geq \lceil \frac{r}{2} \rceil$, and $\alpha'(G) \geq \lceil \frac{r}{2} \rceil + 1 = \lceil \frac{r+2}{2} \rceil$. Now equality holds only if $G - u - v$ is a clique or $K_{2i+1} - tK_2$; hence so is G. ■

Operation 12. Add one vertex of degree $k = 2r$. Subdivide r independent edges and identify the r new vertices. This produces a graph with all the same degrees as before plus one more vertex of degree k.

Note that by the proposition above, if $k = 2r + 1$ is odd, then a k-core must contain $2r$ edges which use each vertex at most twice, since it contains two disjoint independent sets of r edges.

Operation 13. Add two vertices of degree $k = 2r + 1$. Delete $2r$ edges which use each vertex at most twice, add two adjacent vertices, and make each of them adjacent to $2r$ of the endpoints of the deleted edges. This produces a graph with a degree sequence that adds two k's to the degree sequence of the original graph.

Now we can prove the main theorem.

Theorem 14. *A nonincreasing sequence of integers $d_1, \ldots, d_n$ is the degree sequence of some k-monocore graph G, $0 \leq k \leq n-1$, if and only if $k \leq d_i \leq \min\{n-1, k+n-i\}$ and $\sum d_i = 2m$, where m satisfies $\lceil \frac{k \cdot n}{2} \rceil \leq m \leq k \cdot n - \binom{k+1}{2}$.*

Proof. ($\Rightarrow$) The forward direction is just Lemma 9.

($\Leftarrow$) We use induction on k. For $k = 0$, it is obvious. Assume the result holds for $k \geq 1$. By Lemma 10, the result will hold if it holds for sequences with at most $k-1$ integers larger than k. Let D be such a sequence of length n. We may assume that d_1 is $n-1$ or $n-2$, since otherwise we may delete some k's so that this holds, obtain a graph for this shorter sequence, and use the above operations to obtain a graph with the longer sequence.

If $d_1 = n-1$, then the sequence D' formed by deleting v_1 and reducing every other element by one has at most $k-2$ integers larger than $k-1$. Thus it is the degree sequence of a $k-1$-monocore graph H by the induction hypothesis, and $G = H + v$ is k-monocore. If $d_1 = n-2$, then the sequence D' formed by deleting v_1 and reducing all integers but one of the k's by one has at most $k-1$ integers larger than $k-1$. Thus it is the degree sequence of a $k-1$-monocore graph H by the induction hypothesis, and the graph G formed by joining a vertex to the vertices of H with degrees that had been reduced is k-monocore. Thus the result holds for k-monocore graphs by induction. ■

Thus Theorem 6 can be proven as a corollary of this theorem.

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