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MAXCLIQUE AND UNIT DISK CHARACTERIZATIONS OF STRONGLY CHORDAL GRAPHS

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Abstract

Maxcliques (maximal complete subgraphs) and unit disks (closed neighborhoods of vertices) sometime play almost interchangeable roles in graph theory. For instance, interchanging them makes two existing characterizations of chordal graphs into two new characterizations. More intriguingly, these characterizations of chordal graphs can be naturally strengthened to new characterizations of strongly chordal graphs

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1. Maxcliques and Unit Disks

A graph is *chordal* if every cycle of length at least 4 has a *chord* (an edge that connects two nonconsecutive vertices of the cycle); see [2, 12]. But the following unconventional characterization is more relevant here, using the notions of *maxclique* (a maximal complete subgraph) and *unit disk* (a closed neighborhood of a vertex).

> A graph G is chordal if and only if every induced subgraph H of G has a subgraph H' that is simultaneously a maxclique of H and a unit disk of H.

This is equivalent to the traditional characterization of chordal graphs by every induced subgraph H containing a *simplicial vertex* (a vertex v such that the closed neighborhood $N[v] = H'$ induces a complete subgraph of H—in fact, the unit disk H' induces a maxclique of H; see [2, 12]). Unless stated otherwise, all neighborhoods in this paper are with respect to the graph G, thus $N(v) = N_G(v)$ and $N[v] = N_G[v]$.

Unit disks and maxcliques will refer interchangeably to induced subgraphs and their vertex sets, whichever is convenient. It is important to notice that a unit disk can be the closed neighborhood of more than one vertex, and so a graph can have fewer unit disks than it has vertices. In the extreme case, complete graphs consist of a single unit disk (and, of course, a single maxclique).

This paper will study the interchangeability of maxcliques and unit disks for chordal graphs. This will produce several new characterizations of *strongly chordal graphs*—chordal graphs in which every cycle of even length at least 6 has a chord that forms two even cycles with the edges of the original cycle (see [2, 12]). Moreover, each of these new characterizations will be a natural strengthening of a characterization of chordal graphs. Examples 7 and 8 of [8] also exhibit this interchangeability in characterizing strongly chordal graphs.

This interchangeability of maxcliques and unit disks is related to what happens in the theory of "dually chordal graphs", as in [1, 3, 4, 10]. The duality referred to there is "hypergraph duality", which involves interchanging maxcliques with unit disks and also, simultaneously, interchanging intersections with unions, subsets with supersets, and so on. The role of maxcliques and unit disks in hypergraph duality is illustrated by the following relationship, which underlies [10]. In every graph, every unit disk $N[v]$ is the union of maxcliques (namely, all the maxcliques Q that have $v \in V(Q)$) and every maxclique Q is the intersection of unit disks (namely, all the unit disks $N[v]$ that have $v \in V(Q)$).

2. Cycles and Their Chords

Define the *clique strength* of a nonempty $S \subseteq V(G)$, denoted as $\mathrm{cstr}_G(S)$, to be the number of maxcliques of G that contain S, and define a subgraph H of G to be a cstr_G-k subgraph if $\mathrm{cstr}_G(V(H)) \geq k$. (These notions are called *strength* and *strength-k subgraph* in [7, 11].) Since every edge and triangle is trivially a cstr-1 subgraph, G is chordal if and only if every cycle of cstr_G-1 edges either has a cstr_G-1 chord or is a cstr_G-1 triangle. Proposition 1 shows how strongly chordal graphs strengthen this.

Proposition 1 [7]. *A graph G is strongly chordal if and only if, for every $k \geq 1$, every cycle of* cstr_G*-k edges either has a* cstr_G*-k chord or is a* cstr_G*-k triangle.*

Similarly, define the *disk strength* of a nonempty $S \subseteq V(G)$, denoted as $\mathrm{dstr}_G(S)$, to be the number of unit disks of G that contain S, and define a subgraph H of G to be a dstr_G-k *subgraph* if $\mathrm{dstr}_G(V(H)) \geq k$. Since every edge and triangle is trivially a dstr-1 subgraph, G is chordal if and only if every cycle of dstr_G-1 edges either has a dstr_G-1 chord or is a dstr_G-1 triangle. Theorem 2 below will be the dstr_G-k analog to Proposition 1.

Figure 1 illustrates the difference between the clique strength and disk strength of edges. Each triangle in this strongly chordal example has clique strength 1, two triangles have disk strength 3 and four have disk strength 4. Each vertex has clique strength 1, 2, or 3 and disk strength 3, 4, 5, or 6. Disk strengths can be trickier to calculate when the graph has *adjacent twins*, which are vertices $v \neq w$ with $N[v] = N[w]$. In the extreme case of a complete graph, every two vertices are adjacent twins and every subgraph has clique strength 1 and disk strength 1.

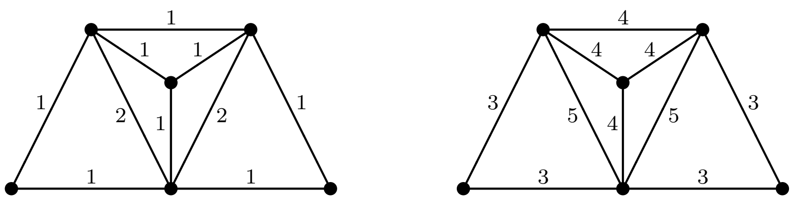

Figure 1. Each edge has its clique strength shown on the left copy and its disk strength shown on the right copy.

The proofs of Theorems 2, 5, and 6 will use two characterizations from Farber [5] (or see [2, 12]). By the first of these, a chordal graph G is strongly chordal if and only if no induced subgraph of G is an n-*sun*, i.e. a subgraph H that consists of a length-n cycle $w_1, \ldots, w_n, w_1$ (so $n \geq 3$) whose vertices induce a complete graph K_n together with additional vertices $u_1, \ldots, u_n$ that have open neighborhoods $N(u_i) = \{w_i, w_{i+1}\}$, when $1 \leq i < n$ and $N(u_n) = \{w_n, w_1\}$. Figure 2 shows examples in which the vertices w_i are pictured as solid and the vertices u_i as hollow.

By the second result from Farber [5], a graph G is strongly chordal if and only if every induced subgraph H of G contains a *simple vertex*, i.e. a vertex v such that $N_H(v) = \{w_1, \ldots, w_\delta\}$ in H where the vertices w_i are linearly ordered so that the unit disks $N_H[w_i]$ of $N_H^2[v]$ satisfy $N_H[v] \subseteq N_H[w_1] \subseteq \cdots \subseteq N_H[w_\delta]$ (where $N_H^2[v]$ denotes the subgraph induced by all the vertices of H that are a distance 2 or less from v in H). Since $N_H[w_i] \subseteq N_H[w_j]$ with $i \neq j$ implies that w_i and w_j are adjacent, simple vertices are always simplicial vertices. Each $N_H[w_i]$

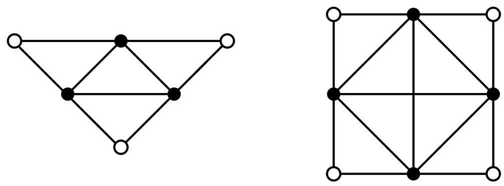

Figure 2. The 3-sun and 4-sun graphs.

is a unit disk of H, but $N_H[w_i]$ is only a maxclique of H if $N_H[w_i] = N_H[v]$ in H. Figure 3 shows an extremely simple example of $N_H^2[v]$ for a simple vertex v with $\delta = 3$.

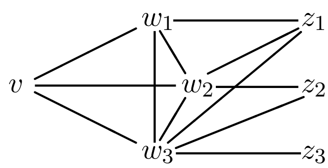

Figure 3. The subgraph $N_H^2[v]$ for a simple vertex v of a graph H.

Theorem 2. *A graph G is strongly chordal if and only if, for every $k \geq 1$, every cycle of* dstr_G*-k edges either has a* dstr_G*-k chord or is a* dstr_G*-k triangle.*

Proof. Argue the "only if" direction by induction on $|V(G)|$. The $|V(G)| = 1$ basis case is trivial since K_1 has no cycles. Suppose G is strongly chordal with $|V(G)| \geq 2$ and, for every $k \geq 1$ and every proper induced subgraph H of G, every cycle of dstr_H-k edges either has a dstr_H-k chord or is a dstr_H-k triangle. Since G is strongly chordal, G contains a simple vertex v with $N(v) = \{w_1, \ldots, w_\delta\}$ and $N[v] \subseteq N[w_1] \subseteq \cdots \subseteq N[w_\delta]$. Suppose $k \geq 1$ and C is a cycle of dstr_G-k edges.

Suppose for the moment that $v \in V(C)$. Say vw_i and vw_j are the edges incident with v along C, and recall that w_i is adjacent to w_j in the maxclique $N[v]$. Since $\mathrm{dstr}_G(vw_i) \geq k$, at least k of the vertices $v, w_1, \ldots, w_\delta$ must have pairwise-distinct closed neighborhoods in G. Therefore, w_iw_j is a dstr_G-k edge that is either a chord of C or is an edge of C where C is a dstr_G-k triangle.

Next suppose instead that $v \notin V(C)$. If no two consecutive vertices of C are in $N(v)$, then C is a cycle of dstr_H-k edges where H is the proper induced subgraph $G - v$ of G, and so the induction hypothesis implies that C either has a dstr_H-k chord (that is also a dstr_G-k chord) or is a dstr_H-k triangle (that is also a dstr_G-k triangle). Now suppose w_i and w_j are two consecutive vertices of C in $N(v)$, say with $i < j$ and with zw_i the edge adjacent to w_iw_j along C. Since $N[w_i] \subseteq N[w_j]$, vertex z is adjacent to w_j and every common neighbor of z and

w_i is also a neighbor of w_j. Since zw_i is a dstr$_G$-k edge, zw_j is a dstr$_G$-k edge that is either chord of C or an edge of the dstr$_G$-k triangle C.

Argue the "if" direction by supposing that G is not strongly chordal. Therefore, G contains an induced subgraph H that is either a length-n cycle with $n \geq 4$ or an n-sun with $n \geq 3$. In the former case, H would be a cycle of dstr$_G$-1 edges that has no dstr$_G$-1 chord and is not a dstr$_G$-1 triangle. In the latter case, the length-n cycle of the n-sun H whose vertices induce a complete subgraph would be a cycle of dstr$_G$-$(n+1)$ edges that has no dstr$_G$-$(n+1)$ chord and is not a dstr$_G$-$(n+1)$ triangle. ■

3. Euler-type Characteristics

For any graph G, let $c_i(G)$ denote the number of cardinality-i subsets of $V(G)$ that are contained in maxcliques of G. Thus $c_1(G)$ is the number of vertices, $c_2(G)$ is the number of edges, $c_3(G)$ is the number of triangles, and so on. Let $\chi(G)$ denote the *clique characteristic* $c_1(G) - c_2(G) + c_3(G) - \cdots$ of G (this alternating sum has also been called the *Euler characteristic*, or simply the *characteristic*, for example in [13]).

For instance, if G is the 3-sun, then $c_1(G) = 6$, $c_2(G) = 9$, and $c_3(G) = 4$, with $c_i(G) = 0$ whenever $i \geq 4$; thus $\chi(G) = 6 - 9 + 4 = 1 = \text{comp}(G)$ (the number of components of G).

Proposition 3 [6]. *A graph is chordal if and only if every induced subgraph H satisfies $\chi(H) = \text{comp}(H)$.*

Similarly, let $d_i(G)$ denote the number of cardinality-i subsets of $V(G)$ that are contained in unit disks of G. Thus, in particular, $d_1(G)$ is the number of vertices in G and $d_2(G)$ is the number of pairs of vertices that are at distance at most two apart in G. Let $\widehat{\chi}(G)$ denote the *disk characteristic* $d_1(G) - d_2(G) + d_3(G) - \cdots$ of G. (The alternating sum $\widehat{\chi}(G)$ resembles—but is very different from—the "neighborhood characteristic" defined in [9], which uses open neighborhoods). Also notice that $d_i(G)$ is not the hypergraph dual of $c_i(G)$—that would be the number of subgraphs of order i that *contain* unit disks of G.)

For instance, if G is the 3-sun, then $d_1(G) = 6$, $d_2(G) = 15$, $d_3(G) = 19$, $d_4(G) = 12$, and $d_5(G) = 3$, with $d_i(G) = 0$ whenever $i \geq 6$. Thus $\widehat{\chi}(G) = 6 - 15 + 19 - 12 + 3 = 1 = \text{comp}(G)$.

Let $N^2(v) = N^2[v] - \{v\}$. A set $S \subset V(G)$ is *dominated* by a vertex v if $S \subseteq N[v]$ (whether or not $v \in S$).

Theorem 4. *A graph is chordal if and only if every induced subgraph H satisfies $\widehat{\chi}(H) = \text{comp}(H)$.*

Proof. Argue the "only if" direction by induction on $|V(G)|$. The $|V(G)| = 1$ basis case follows from $\widehat{\chi}(K_1) = 1$. Suppose that G is chordal—and so every induced subgraph of G is chordal—with $|V(G)| \geq 2$ and $\widehat{\chi}(H) = \text{comp}(H)$ for every proper induced subgraph H of G (toward showing $\widehat{\chi}(G) = \text{comp}(G)$).

Let v be a simplicial vertex of G with degree $\delta \geq 0$, let $H = G - v$, and let $\pi = |N^2(v)| \geq \delta$. For each $j \geq 1$, the subsets of $N^2[v]$ of cardinality j that are not subsets of $V(H)$ are the $\binom{\pi}{j-1}$ subsets that contain v. Furthermore, each of these $\binom{\pi}{j-1}$ subsets is contained in a unit disk of G if and only if it is dominated by a neighbor of v. For each $i \geq 0$, let Δ_i denote the number of subsets of $N^2(v)$ of cardinality i that are not dominated by any neighbor of v (so always $\Delta_0 = 0 = \Delta_1$). Notice that if $S \subseteq N^2(v)$ is not dominated by any neighbor of v and if $S \subseteq S^+ \subseteq N^2(v)$, then S^+ is not dominated by any neighbor of v. This implies $\sum_{i\geq 1}(-1)^{i+1}\Delta_i = 0$, because if $S \subseteq N^2(v)$ is not dominated by any neighbor of v and if $s = |S| \geq 2$, then S contributes $\sum_{i=s}^{\pi}(-1)^{i+1}\binom{\pi-s}{i-s} = 0$ to $\sum_{i\geq 1}(-1)^{i+1}\Delta_i$. Also, for each $i \geq 1$,

$$d_i(G) = d_i(H) + \left[\binom{\pi}{i-1} - \Delta_{i-1}\right],$$

because $d_i(H)$ is the number of cardinality-i subsets of $V(G)$ that are contained in unit disks of G and do not contain v, while $\binom{\pi}{i-1} - \Delta_{i-1}$ is the number of cardinality-i subsets of $V(G)$ that are contained in unit disks of G and do contain v. Thus $\widehat{\chi}(G) = \sum_{i\geq 1}(-1)^{i+1}d_i(G)$ is equal to

$$\sum\nolimits_{i\geq 1}(-1)^{i+1}d_i(H) + \sum\nolimits_{i\geq 1}(-1)^{i+1}\binom{\pi}{i-1} - \sum\nolimits_{i\geq 1}(-1)^{i+1}\Delta_i.$$

Therefore, either $\widehat{\chi}(G) = \widehat{\chi}(H) + 0 - 0$ (when $\pi > 0$) or $\widehat{\chi}(G) = \widehat{\chi}(H) + 1 - 0$ (when $\pi = 0$ and v is an isolated vertex). Either way, the induction hypothesis $\widehat{\chi}(H) = \text{comp}(H)$ implies that $\widehat{\chi}(G) = \text{comp}(G)$.

Argue the "if" direction by supposing that G is not chordal. Therefore, G contains an induced length-n cycle H with $n \geq 4$. If $n = 4$, then $\widehat{\chi}(H) = 4 - 6 + 4 - 0 + \cdots = 2$. If $n > 4$, then $\widehat{\chi}(H) = n - 2n + n - 0 + \cdots = 0$. In either case, $\widehat{\chi}(H) \neq \text{comp}(H)$. ■

Much as the simple cstr_G-1 and dstr_G-1 characterizations of chordal graphs can be transformed into the cstr_G-k and dstr_G-k characterizations of strongly chordal graphs in Proposition 1 and Theorem 2, the characterizations of chordal graphs in Proposition 3 and Theorem 4 can be transformed into Theorems 5 and 6, characterizing strongly chordal graphs in terms of suitably generalized clique characteristic and disk characteristic parameters.

For any graph G and $k \geq 1$, let $c_i^{(k)}(G)$ denote the number of cardinality-i subsets $S \subset V(G)$ that have $\text{cstr}_G(S) \geq k$. Thus $c_1^{(k)}(G)$ is the number of cstr_G-k

vertices, $c_2^{(k)}(G)$ is the number of cstr_G-k edges, $c_3^{(k)}(G)$ is the number of cstr_G-k triangles, and so on. Let $\chi^{(k)}(G) = c_1^{(k)}(G) - c_2^{(k)}(G) + c_3^{(k)}(G) - \cdots$, and let $G^{(k)}$ denote the subgraph of G formed by the cstr_G-k vertices and cstr_G-k edges of G. Thus, in particular, $\chi^{(1)}(G) = \chi(G)$ and $G^{(1)} = G$. If $V(G^{(i)}) = \emptyset$, set $\mathrm{comp}(G^{(i)}) = 0$.

For instance, if G is the 3-sun, then $c_1^{(2)}(G) = 3$ and $c_2^{(2)}(G) = 3$, with $c_i^{(2)}(G) = 0$ whenever $i \geq 3$. Thus $\chi^{(2)}(G) = 3 - 3 = 0$ (with the triangle in $G^{(2)} \cong K_3$ not counted in calculating $\chi^{(2)}(G)$ because $c_3^{(2)}(G) = 0$). Similarly, $\chi^{(3)}(G) = 3 - 0 = 3$ (with $G^{(3)} \cong 3K_1$, the edgeless graph on three vertices) and $\chi^{(k)}(G) = 0$ whenever $k \geq 4$ (with $V(G^{(k)}) = \emptyset$).

Theorem 5. *A graph is strongly chordal if and only if, for every $k \geq 1$, every induced subgraph H satisfies $\chi^{(k)}(H) = \mathrm{comp}(H^{(k)})$.*

Proof. Argue the "only if" direction by induction on $|V(G)|$. The $|V(G)| = 1$ basis case follows from $\chi^{(1)}(K_1) = 1$ and $\chi^{(k)}(K_1) = 0$ whenever $k \geq 2$. Suppose G is strongly chordal—and so every induced subgraph of G is strongly chordal—with $|V(G)| \geq 2$ and, for every $k \geq 1$, suppose $\chi^{(k)}(H) = \mathrm{comp}(H^{(k)})$ for every proper induced subgraph H of G (toward showing $\chi^{(k)}(G) = \mathrm{comp}(G^{(k)})$). Since the conclusion for $k = 1$ follows from Proposition 3, assume $k \geq 2$. Let v be a simple vertex of G, let $H = G - v$, and order the vertices of $N(v) = \{w_1, \ldots, w_\delta\}$ so that $N[v] \subseteq N[w_1] \subseteq \cdots \subseteq N[w_\delta]$, which also ensures that $\mathrm{cstr}_G(v) \leq \mathrm{cstr}_G(w_1) \leq \cdots \leq \mathrm{cstr}_G(w_\delta)$.

If $k > \mathrm{cstr}_G(w_\delta)$, then $H^{(k)} = G^{(k)}$ with $\mathrm{comp}(H^{(k)}) = \mathrm{comp}(G^{(k)})$ and $\chi^{(k)}(G) = \chi^{(k)}(H)$. Hence the induction hypothesis implies $\chi^{(k)}(G) = \chi^{(k)}(H) = \mathrm{comp}(H^{(k)}) = \mathrm{comp}(G^{(k)})$. Because of this, assume for the remainder of the proof that $2 \leq k \leq \mathrm{cstr}_G(w_\delta)$.

Suppose for the moment that $N[v] = N[w_1]$. This makes $N(v)$ a maxclique of H and $\mathrm{cstr}_G(S) = \mathrm{cstr}_H(S)$ for every $S \subseteq V(H)$. Therefore $H^{(k)} = G^{(k)}$ and $\chi^{(k)}(G) = \chi^{(k)}(H)$. The induction hypothesis now implies $\chi^{(k)}(G) = \chi^{(k)}(H) = \mathrm{comp}(H^{(k)}) = \mathrm{comp}(G^{(k)})$. Because of this, assume for the remainder of the proof that $N[v] \neq N[w_1]$.

The $N[v] \neq N[w_1]$ assumption implies that G has exactly one more maxclique (namely, $N[v]$) than H. Moreover, the sets $S \subseteq V(H)$ that have different clique strengths in G and H are precisely the sets $S \subset N(v)$, and these have $\mathrm{cstr}_G(S) = \mathrm{cstr}_H(S) + 1$. Hence, $G^{(k)}$ can be constructed from $H^{(k)}$ by inserting the vertices w_i and edges w_iw_j that have clique strength k in G.

If there is no i for which $\mathrm{cstr}_G(w_i) = k$, then $c_j^{(k)}(G) = c_j^{(k)}(H)$ for every value of j (since each side of the equality counts the number of subgraphs of order j that are in *at least* k maxcliques). Therefore $\chi^{(k)}(G) = \chi^{(k)}(H)$ and $\mathrm{comp}(G^{(k)}) = \mathrm{comp}(H^{(k)})$, and the induction hypothesis implies $\chi^{(k)}(G) = \chi^{(k)}(H) = \mathrm{comp}(H^{(k)})$

$= \mathrm{comp}(G^{(k)})$. Because of this, assume for the remainder of the proof that $\mathrm{cstr}_G(w_i) = k$ holds on a nonempty subinterval of $[1, \delta]$, and let $p = |\{w_i \in N(v) : \mathrm{cstr}_G(w_i) = k\}| \geq 1$.

Case 1. $k < \mathrm{cstr}_G(w_\delta)$. The construction of $G^{(k)}$ from $H^{(k)}$ by inserting the vertices w_i and edges w_iw_j that have clique strength k in G implies that $\mathrm{comp}(G^{(k)}) = \mathrm{comp}(H^{(k)})$ (since each such w_i and w_j is adjacent to w_δ in G). Let $q = |\{w_i \in N(v) : k < \mathrm{cstr}_G(w_i) \leq \mathrm{cstr}_G(w_\delta)\}| \geq 1$. For each $j \in \{1, \ldots, p+q\}$, there are exactly $\binom{p+q}{j} - \binom{q}{j}$ sets $S \subseteq N(v)$ that have $|S| = j$ and $\mathrm{cstr}_G(S) = k$. Therefore, $\chi^{(k)}(G) = \chi^{(k)}(H) + \left[\binom{p+q}{1} - \binom{p+q}{2} + \cdots - (-1)^{p+q}\binom{p+q}{p+q}\right] - \left[\binom{q}{1} - \binom{q}{2} + \cdots - (-1)^q\binom{q}{q}\right] = \chi^{(k)}(H) + 1 - 1$, and so $\chi^{(k)}(G) = \chi^{(k)}(H)$. The induction hypothesis now implies that $\chi^{(k)}(G) = \chi^{(k)}(H) = \mathrm{comp}(H^{(k)}) = \mathrm{comp}(G^{(k)})$.

Case 2. $k = \mathrm{cstr}_G(w_\delta)$. The construction of $G^{(k)}$ from $H^{(k)}$ by inserting the vertices w_i and edges w_iw_j that have clique strength k in G implies that $\mathrm{comp}(G^{(k)}) = \mathrm{comp}(H^{(k)}) + 1$ (with $\{w_i : \mathrm{cstr}_G(w_i) = k\}$ inducing the one additional component of $G^{(k)}$). For each $j \in \{1, \ldots, p\}$, there are exactly $\binom{p}{j}$ sets $S \subseteq N(v)$ that have $|S| = j$ and $\mathrm{cstr}_G(S) = k$. Therefore, $\chi^{(k)}(G) = \chi^{(k)}(H) + \binom{p}{1} - \binom{p}{2} + \cdots - (-1)^p\binom{p}{p} = \chi^{(k)}(H) + 1$. The induction hypothesis now implies that $\chi^{(k)}(G) = \chi^{(k)}(H) + 1 = \mathrm{comp}(H^{(k)}) + 1 = \mathrm{comp}(G^{(k)})$.

Therefore, $\chi^{(k)}(G) = \mathrm{comp}(G^{(k)})$ for all strongly chordal graphs G and all k, completing the proof of the "only if" direction.

Argue the "if" direction by supposing that G is not strongly chordal. Therefore, G contains an induced subgraph H that is either a length-n cycle with $n \geq 4$ or an n-sun with $n \geq 3$. In the former case, $\chi^{(1)}(H) = n - n + 0 \neq \mathrm{comp}(H^{(1)}) = 1$ (H has no triangle, since $n \geq 4$). In the latter case, $H^{(2)}$ is a length-n cycle and $\chi^{(2)}(H) = n - n + 0 \neq \mathrm{comp}(H^{(2)}) = 1$ (H has no cstr_H-2 triangle, even when $n = 3$). ■

Finally, Theorem 5 can be made into an additional characterization of strongly chordal graphs by interchanging maxcliques with unit disks. Let $d_i^{(k)}(G)$ denote the number of cardinality-i subsets $S \subset V(G)$ that have $\mathrm{dstr}_k(S) \geq k$. Let $\widehat{\chi}^{(k)}(G) = d_1^{(k)}(G) - d_2^{(k)}(G) + d_3^{(k)}(G) - \cdots$, and let $\widehat{G}^{(k)}$ denote the subgraph of G formed by the dstr_G-k vertices and dstr_G-k edges of G. Thus, in particular, $\widehat{\chi}^{(1)}(G) = \widehat{\chi}(G)$ and $\widehat{G}^{(1)} = G$. If $V(\widehat{G}^{(i)}) = \emptyset$, set $\mathrm{comp}(\widehat{G}^{(i)}) = 0$.

For instance, if G is the 3-sun, then $d_1^{(2)}(G) = 6$, $d_2^{(2)}(G) = 12$, $d_3^{(2)}(G) = 10$, and $d_4^{(2)}(G) = 3$, with $d_i^{(2)}(G) = 0$ whenever $i \geq 5$. Thus $\widehat{\chi}^{(2)}(G) = 6 - 12 + 10 - 3 = 1$ (with $\widehat{G}^{(2)} \cong G$). Similarly, $\widehat{\chi}^{(3)}(G) = 6 - 9 + 4 = 1$ (with $\widehat{G}^{(3)} \cong G$), $\widehat{\chi}^{(4)}(G) = 3 - 3 = 0$ (with $\widehat{G}^{(4)} \cong K_3$), $\widehat{\chi}^{(5)}(G) = 3 - 0 = 3$ (with $\widehat{G}^{(5)} \cong 3K_1$), and $\widehat{\chi}^{(k)}(G) = 0$ whenever $k \geq 6$ (with $V(\widehat{G}^{(k)}) = \emptyset$ and so $\mathrm{comp}(\widehat{G}^{(k)}) = 0$).

Theorem 6. *A graph is strongly chordal if and only if, for every $k \geq 1$, every induced subgraph H satisfies $\widehat{\chi}^{(k)}(H) = \mathrm{comp}(\widehat{H}^{(k)})$.*

Proof. First suppose that H is an induced subgraph of a strongly chordal graph G. Since Theorem 4 is the $k = 1$ case, assume $k \geq 2$.

Partition $V(H)$ into sets $A_1, \ldots, A_m$ such that two vertices of H have the same closed neighborhood if and only if they are in the same set A_i. Let H' be the graph obtained from H by adding, for each $i \in \{1, \ldots, m\}$, a vertex u_i that has $N_{H'}(u_i) = A_i$. Thus each u_i is a simple vertex of H', and so H' is strongly chordal. Let H^* be the square of H' (so $V(H^*) = V(H')$ with vertices adjacent in H^* if and only if they are at distance at most 2 in H'). Thus H^* is also strongly chordal (since H' is strongly chordal; see [2, 12]).

We next prove the claim that the maxcliques of H^* are the closed neighborhoods in H' of the vertices in $V(H)$. Let $v \in V(H)$. Take j such that $v \in N_{H'}(u_j)$. Hence u_j is a simplicial vertex of H^* and $N_{H^*}[u_j] = N_{H'}[v]$. Therefore, $N_{H'}[v]$ is a maxclique of H^*.

Conversely, let Q be a maxclique of H^*. Since every two vertices of Q are at distance at most 2 in H', the closed neighborhoods in H' of the vertices in Q are pairwise intersecting. Thus H' contains a vertex w such that $Q \subseteq N_{H'}[w]$ (since H' being strongly chordal implies H is dually chordal, and so the family of unit disks of H' is Helly; see [2]). Since $N_{H'}[w]$ is a complete subgraph of H^*, the maximality of Q implies that $Q = N_{H'}[w]$. If $w \in V(H)$, then the claim holds. If instead $w = u_i$ for some i, then there is a vertex v in $N_{H'}(w)$, and so $Q = N_{H'}[w] \subseteq N_{H'}[v]$ implies $Q = N_{H'}[v]$.

Therefore, the maxcliques of H^* are the closed neighborhoods in H' of the vertices in $V(H)$, as claimed. The construction of H^* and the assumption $k \geq 2$ imply that $d_i^{(k)}(H) = c_i^{(k)}(H^*)$ for all $i \geq 1$, which then implies both that $\widehat{\chi}^{(k)}(H) = \chi^{(k)}(H^*)$ and that $\widehat{H}^{(k)} = H^{*(k)}$. Therefore, by Theorem 5, $\widehat{\chi}^{(k)}(H) = \chi^{(k)}(H^*) = \mathrm{comp}(H^{*(k)}) = \mathrm{comp}(\widehat{H}^{(k)})$.

To prove the "if" direction, suppose that G is not strongly chordal. Therefore, G contains an induced subgraph H that is either a length-n cycle with $n \geq 4$ or an n-sun with $n \geq 3$. If H is an n-cycle, then either $n = 4$ and $\widehat{\chi}^{(1)}(H) = 4-6+4 = 2 \neq 1 = \mathrm{comp}(\widehat{H}^{(1)}) = \mathrm{comp}(C_4)$ or $n \geq 5$ and $\widehat{\chi}^{(1)}(H) = n-2n+n = 0 \neq 1 = \mathrm{comp}(\widehat{H}^{(1)}) = \mathrm{comp}(C_n)$. For the other possibility, if H is an n-sun, then $\widehat{\chi}^{(n+1)}(H) = n - n = 0 \neq 1 = \mathrm{comp}(\widehat{H}^{(n+1)}) = \mathrm{comp}(C_n)$. ■

References

[1] A. Brandstädt, F. Dragan, V. Chepoi, and V. Voloshin, *Dually chordal graphs*, SIAM J. Discrete Math. **11** (1998) 437–455.
doi:10.1137/S0895480193253415

[2] A. Brandstädt, V.B. Le, and J.P. Spinrad, Graph Classes: A Survey (Society for Industrial and Applied Mathematics, Philadelphia, 1999).
doi:10.1137/1.9780898719796

[3] P. De Caria and M. Gutierrez, *On minimal vertex separators of dually chordal graphs: properties and characterizations*, Discrete Appl. Math. **160** (2012) 2627–2635.
doi:10.1016/j.dam.2012.02.022

[4] P. De Caria and M. Gutierrez, *On the correspondence between tree representations of chordal and dually chordal graphs*, Discrete Appl. Math. **164** (2014) 500–511.
doi:10.1016/j.dam.2013.07.011

[5] M. Farber, *Characterizations of strongly chordal graphs*, Discrete Math. **43** (1983) 173–189.
doi:10.1016/0012-365X(83)90154-1

[6] T.A. McKee, *How chordal graphs work*, Bull. Inst. Combin. Appl. **9** (1993) 27–39.

[7] T.A. McKee, *A new characterization of strongly chordal graphs*, Discrete Math. **205** (1999) 245–247.
doi:10.1016/S0012-365X(99)00107-7

[8] T.A. McKee, *Subgraph trees in graph theory*, Discrete Math. **270** (2003) 3–12.
doi:10.1016/S0012-365X(03)00161-4

[9] T.A. McKee, *The neighborhood characteristic parameter for graphs*, Electron. J. Combin. **10** (2003) #R20.

[10] T.A. McKee, *When fundamental cycles span cliques*, Congr. Numer. **191** (2008) 213–218.

[11] T.A. McKee, *Simplicial and nonsimplicial complete subgraphs*, Discuss. Math. Graph Theory **31** (2011) 577–586.
doi:10.7151/dmgt.1566

[12] T.A. McKee and F.R. McMorris, Topics in Intersection Graph Theory (Society for Industrial and Applied Mathematics, Philadelphia, 1999).
doi:10.1137/1.9780898719802

[13] T.A. McKee and E. Prisner, *An approach to graph-theoretic homology*, Combinatorics, Graph Theory and Algorithms Y. Alavi, *et al.* Eds, New Issues Press, Kalamazoo, MI (1999) **2** 631–640.

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