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ON TWIN EDGE COLORINGS OF GRAPHS

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Abstract

A twin edge k-coloring of a graph G is a proper edge coloring of G with the elements of $\mathbb{Z}_k$ so that the induced vertex coloring in which the color of a vertex v in G is the sum (in $\mathbb{Z}_k$) of the colors of the edges incident with v is a proper vertex coloring. The minimum k for which G has a twin edge k-coloring is called the twin chromatic index of G. Among the results presented are formulas for the twin chromatic index of each complete graph and each complete bipartite graph.

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1. Introduction

In 1968, Rosa [13] introduced a vertex labeling that induces an *edge-distinguishing labeling* defined by subtracting labels. In particular, for a graph G of size m, a vertex labeling (an injective function) $f : V(G) \to \{0, 1, \ldots, m\}$ was called a β-*valuation* by Rosa if the induced edge labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ defined by $f'(uv) = |f(u) - f(v)|$ was bijective. In 1972 Golomb [8] called a β-valuation a *graceful labeling* and a graph possessing a graceful labeling a *graceful graph.* It is this terminology that has become standard. Much research has been done on graceful graphs. A popular conjecture in graph theory, due to Anton Kotzig and Gerhard Ringel, is the following.

The Graceful Tree Conjecture. *Every nontrivial tree is graceful.*

In 1991 Gnana Jothi [7] introduced a concept that, in a certain sense, reverses the roles of vertices and edges in graceful labelings (see also [6]). For a connected graph G of order $n \geq 3$, let $f : E(G) \to \mathbb{Z}_n$ be an edge labeling of G that induces a bijective function $f' : V(G) \to \mathbb{Z}_n$ defined by $f'(v) = \sum_{e \in E_v} f(e)$ for each vertex v of G, where E_v is the set of edges of G incident with a vertex v. Such a labeling f is called a *modular edge-graceful labeling*, while a graph possessing such a labeling is called *modular edge-graceful* (see [10]). Verifying a conjecture by Gnana Jothi on trees, Jones, Kolasinski and Zhang [11] showed not only that every tree of order $n \geq 3$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$ but a connected graph of order $n \geq 3$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$. These concepts have been studied in greater detail by Jones [9]. A generalization of this concept has been introduced recently by Anholcer, Cichacz and Milanič in [2].

Prior to Jothi's paper, an edge labeling (with positive integers) of a connected graph G was introduced in 1986 [3] for the purpose of producing a weighted graph whose degrees (obtained by adding the weights of the incident edges of each vertex) were distinct. Such a weighted graph was called *irregular*. This concept was later looked at in another manner. For the set $\mathbb{N}$ of positive integers, an edge coloring $c : E(G) \to \mathbb{N}$, where adjacent edges may be colored the same, is said to be *vertex-distinguishing* if the coloring $c' : V(G) \to \mathbb{N}$ induced by c and defined by $c'(v) = \sum_{e \in E_v} c(e)$ has the property that $c'(x) \neq c'(y)$ for every two distinct vertices x and y of G. The research in [3] dealt with minimizing the largest color assigned to the edges of the graph to produce an irregular graph. Vertex-distinguishing edge colorings have received increased attention during the past 25 years (see [5, pp. 370-385]).

A *neighbor-distinguishing coloring* of a graph G is a coloring in which every pair of adjacent vertices of G are colored differently. Such a coloring is more commonly called a *proper vertex coloring*. The minimum number of colors needed in a proper vertex coloring of a graph G is the chromatic number of G and denoted by $\chi(G)$. A number of neighbor-distinguishing vertex colorings different from standard proper colorings have been introduced in the literature (see [5, pp. 383-391], for example).

In 2005 non-proper edge colorings of graphs were studied that induce a proper vertex coloring [1]. In particular, for $k \in \mathbb{N}$, let $c : E(G) \to \{1, 2, \ldots, k\}$ be an edge coloring of G (where adjacent edges may be assigned the same color). A vertex coloring $c' : V(G) \to \mathbb{N}$ is defined where $c'(v)$ is the sum of the colors of the edges incident with v. If c' is a proper vertex coloring of G, then c is called a *neighbor-distinguishing edge coloring* of G (see [5, p. 385]). A major conjecture in this area is the following [12].

The 1-2-3 Conjecture. *For every connected graph G of order at least 3, there exists a neighbor-distinguishing edge coloring of G using only the colors $1, 2, 3$.*

Among the various edge colorings studied in graph theory, the best known and most studied are proper edge colorings. In a *proper edge coloring* of a graph G, each edge of G is assigned a color from a given set of colors where adjacent edges are colored differently. The minimum number of colors needed in a proper edge coloring of G is called the *chromatic index* of G and is denoted by $\chi'(G)$. The classic theorem in this connection is due to Vizing [14] who proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for every nonempty graph G.

A related and also well-studied graph coloring is the so-called *total coloring* of a graph G that assigns colors to both the vertices and edges of G so that not only the vertex coloring and edge coloring are proper but no vertex and an incident edge are assigned the same color. The minimum number of colors required for a total coloring of G is the *total chromatic number* of G, denoted by $\chi''(G)$. It then follows that $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. A well-known conjecture in this area is due independently to Behzad and Vizing (see [5, p. 282]).

The Total Coloring Conjecture. *For every graph G, $\chi''(G) \leq 2 + \Delta(G)$.*

Inspired by the graph colorings described above, we introduce a proper edge coloring of a graph that induces a proper vertex coloring where the colors belong to $\mathbb{Z}_k$ for some integer $k \geq 2$. We refer to the books [4, 5] for graph theory notation and terminology not described in this paper. All graphs under consideration here are connected graphs of order at least 3.

2. Twin Chromatic Index

For a connected graph G of order at least 3, a proper edge coloring $c : E(G) \to \mathbb{Z}_k$ for some integer $k \geq 2$ is sought for which the induced vertex coloring $c' : V(G) \to \mathbb{Z}_k$ defined by

$$c'(v) = \sum\nolimits_{e \in E_v} c(e) \ \text{ in } \mathbb{Z}_k,$$

(where the indicated sum is computed in $\mathbb{Z}_k$) results in a proper vertex coloring of G. We refer to such a coloring as a *twin edge k-coloring* or simply a *twin edge coloring* of G. The minimum k for which G has a twin edge k-coloring is called the *twin chromatic index* of G and is denoted by $\chi'_t(G)$. Since a twin edge coloring is not only a proper edge coloring of G but induces a proper vertex coloring of G, it follows that

$$\chi'_t(G) \geq \max\{\chi(G), \chi'(G)\}.$$

Since $\max\{\chi(G), \chi'(G)\} = \chi'(G)$ except when G is a complete graph of even order, we have $\chi'_t(G) \geq \chi'(G)$ except possibly when G is a complete graph of even order.

While $\chi'_t(G)$ does not exist if G is the connected graph of order 2, every connected graph of order at least 3 has a twin edge coloring. To see this, let G be a connected graph of size $m \geq 2$. If $m = 2$, then assign the colors 1 and 2 in $\mathbb{Z}_3$ to the two edges of G. If $m \geq 3$, then assign the m elements $0, 1, 2, 4, \ldots, 2^{m-2} \in \mathbb{Z}_{2^{m-1}}$ to the m edges of G in a one-to-one manner so that the color 0 is assigned to a pendant edge if G has such an edge. Hence the sets of edges colored by nonzero elements in $\mathbb{Z}_{2^{m-1}}$ that are incident with every two adjacent vertices are distinct. Since the base 2 representations of the colors of these vertices are different, it follows that adjacent vertices are assigned distinct colors in $\mathbb{Z}_{2^{m-1}}$. Thus, this coloring is a twin edge coloring. This observation yields the following.

Proposition 2.1. *If G is a connected graph of order at least 3 and size m, then $\chi'_t(G)$ exists. Furthermore, $\chi'_t(G) \leq 2^{m-1}$ if $m \geq 3$.*

To illustrate the concept of twin edge colorings, we determine the twin chromatic indexes of two familiar classes of graphs, namely paths and cycles. We begin with paths.

Proposition 2.2. *If P_n is a path of order $n \geq 3$, then $\chi'_t(P_n) = 3$.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n \geq 3$ where $e_i = v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$. Since $\chi'(P_n) = 2$, it follows that $\chi'_t(P_n) \geq \chi'(P_n) = 2$. First, we show that $\chi'_t(P_n) \neq 2$. Let c be a proper edge coloring of P_n using the colors of $\mathbb{Z}_2$. Then $c(e_i) = 1 \in \mathbb{Z}_2$ for some $i \in \{1, 2, \ldots, n-1\}$ and so $c(e_{i-1}) = 0$ if $i \geq 2$ and $c(e_{i+1}) = 0$ if $i \leq n-2$. However then, $c'(v_i) = c'(v_{i+1}) = 1$ and so c is not a twin edge 2-coloring. Thus, as claimed, $\chi'_t(P_n) \geq 3$. It remains to show that P_n has a twin edge 3-coloring. A coloring $c : E(P_n) \to \mathbb{Z}_3$ is defined as follows.

- For $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$, let $c(e_j) = r$ if $j \equiv r \pmod 3$ for $r = 0, 1, 2$. For example, if $n = 6$, then $(c(e_1), c(e_2), \ldots, c(e_5)) = (1, 2, 0, 1, 2)$; while if $n = 7$, then $(c(e_1), c(e_2), \ldots, c(e_6)) = (1, 2, 0, 1, 2, 0)$. If $n \equiv 0 \pmod 3$, then for $1 \leq i \leq n$,

$$c'(v_i) = \begin{cases} 0 & \text{if } i \equiv 2 \pmod 3, \\ 1 & \text{if } i \equiv 1 \pmod 3, \\ 2 & \text{if } i \equiv 0 \pmod 3. \end{cases} \tag{1}$$

If $n \equiv 1 \pmod 3$, then $c'(v_i)$ is given in (1) for $1 \le i \le n-1$ and $c'(v_n) = 0$. Hence $(c'(v_1), c'(v_2), \ldots, c'(v_6)) = (1,0,2,1,0,2)$ and $(c'(v_1), c'(v_2), \ldots, c'(v_7)) = (1,0,2,1,0,2,0)$.

• For $n \equiv 2 \pmod 3$, let $c(e_j) = 2 + r$ if $j \equiv r \pmod 3$ for $r = 0,1,2$. Then $c'(v_1) = c'(v_n) = 0$ and for $2 \le i \le n-1$,

$$c'(v_i) = \begin{cases} 0 & \text{if } i \equiv 0 \pmod 3, \\ 1 & \text{if } i \equiv 2 \pmod 3, \\ 2 & \text{if } i \equiv 1 \pmod 3. \end{cases}$$

For example, if $n = 8$, then $(c(e_1), c(e_2), \ldots, c(e_7)) = (0,1,2,0,1,2,0)$ and $(c'(v_1), c'(v_2), \ldots, c'(v_8)) = (0,1,0,2,1,0,2,0)$. Therefore, $\chi'_t(P_n) \ge 3$ and so $\chi'_t(P_n) = 3$ for $n \ge 3$. ∎

To determine the twin chromatic indexes of cycles, the following observation will be useful.

Observation 2.3. *If a connected graph G contains two adjacent vertices of degree $\Delta(G)$, then $\chi'_t(G) \ge 1 + \Delta(G)$.*

Proposition 2.4. *If C_n is a cycle of order $n \ge 3$, then*

$$\chi'_t(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3, \\ 4 & \text{if } n \not\equiv 0 \pmod 3 \text{ and } n \ne 5, \\ 5 & \text{if } n = 5. \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ where $e_i = v_i v_{i+1}$ for $i = 1,2,\ldots,n$ and $e_{n+1} = e_1$. By Observation 2.3, $\chi'_t(C_n) \ge 3$. First, suppose that $n \equiv 0 \pmod 3$ and so $n = 3k$ for some positive integer k. Define the coloring $c : E(C_n) \to \mathbb{Z}_3$ by $c(e_i) \equiv 2 + r \pmod 3$ if $i \equiv r \pmod 3$ for $r = 0,1,2$. Then for $1 \le i \le n$,

$$c'(v_i) = \begin{cases} 0 & \text{if } i \equiv 0 \pmod 3, \\ 1 & \text{if } i \equiv 2 \pmod 3, \\ 2 & \text{if } i \equiv 1 \pmod 3. \end{cases}$$

For example, if $n = 6$, then $(c(e_1), c(e_2), \ldots, c(e_6)) = (0,1,2,0,1,2)$ and $(c'(v_1), c'(v_2), \ldots, c'(v_6)) = (2,1,0,2,1,0)$. Hence $\chi'_t(C_n) = 3$ when $n \equiv 0 \pmod 3$.

Next, suppose that $n \not\equiv 0 \pmod 3$ and $n \ne 5$. First, we make an observation, namely, if c is a twin edge coloring of C_n and $|i - j| = 2$, then $c(e_i) \ne c(e_j)$. Suppose, say, that $c(e_1) = c(e_3)$. However then, $c'(v_2) = c(e_1) + c(e_2) = c(e_2) + c(e_3) = c'(v_3)$, which is impossible. This implies that if $n \not\equiv 0 \pmod 3$, then $\chi'_t(C_n) \ge 4$. To show that $\chi'_t(C_n) \le 4$, define the coloring $c : E(C_n) \to \mathbb{Z}_4$ as follows.

• For $n \equiv 1 \pmod 3$, let $c(e_i) \equiv 2+r \pmod 3$ if $i \equiv r \pmod 3$ for $r = 0, 1, 2$ and $1 \le i \le n-1$ and $c(e_n) = 3$. Then $c'(v_1) = 3$, $c'(v_n) = 1$ and for $2 \le i \le n-1$,

$$c'(v_i) = \begin{cases} 1 & \text{if } i \equiv 2 \pmod 3, \\ 2 & \text{if } i \equiv 1 \pmod 3, \\ 3 & \text{if } i \equiv 0 \pmod 3. \end{cases} \tag{2}$$

(In particular, $c'(v_2) = 1$ and $c'(v_{n-1}) = 3$.) For example, if $n = 7$, then $(c(e_1), c(e_2), \ldots, c(e_7)) = (0, 1, 2, 0, 1, 2, 3)$ and $(c'(v_1), c'(v_2), \ldots, c'(v_7)) = (3, 1, 3, 2, 1, 3, 1)$. Hence $\chi'_t(C_n) = 4$ when $n \equiv 1 \pmod 3$.

• Let $n \equiv 2 \pmod 3$ and $n \ge 8$. If $n = 8$, let $(c(e_1), c(e_2), \ldots, c(e_8)) = (0, 1, 2, 3, 0, 1, 2, 3)$; while if $n \ge 11$, let $c(e_i) \equiv 2 + r \pmod 3$ if $i \equiv r \pmod 3$ for $r = 0, 1, 2$ and $1 \le i \le n-9$ and let $(c(e_{n-8}), c(e_{n-7}), \ldots, c(e_n)) = (0, 1, 2, 3, 0, 1, 2, 3)$.

Consequently, if $n = 8$, then $(c'(v_1), c'(v_2), \ldots, c'(v_8)) = (3, 1, 3, 1, 3, 1, 3, 1)$; while if $n \ge 11$, then $c'(v_1) = 3$, $c'(v_i)$ is the same as in (2) for $2 \le i \le n-9$ and $(c'(v_{n-8}), c'(v_{n-7}), \ldots, c'(v_n)) = (3, 1, 3, 1, 3, 1, 3, 1)$. For example, if $n = 11$, then $(c(e_1), c(e_2), \ldots, c(e_{11})) = (0, 1, 2, 0, 1, 2, 3, 0, 1, 2, 3)$ and $(c'(v_1), c'(v_2), \ldots, c'(v_{11})) = (3, 1, 3, 2, 1, 3, 1, 3, 1, 3, 1)$. Hence $\chi'_t(C_n) = 4$ when $n \equiv 2 \pmod 3$.

Finally, we show that $\chi'_t(C_5) = 5$. We have already observed that $\chi'_t(C_5) \ge 3$. Let $C_5 = (v_0, v_1, v_2, v_3, v_4, v_5 = v_0)$ and let $c : E(C_5) \to \mathbb{Z}_5$ be defined by $c(v_i v_{i+1}) = i$ for $0 \le i \le 4$. Since $c'(v_0) = 4$, $c'(v_1) = 1$, $c'(v_2) = 3$, $c'(v_3) = 0$ and $c'(c_4) = 2$, it follows that c is a twin edge 5-coloring of C_5 and so $\chi'_t(C_5) \le 5$. We now show that $\chi'_t(C_5) \ge 5$. Suppose that there is a twin edge k-coloring where $k = 3$ or $k = 4$. Then some element $a \in \mathbb{Z}_k$ must be used twice, say $c(v_0 v_1) = c(v_2 v_3) = a$. Suppose that $c(v_1 v_2) = b$, where $b \ne a$. Then $c'(v_1) = c'(v_2) = a + b$, which is a contradiction. Thus, $\chi'_t(C_5) = 5$. ∎

3. Complete Graphs

We now investigate twin edge colorings of complete graphs K_n starting with the case n being odd. The following observation will be useful later.

Observation 3.1. *Let $n \ge 2$ be an integer. If n is odd, then $\binom{n}{2} = 0$ in $\mathbb{Z}_n$ and if n is even, then $\binom{n}{2} = \frac{n}{2}$ in $\mathbb{Z}_n$.*

Lemma 3.2. *If $n \ge 3$ is an odd integer, then $\chi'_t(K_n) = n$.*

Proof. By Observation 2.3, $\chi'_t(K_n) \ge 1 + \Delta(K_n) = n$. To show that $\chi'_t(K_n) \le n$, let $V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and arrange the vertices $v_0, v_1, \ldots, v_{n-1}$ consecutively in a regular n-gon and join every two vertices by a straight line segment, producing K_n. For each i $(0 \le i \le n-1)$, assign to $v_{i-1}v_{i+1}$ and those edges parallel to $v_{i-1}v_{i+1}$ the color i. Then v_i has the color $\binom{n}{2} - i$, resulting in a proper vertex coloring of K_n. Thus $\chi'_t(K_n) = n$. ■

When $n \geq 4$ is even, however, $\chi'_t(K_n) \neq n$.

Lemma 3.3. *If $n \geq 4$ is an even integer, then $\chi'_t(K_n) \geq n + 1$.*

Proof. Since $\chi'_t(K_n) \geq 1 + \Delta(K_n) = n$ by Observation 2.3, it remains to show that $\chi'_t(K_n) \neq n$. Assume, to the contrary, that $\chi'_t(K_n) = n$. Then there is a proper edge coloring of K_n using the colors in $\mathbb{Z}_n$ that results in a proper vertex coloring of K_n. Since every vertex of K_n has degree $n - 1$, the edges incident with each vertex of K_n are colored with an $(n-1)$-element subset of $\mathbb{Z}_n$. For example, if v is a vertex of K_n, then there is exactly one element $a \in \mathbb{Z}_n$ that is not used in coloring the edges incident with v. Consequently, at most $\frac{n}{2} - 1$ edges of K_n are colored a, implying that there exists some other vertex u of K_n none of whose incident edges are colored a. However then, $c'(u) = c'(v) = \binom{n}{2} - a$, which is impossible since u and v are adjacent in K_n. Thus $\chi'_t(K_n) \geq n + 1$. ■

If $n \geq 4$ is an even integer, then either $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$. We consider these two situations, beginning with $n \equiv 0 \pmod 4$.

Lemma 3.4. *If $n \geq 4$ is an integer with $n \equiv 0 \pmod 4$, then $\chi'_t(K_n) = n + 1$.*

Proof. By Lemma 3.3, it suffices to show that K_n has a twin edge $(n+1)$-coloring. Let $V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and arrange the vertices $v_0, v_1, \ldots, v_{n-1}$ consecutively in a regular n-gon and join every two vertices by a straight line segment, thereby producing K_n.

Since $n \equiv 0 \pmod 4$ and $n \geq 4$, it follows that $n = 4k$ for some positive integer k. For $k = 1$, the coloring $c : E(K_4) \to \mathbb{Z}_5$ defined by $c(v_0v_1) = c(v_2v_3) = 0$, $c(v_0v_2) = 1$, $c(v_0v_3) = 2$, $c(v_1v_2) = 3$ and $c(v_1v_3) = 4$ is a twin edge 5-coloring of K_4 and so we may assume that $k \geq 2$. First, let $M_0, M_1, \ldots, M_{2k-1}$ be $2k$ pairwise edge-disjoint matchings of size $2k - 1$ in K_{4k} where each matching M_i $(0 \leq i \leq 2k-1)$ consists of those $2k - 1$ edges perpendicular to v_iv_{2k+i}. Then $H = K_{4k} - \left(\bigcup_{i=0}^{2k-1} M_i\right)$ is therefore a $(2k)$-regular graph. The graph H has a 1-factorization $\{F_1, F_2, \ldots, F_{2k}\}$ where F_i $(1 \leq i \leq 2k)$ consists of the edge v_iv_{i+1} and those edges parallel to v_iv_{i+1}. Let $X_1 = \{v_0v_{2k-1}, v_1v_{2k-2}, \ldots, v_{k-1}v_k\}$ and $X'_1 = \{v_{2k}v_{4k-1}, v_{2k+1}v_{4k-2}, \ldots, v_{3k-1}v_{3k}\}$. Thus $|X_1| = |X'_1| = k$ and $E(F_{k-1}) = X_1 \cup X'_1$. Define a coloring $c : E(K_{4k}) \to \mathbb{Z}_{4k+1}$ as follows. If $k = 2$, let

$$c(e) = \begin{cases} 0 & \text{if } e \in X'_1, \\ i - 1 & \text{if } e \in E(F_i) \text{ where } 2 \leq i \leq 2k, \\ 2k & \text{if } e \in X_1, \\ 2k + j + 1 & \text{if } e \in M_j \text{ where } 0 \leq j \leq 2k - 1. \end{cases}$$

If $k \geq 3$, let

$$c(e) = \begin{cases} 0 & \text{if } e \in X_1', \\ i & \text{if } e \in E(F_i) \text{ where } 1 \leq i \leq k-2, \\ i-1 & \text{if } e \in E(F_i) \text{ where } k \leq i \leq 2k, \\ 2k & \text{if } e \in X_1, \\ 2k+j+1 & \text{if } e \in M_j \text{ where } 0 \leq j \leq 2k-1. \end{cases}$$

Then c is a proper edge coloring. For $0 \leq i \leq 2k-1$,

$$c'(v_i) = \left[\binom{4k+1}{2} - 2k\right] - (2k+i+1) + 2k = -(2k+i+1) \text{ in } \mathbb{Z}_{4k+1};$$

while for $2k \leq i \leq 4k-1$,

$$c'(v_i) = \left[\binom{4k+1}{2} - 2k\right] - (i+1) + 0 = -(2k+i+1) \text{ in } \mathbb{Z}_{4k+1}.$$

Thus $(c'(v_0), c'(v_1), \ldots, c'(v_{4k-1})) = (2k, 2k-1, \ldots, 1, 0, 4k, 4k-1, \ldots, 2k+2)$. That is, each color in $\mathbb{Z}_{4k+1}$ (except $2k+1$) is used exactly once. Therefore, $c' : V(K_{4k}) \to \mathbb{Z}_{4k+1}$ is a proper vertex coloring of G and so $\chi_t'(K_n) = n+1$. ■

Lemma 3.5. *If $n \geq 6$ is an integer with $n \equiv 2 \pmod 4$, then $\chi_t'(K_n) = n+1$.*

Proof. Since $\chi_t'(K_n) \geq n+1$ by Lemma 3.3, it suffices to show that K_n has a twin edge $(n+1)$-coloring when $n \geq 6$ with $n \equiv 2 \pmod 4$. Let $n = 4k+2$ for some positive integer k and let $V(K_{4k+2}) = \{v_0, v_1, \ldots, v_{4k+1}\}$. Arrange the vertices $v_1, v_2, \ldots, v_{4k+1}$ consecutively in a regular $(4k+1)$-gon, place v_0 in the center of the $(4k+1)$-gon and then join every two vertices by a straight line segment, thereby producing K_{4k+2}.

Let $\mathcal{F} = \{F_1, F_2, \ldots, F_{4k+1}\}$ be the 1-factorization of K_{4k+2}, in which F_i is the 1-factor of K_{4k+2} that consists of the edge v_0v_{2k+1+i} and the $2k$ edges perpendicular to v_0v_{2k+1+i} when $1 \leq i \leq 2k$ and F_i consists of the edge v_0v_{i-2k} and the $2k$ edges perpendicular to v_0v_{i-2k} where $2k+1 \leq i \leq 4k+1$. Also, let $M_i = E(F_i)$ $(1 \leq i \leq 4k+1)$ denote the perfect matching of K_{4k+2} resulting from F_i. Observe that the edge v_iv_{i+1} belongs to M_i for $1 \leq i \leq 4k$ and $v_{4k+1}v_1 \in M_{4k+1}$.

We now define an edge coloring c_1 (described below) that assigns the $4k+1$ colors in $\mathbb{Z}_{4k+3} - \{0, 1\}$ to the $4k+1$ matchings $M_1, M_2, \ldots, M_{4k+1}$ such that

(i) c_1 assigns exactly one color to all edges in M_i for each i $(1 \leq i \leq 4k+1)$ and

(ii) $c_1(e) \neq c_1(f)$ if $e \in M_i$, $f \in M_j$ where $i \neq j$.

- For an even integer i with $2 \le i \le 4k$, let

$$c_1(e) = \begin{cases} (2k+3) - i & \text{if } e \in M_i \text{ and } 2 \le i \le 2k, \\ i - 2k & \text{if } e \in M_i \text{ and } 2k+2 \le i \le 4k. \end{cases}$$

- For $i = 1$ or $i = 2k+1$, let

$$c_1(e) = \begin{cases} 2k+3 & \text{if } e \in M_1, \\ 2k+2 & \text{if } e \in M_{2k+1}. \end{cases}$$

- For the remaining $2k-1$ matchings $M_3, M_5, \ldots, M_{2k-1}$ and $M_{2k+3}, M_{2k+5}, \ldots, M_{4k+1}$, the coloring c_1 assigns the remaining $2k-1$ colors $2k+4, 2k+5, \ldots, 4k+2$ to these matchings in an arbitrary way such that distinct colors are assigned to the edges in distinct matchings.

Hence, the $2k$ colors $2, 3, \ldots, 2k+1$ in $\mathbb{Z}_{4k+3}$ are used to color the edges in the $2k$ matchings $M_2, M_4, \ldots, M_{4k}$; while the $2k+1$ colors $2k+2, 2k+3, \ldots, 4k+2$ in $\mathbb{Z}_{4k+3}$ are used to color the edges in $2k+1$ matchings $M_1, M_3, \ldots, M_{4k+1}$. Therefore, c_1 is a proper edge coloring of K_{4k+2}. Since the colors 0 and 1 are not used,

$$c_1'(v) = 2 + 3 + \cdots + (4k+2) = \binom{4k+3}{2} - 0 - 1$$

for each $v \in V(K_{4k+2})$.

Next, we define a new edge coloring $c : E(K_{4k+2}) \to \mathbb{Z}_{4k+3}$ from the coloring c_1 as follows. First, we partition M_{2k+1} into two sets X and Y where

$$X = \{v_i v_{4k+3-i} : \ i \text{ is odd and } 3 \le i \le 2k+1\},$$
$$Y = \{v_0 v_1\} \cup \{v_i v_{4k+3-i} : \ i \text{ is even and } 2 \le i \le 2k\}.$$

For each $e \in E(K_{4k+2})$, let

$$c(e) = \begin{cases} 0 & \text{if } e \in \{v_i v_{i+1} : \ i \text{ is even and } 0 \le i \le 4k\}, \\ 1 & \text{if } e \in \{v_1 v_2\} \cup X, \\ c_1(e) & \text{otherwise.} \end{cases} \tag{3}$$

Let $b = \binom{4k+3}{2} - 1$, where then $b = -1$ in $\mathbb{Z}_{4k+3}$ and let $c' : V(K_{4k+2}) \to \mathbb{Z}_{4k+3}$ be the vertex coloring induced by c. Then

- For $i = 0, 1, 2$,

$$c'(v_0) = b - (2k+2),$$
$$c'(v_1) = b - (2k+2) - (2k+3) + 1 = b - 1,$$
$$c'(v_2) = b - (2k+3) - (2k+1) + 1 = b - 0.$$

- For $3 \le i \le 2k+1$,

$$c'(v_i) = \begin{cases} b - (2k+3-i) & \text{if } i \text{ is even,} \\ b - (2k+2) - (2k+4-i) + 1 = b - (2-i) & \text{if } i \text{ is odd.} \end{cases}$$

- For $2k+2 \le i \le 4k+1$,

$$c'(v_i) = \begin{cases} b - (2k+2) - (i-2k) + 1 = b - (i+1) & \text{if } i \text{ is even,} \\ b - (i-1-2k) & \text{if } i \text{ is odd.} \end{cases}$$

For each i with $0 \le i \le 4k+1$, let $c'(v_i) = b - a_i$ for $0 \le i \le 4k+1$. If $s_{c'} = (a_0, a_1, \ldots, a_{4k+1})$ (where $a_i = b - c'(v_i)$ for $0 \le i \le 4k+1$), then

$$\begin{aligned} s_{c'} = (&2k+2, 1, 0, 4k+2, 2k-1, 4k, 2k-3, \ldots, 2k+8, 5, \\ &2k+6, 3, 2k+4 = b - c'(v_{2k+1}), 2k+3, 2, \\ &2k+5, 4, \ldots, 4k-1, 2k-2, 4k+1, 2k). \end{aligned}$$

For example, the sequences $s_{c'}$ for $n = 6, 10, 14$ are the following:

$$\begin{aligned} &(4,1,0,6,5,2) \ \text{ for } n = 6 \text{ and } k = 1, \\ &(6,1,0,10,3,8,7,2,9,4) \ \text{ for } n = 10 \text{ and } k = 2, \\ &(8,1,0,14,5,12,3,10,9,2,11,4,13,6) \ \text{ for } n = 14 \text{ and } k = 3. \end{aligned}$$

In conclusion, we observe that $\{c'(v) : v \in V(K_{4k+2})\} = \{b - i : 0 \le i \le 4k+2, i \ne 2k+1\}$ and so c is a twin edge $(4k+3)$-coloring of K_{4k+2}. ■

In summary, we have the following.

Theorem 3.6. *For each integer $n \ge 3$,*

$$\chi'_t(K_n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n+1 & \text{if } n \text{ is even.} \end{cases}$$

4. Complete Bipartite Graphs

In this section we determine the twin chromatic indexes of the complete bipartite graphs $K_{a,b}$ where $1 \le a \le b$, beginning with stars. For a star $K_{1,b}$ ($b \ge 2$), a twin edge coloring is the same as a modular edge-graceful labeling (see [9]) and so we have the following result (see also Lemma 2 in [2]).

Proposition 4.1. *If $K_{1,b}$ is a star of order $b \ge 2$, then*

$$\chi'_t(K_{1,b}) = \begin{cases} b+1 & \text{if } b \not\equiv 1 \pmod 4, \\ b+2 & \text{if } b \equiv 1 \pmod 4. \end{cases}$$

We now determine $\chi'_t(K_{a,b})$ where $2 \leq a \leq b$ and $b \in \{a, a+1\}$.

Lemma 4.2. *If $a \geq 2$ and b are integers with $b \in \{a, a+1\}$, then $\chi'_t(K_{a,b}) = a+2$.*

Proof. Let $U = \{u_1, u_2, \ldots, u_a\}$ and $V = \{v_1, v_2, \ldots, v_b\}$ be the partite sets of $K_{a,b}$. We consider two cases, according to whether $b = a$ or $b = a+1$.

Case 1. $b = a$. By Observation 2.3, $\chi'_t(K_{a,a}) \geq a+1$. First, we show that $\chi'_t(K_{a,a}) \neq a+1$. Suppose that $\chi'_t(K_{a,a}) = a+1$. Then there is a twin edge $(a+1)$-coloring c of $K_{a,a}$ using the colors in $\mathbb{Z}_{a+1}$. Hence c assigns exactly a colors to the a incident edges of each vertex of $K_{a,a}$. Consider u_1 and let $t \in \mathbb{Z}_{a+1}$ such that c assigns the colors in $\mathbb{Z}_{a+1} - \{t\}$ to the edges incident with u_1 (and so no edge incident with u_1 is colored t). We claim that for each vertex v_j $(1 \leq j \leq a)$, there is an edge incident with v_j that is colored t; for otherwise, we may assume that no edge incident with v_1 is colored t. However then, c assigns the colors in $\mathbb{Z}_{a+1} - \{t\}$ to the edges incident with v_1 and so $c'(u_1) = c'(v_1)$, which is impossible. Thus, as claimed, there is an edge incident with v_j that is colored t for $j = 1, 2, \ldots, a$. Hence there are at least a edges of $K_{a,a}$ that are colored t. Since no edge incident with u_1 is colored t, it follows that at least two edges colored t are incident with the same vertex in U, which is a contradiction. Therefore, $\chi'_t(K_{a,a}) \neq a+1$ and so $\chi'_t(K_{a,a}) \geq a+2$.

Next, we show that $K_{a,a}$ has a twin edge $(a+2)$-coloring. Since $K_{a,a}$ is bipartite and a-regular, $K_{a,a}$ is 1-factorable. Let $\{F_0, F_1, \ldots, F_{a-1}\}$ be a 1-factorization of $K_{a,a}$ where

$$E(F_i) = \{u_j v_{j+i} : 1 \leq j \leq a\} \text{ for } 0 \leq i \leq a-1$$

(all subscripts are expressed as integers modulo a). For example, $E(F_0) = \{u_j v_j : 1 \leq j \leq a\}$, $E(F_1) = \{u_j v_{j+1} : 1 \leq j \leq a\}$ and $E(F_{a-1}) = \{u_j v_{j+(a-1)} : 1 \leq j \leq a\}$. We consider two cases, according to whether a is odd or a is even.

Subcase 1.1 a is odd. Then $a = 2k+1$ for some positive integer k. Let M_a and M_{a+1} be the following matchings in $K_{a,a}$:

$$M_a = \{u_1v_1, u_3v_2, u_4v_4, u_6v_6, \ldots, u_{2k}v_{2k}\},$$
$$M_{a+1} = \{u_1v_2, u_3v_3, u_4v_5, u_6v_7, \ldots\, u_{2k}v_{2k+1}\}.$$

Thus $|M_a| = |M_{a+1}| = k+1$. For each i with $0 \leq i \leq a-1$, let $M_i = E(F_i) - (M_a \cup M_{a+1})$. Define a proper edge coloring $c : E(K_{a,a}) \to \mathbb{Z}_{a+2}$ by $c(e) = i$ if $e \in M_i$ for $0 \leq i \leq a+1$. Since $c'(u) = \binom{a}{2}$ or $c'(u) = \binom{a}{2} - 4$ if $u \in U$ and $c'(v) = \binom{a}{2} - 1$ or $c'(v) = \binom{a}{2} - 2$ if $v \in V$, it follows that c is a twin edge $(a+2)$-coloring. Therefore, $\chi'_t(K_{a,a}) = a+2$.

Subcase 1.2 a is even. Then $a = 2k \geq 2$ for some positive integer k. Let M_a and M_{a+1} be the following matchings in $K_{a,a}$:

$$M_a = \{u_1v_1, u_3v_3, \ldots, u_{2k-1}v_{2k-1}\},$$
$$M_{a+1} = \{u_1v_2, u_3v_4, \ldots, u_{2k-1}v_{2k}\}.$$

Thus $|M_a| = |M_{a+1}| = k$. For each i with $0 \le i \le a-1$, let $M_i = E(F_i) - (M_a \cup M_{a+1})$. Define a proper edge coloring $c : E(K_{a,a}) \to \mathbb{Z}_{a+2}$ by $c(e) = i$ if $e \in M_i$ for $0 \le i \le a+1$. Since $c'(u) = \binom{a}{2}$ or $c'(u) = \binom{a}{2} - 4$ if $u \in U$ and $c'(v) = \binom{a}{2} - 2$ if $v \in V$, it follows that c is a twin edge $(a+2)$-coloring. Therefore, $\chi'_t(K_{a,a}) = a+2$.

Case 2. $b = a+1$. Since $\Delta(K_{a,a+1}) = a+1$, it follows that $\chi'_t(K_{a,a+1}) \ge a+1$. First, we show that $\chi'_t(K_{a,a+1}) \ne a+1$. Suppose that $\chi'_t(K_{a,a+1}) = a+1$. Then there is a twin edge $(a+1)$-coloring c of $K_{a,a+1}$ using colors in $\mathbb{Z}_{a+1}$. Since $\deg u_i = a+1$ for $1 \le i \le a$, it follows that c assigns all colors in $\mathbb{Z}_{a+1}$ to the $a+1$ edges incident with each vertex u_i. Thus, a edges in $K_{a,a+1}$ are colored 0. Since $|V| = a+1$, some vertex in V is not incident with any edge colored 0, say v_1. Consequently, c assigns the a colors in $\mathbb{Z}_{a+1} - \{0\}$ to the a edges incident with v_1. However then, $c'(v_1) = c'(u_i) = \binom{a+1}{2}$ for $1 \le i \le a$, which is impossible. Therefore, $\chi'_t(K_{a,a+1}) \ne a+1$ and $\chi'_t(K_{a,a+1}) \ge a+2$.

Next, we show that $K_{a,a+1}$ has a twin edge $(a+2)$-coloring. Define a proper edge coloring $c : E(K_{a,a+1}) \to \mathbb{Z}_{a+2}$ using only the colors in $\mathbb{Z}_{a+2} - \{0\}$ as follows. For each i with $1 \le i \le a$, let $c(u_i v_{i+j}) = j+1$ for each j with $0 \le j \le a$. In particular, $c(u_i v_i) = 1$ for $1 \le i \le a$. Thus, $c'(u_i) = \binom{a+2}{2}$ for $1 \le i \le a$. Furthermore, $c'(v_j) = \binom{a+2}{2} - (j+1)$ for $1 \le j \le a$ and $c'(v_{a+1}) = \binom{a+2}{2} - 1$. Since $c'(v_j) \ne \binom{a+2}{2}$ in $\mathbb{Z}_{a+2}$ for $1 \le j \le a+1$, it follows that c' is a proper vertex coloring of $K_{a,a+1}$. Therefore, $\chi'_t(K_{a,a+1}) = a+2$. ■

Finally, we determine $\chi'_t(K_{a,b})$ for all integers a and b with $a \ge 2$ and $b \ge a+2$.

Lemma 4.3. *If $a \ge 2$ and b are integers with $b \ge a+2$, then $\chi'_t(K_{a,b}) = b$.*

Proof. Since $\chi'_t(K_{a,b}) \ge \chi'(K_{a,b}) = \Delta(K_{a,b}) = b$, it suffices to show that $K_{a,b}$ has a twin edge b-coloring. Let $U = \{u_1, u_2, \ldots, u_a\}$ and $V = \{v_1, v_2, \ldots, v_b\}$ be the partite sets of $K_{a,b}$. Suppose that

$$\sum_{i=0}^{a-1} i = \binom{a}{2} \equiv k \pmod{b} \text{ and } \sum_{i=0}^{b-1} i = \binom{b}{2} \equiv \ell \pmod{b},$$

where $0 \le k, \ell \le b-1$. We consider two cases, depending on whether a and b are relatively prime.

Case 1. a and b are not relatively prime. Then $d = \gcd(a,b) \ge 2$ and $b = pd$ for some $p \in \mathbb{N}$. For $0 \le i \le d-1$, let $X_i = \{i, i+a, i+2a, \ldots, i+(p-1)a\}$ be a subset of $\mathbb{Z}_b$. In fact, $X_0, X_1, \ldots, X_{d-1}$ are the cosets of the subgroup $X_0 = \{0, a, 2a, \ldots, (p-1)a\}$ in the group $\mathbb{Z}_b$. Hence $\mathcal{X} = \{X_0, X_1, \ldots, X_{d-1}\}$ is a partition of $\mathbb{Z}_b$. Next, let $X, X' \in \mathcal{X}$ such that $k \in X$ and $\ell \in X'$. We define a coloring $c : E(K_{a,b}) \to \mathbb{Z}_b$, according to whether $X \ne X'$ or $X = X'$.

Subcase 1.1. $X \neq X'$. For $1 \leq i \leq a$ and $1 \leq j \leq b$, define $c(u_iv_j) = i + j - 2 \in \mathbb{Z}_b$. Then $c'(u_i) = \ell \in X'$ for $1 \leq i \leq a$ and $c'(v_j) = k + (j-1)a \in X$ for $1 \leq j \leq b$. Since $X \cap X' = \emptyset$, it follows that $c'(u_i) \neq c'(v_j)$ for all i, j with $1 \leq i \leq a$ and $1 \leq j \leq b$. Thus c' is a proper vertex coloring.

Subcase 1.2. $X = X'$. Since $d \geq 2$, it follows that $k+1 \notin X$, say $k+1 \in X'' \in \mathcal{X}$. For $1 \leq i \leq a-1$ and $1 \leq j \leq b$, define $c(u_iv_j) = i+j-2 \in \mathbb{Z}_b$. Furthermore, define $c(u_av_j) = a + j - 1 \in \mathbb{Z}_b$ for $1 \leq j \leq b$. Then $c'(u_i) = \ell \in X$ for $1 \leq i \leq a$ and $c'(v_j) = (k+1) + (j-1)a \in X''$ for $1 \leq j \leq b$. Since $X \cap X'' = \emptyset$, it follows that $c'(u_i) \neq c'(v_j)$ for all i, j with $1 \leq i \leq a$ and $1 \leq j \leq b$. Thus c' is a proper vertex coloring.

Case 2. a and b are relatively prime. Note that $\mathbb{Z}_b = \{\ell, \ell+a, \ldots, \ell+(b-1)a\}$. We start with a proper edge coloring $c_1 : E(K_{a,b}) \to \mathbb{Z}_b$ defined by $c_1(u_iv_j) = i + j - 2$ for $1 \leq i \leq a$ and $1 \leq j \leq b$. Then $c_1'(u_i) = \ell$ for $1 \leq i \leq a$ and $c_1'(v_j) = k + (j-1)a \in X$ for $1 \leq j \leq b$. Since a and b are relatively prime, $\{c_1'(v_j) : 1 \leq j \leq b\} = \mathbb{Z}_b$. Therefore, there exists exactly one integer t with $1 \leq t \leq b$ such that $c_1'(v_t) = \ell$. Thus c_1' is not a proper vertex coloring. We now produce a twin edge b-coloring c from c_1 as follows. Let $r = \lceil a/2 \rceil$ and $s = t + \lfloor (b-1)/2 \rfloor$ in $\mathbb{Z}_b$, where then $1 \leq s \leq b$ and $s \neq t$, and let c be the coloring obtained from c_1 by interchanging the colors of the edges u_rv_t and u_rv_s in c_1; that is,

$$c(e) = \begin{cases} c_1(e) & \text{if } e \in E(K_{a,b}) - \{u_rv_t, u_rv_s\}, \\ c_1(u_rv_s) & \text{if } e = u_rv_t, \\ c_1(u_rv_t) & \text{if } e = u_rv_s. \end{cases}$$

We show that $c'(u_i) = \ell$ for $1 \leq i \leq a$ and $c'(v_j) \neq \ell$ for $1 \leq j \leq b$.

By the defining property of c, it follows that $c'(u_i) = c_1'(u_i) = \ell$ and $c'(v_j) = c_1'(v_j) \neq \ell$ for $1 \leq j \leq b$ and $j \neq s, t$. Thus, it remains to show that $c'(v_t) \neq \ell$ and $c'(v_s) \neq \ell$. Since $\ell = k + (t-1)a$ and $s = t + \lfloor (b-1)/2 \rfloor$, it follows that

$$\begin{aligned} c'(v_t) &= c_1'(v_t) - c_1(u_rv_t) + c_1(u_rv_s) = \ell - (r+t-2) + (r+s-2) \\ &= \ell - t + s = \ell - t + [t + \lfloor (b-1)/2 \rfloor] = \ell + \lfloor (b-1)/2 \rfloor \\ c'(v_s) &= c_1'(v_s) - c_1(u_rv_s) + c_1(u_rv_t) = [k + (s-1)a] - (r+s-2) + (r+t-2) \\ &= [k + (s-1)a] - s + t = [k + (s-1)a] - \lfloor (b-1)/2 \rfloor \\ &= k + (t + \lfloor (b-1)/2 \rfloor - 1)a - \lfloor (b-1)/2 \rfloor = \ell + (a-1)\lfloor (b-1)/2 \rfloor. \end{aligned}$$

We consider two cases, according to whether b is odd or b is even.

Subcase 2.1. *b is odd.* Then $\lfloor (b-1)/2 \rfloor = \frac{b-1}{2}$. We claim that

$$c'(v_t) = \ell + \frac{b-1}{2} \neq \ell \text{ in } \mathbb{Z}_b, \tag{4}$$

$$c'(v_s) = \ell + (a-1)\frac{b-1}{2} \neq \ell \text{ in } \mathbb{Z}_b. \tag{5}$$

Since b is odd, $\ell = 0$ in $\mathbb{Z}_b$ by Observation 3.1, while $\frac{b-1}{2} \neq 0$ in $\mathbb{Z}_b$, which implies that (4) holds. To verify (5), we show that $\frac{b-1}{2}(a-1) \not\equiv 0 \pmod{b}$. If this were not the case, then $\frac{b-1}{2}(a-1) = bx$ for some integer x. This implies that $2bx = (b-1)(a-1) = a-1$ in $\mathbb{Z}_b$ or $a-1 \equiv 0 \pmod{b}$. However then, $b \mid (a-1)$, which is impossible.

Subcase 2.2. b is even. Then $\lfloor (b-1)/2 \rfloor = \frac{b}{2} - 1$ and $\ell = \frac{b}{2}$ in $\mathbb{Z}_b$ by Observation 3.1. Since a and b are relatively prime, it follows that $a \geq 3$ is odd and so $b \geq a+3 \geq 6$. We claim that

$$c'(v_t) = \ell + \left(\frac{b}{2} - 1\right) = b - 1 \neq \ell \text{ in } \mathbb{Z}_b, \tag{6}$$

$$c'(v_s) = \ell + \left(\frac{b}{2} - 1\right)(a-1) \neq \ell \text{ in } \mathbb{Z}_b \,. \tag{7}$$

Since $\ell = \frac{b}{2}$ in $\mathbb{Z}_b$ and $b - 1 \neq \frac{b}{2}$ in $\mathbb{Z}_b$, it follows that (6) holds. To verify (7), we show that $\left(\frac{b}{2} - 1\right)(a-1) \not\equiv 0 \pmod{b}$. If this were not the case, then $\frac{b-2}{2}(a-1) = bx$ for some positive integer x. Since b is even, $b = 2y$ for some integer $y \geq 3$. Then $a = 2\frac{xy}{y-1} + 1$. Since a is an integer and $y \geq 3$, it follows that $(y-1) \nmid y$ and so $(y-1) \mid x$. Let $x = (y-1)z$ for some positive integer z. However then, $a = 2yz + 1 = bz + 1$, which is impossible.

Thus c' is a proper vertex coloring of $K_{a,b}$ and so $\chi'_t(K_{a,b}) = b$. ■

In summary, we have the following.

Theorem 4.4. *For positive integers a and b with $a \leq b$,*

$$\chi'_t(K_{a,b}) = \begin{cases} b & \text{if } b \geq a+2 \text{ and } a \geq 2, \\ b+1 & \text{if either } a = 1 \text{ and } b \not\equiv 1 \pmod{4} \text{ or } b = a+1 \geq 3, \\ b+2 & \text{if either } a = 1 \text{ and } b \equiv 1 \pmod{4} \text{ or } b = a \geq 2. \end{cases}$$

For every connected graph G for which the twin chromatic index has been determined, we have seen that $\chi'_t(G) = \Delta(G) + i$ for some $i \in \{0, 1, 2, 3\}$. This leads us to conclude this paper by stating the following problem.

Problem 4.5. *Is $\chi'_t(G) \leq \Delta(G) + 3$ for every connected graph G of order at least 3?*

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