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ON THE INDEPENDENCE NUMBER OF EDGE CHROMATIC CRITICAL GRAPHS*

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Abstract

In 1968, Vizing conjectured that for any edge chromatic critical graph $G = (V, E)$ with maximum degree Δ and independence number $\alpha(G)$, $\alpha(G) \leq \frac{|V|}{2}$. It is known that $\alpha(G) < \frac{3\Delta-2}{5\Delta-2}|V|$. In this paper we improve this bound when $\Delta \geq 4$. Our precise result depends on the number n_2 of 2-vertices in G, but in particular we prove that $\alpha(G) \leq \frac{3\Delta-3}{5\Delta-3}|V|$ when $\Delta \geq 5$ and $n_2 \leq 2(\Delta - 1)$.

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1. Introduction

Throughout this paper, let $G = (V(G), E(G))$ be a simple graph with n vertices and m edges. We write $d(x)$ and $N(x)$ to denote the degree and the set of neighbors of a vertex $x \in V(G)$, and we write $N(x, y) = N(x) \cup N(y)$ and $N(X) = \bigcup_{x\in X} N(x)$ for $X \subseteq V(G)$. A *k-vertex*, ($\geq k$)*-vertex* or ($\leq k$)*-vertex*

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is a vertex of degree k, at least k or at most k, respectively. We call k-vertices adjacent to x *k-neighbors* of x and define $N_k(x)$ to be the set of k-neighbors of x and $d_k(x)$ to be the number of k-neighbors of x. Similarly, we define *$(\geq k)$-neighbors, $(\leq k)$-neighbors* and use corresponding notations $N_{\geq k}(x)$ and $N_{\leq k}(x)$, and $d_{\geq k}(x)$ and $d_{\leq k}(x)$, respectively. Let $\Delta(G)$, $\delta(G)$ (or Δ, δ) and $\alpha(G)$ be the maximum degree, minimum degree and independence number of G, respectively.

An *edge coloring* of a graph is a function assigning values (colors) to the edges of the graph in such a way that any two adjacent edges receive different colors. A graph is *edge k-colorable*, if there is an edge coloring of the graph with colors from $\{1, 2, \dots, k\}$, and the smallest k such that G is edge k-colorable is called the *edge chromatic number*, denoted by $\chi'(G)$. In 1964, Vizing [8] proved a theorem which states that if G is a graph of maximum degree Δ, then the edge chromatic number $\chi'(G)$ of G is either Δ or $\Delta + 1$. A graph G is said to be of *class one* if $\chi'(G) = \Delta$, and of *class two* if $\chi'(G) = \Delta + 1$. G is said to be *edge chromatic critical* if it is connected, class two and $\chi'(G - e) < \chi'(G)$ for every edge $e \in E(G)$. An edge chromatic critical graph G of maximum degree Δ is called an (*edge chromatic*) *Δ-critical graph.* The following conjecture about Δ-critical graphs was proposed by Vizing in 1968.

Conjecture 1.1 [9]**.** *Let G be a Δ-critical graph with n vertices. Then*

$$\alpha(G) \leq \frac{n}{2}.$$

Conjecture 1.1 is still open so far. The following are some results towards this conjecture.

Theorem 1.2 [1]**.** *Let G be a Δ-critical graph with n vertices. Then*

(i) $\alpha(G) \leq \frac{(2k-1)\Delta - k(k-1)}{(3k-1)\Delta - k(k-1)} n$, $k = \left\lfloor \sqrt{\Delta(G) + \frac{1}{4}} + \frac{1}{2} \right\rfloor$.

(ii) $\alpha(G) < \frac{2n}{3}$.

(iii) $$\alpha(G) \leq \begin{cases} \frac{3\Delta-2}{5\Delta-2} n & if\ 3 \leq \Delta \leq 6, \\ \frac{5\Delta-6}{8\Delta-6} n & if\ \ 7 \leq \Delta \leq 10. \end{cases}$$

In 2004, Grünewald and Steffen [2] verified this conjecture for critical graphs with many edges and in particular, they verified the conjecture for overfull critical graphs.

Luo and Zhao proved the conjecture for critical graphs with large maximum degrees.

Theorem 1.3 [3]**.** *Let G be a Δ-critical graph with n vertices and $\Delta \geq \frac{n}{2}$. Then $\alpha(G) \leq \frac{n}{2}$.*

The following result is due to Woodall, and improves the bound in Theorem 1.2.

Theorem 1.4 [10]**.** *Let G be a Δ-critical graph with n vertices. Then*

$$\alpha(G) < \frac{3\Delta - 2}{5\Delta - 2} n.$$

Other results about this conjecture can be found in [4,6].

In this paper, we will use the properties of critical graphs to improve the above bound by the following result.

Theorem 1.5. *Let $G = (V, E)$ be a Δ-critical graph of order n with $\Delta \geq 4$, let n_2 denote the number of 2-vertices of G and define*

$$\beta = \begin{cases} \frac{2}{3} & \text{if } \Delta = 4, \\ 1 & \text{if } \Delta \geq 5, \end{cases} \qquad \gamma = \max\{2\beta n_2 - 4(\Delta - 1), 0\}.$$

Then $\alpha(G) \leq \frac{(3\Delta - 2 - \beta)n + \gamma}{5\Delta - 2 - \beta}$.

2. Lemmas

Lemma 2.1 (Vizing's Adjacency Lemma, or VAL [9])**.** *Let x be a vertex of a Δ-critical graph. Then*

(i) *if $d_k(x) \geq 1$, then $d_\Delta(x) \geq \Delta - k + 1$,*

(ii) *$d_\Delta(x) \geq 2$.*

Lemma 2.1 implies the following corollary.

Corollary 2.2. *Let xy be an edge in a Δ-critical graph. Then*

(i) *$d(x) \geq 2$,*

(ii) *$d(x) + d(y) \geq \Delta + 2$.*

Lemma 2.3 [7, 12]**.** *Let G be a Δ-critical graph, $xy \in E(G)$, and $d(x) + d(y) = \Delta + 2$. Then the following hold:*

(i) *every vertex of $N(x, y) \setminus \{x, y\}$ is a Δ-vertex,*

(ii) *every vertex of $N(N(x, y)) \setminus \{x, y\}$ is of degree at least $\Delta - 1$,*

(iii) *if $d(x), d(y) < \Delta$, then every vertex of $N(N(x, y)) \setminus \{x, y\}$ is a Δ-vertex.*

Lemma 2.4 [5]**.** *Let G be a Δ-critical graph with $\Delta \geq 5$ and x be a 3-vertex. Then there are at least two Δ-vertices in $N(x)$ which are not adjacent to any $(\leq \Delta - 2)$-vertex except x.*

Let G be a Δ-critical graph, $xy \in E(G)$. Let $\sigma(x,y)$ denote the number of vertices in $N(y)\backslash x$ that have degree at least $2\Delta - d(x) - d(y) + 2$. By Corollary 2.2 and Lemma 2.1, $2\Delta - d(x) - d(y) + 2 \leq \Delta$, and y has at least $\Delta - d(x) + 1$ neighbors different from x with degree Δ; thus

$$\sigma(x,y) \geq \Delta - d(x) + 1. \tag{1}$$

Lemma 2.5 [11]**.** *Let x be a vertex in a Δ-critical graph G, and let*

$$p_{min} = \min_{y\in N(x)}\{\sigma(x,y) - \Delta + d(x) - 1\} \tag{2}$$

and

$$p = \min\left\{p_{min}, \left\lfloor \frac{d(x)}{2} \right\rfloor - 1\right\}. \tag{3}$$

Then x has at least $d(x) - p - 1$ neighbors y for which $\sigma(x,y) \geq \Delta - p - 1$.

3. Proof of Theorem 1.5

The main idea of the proof and many details are the same as those presented in [10].

Let G be a Δ-critical graph. Let $S \subset V$ be an independent set, and let $T = V \setminus S$. Let $S_2 = \{x \in S : d(x) = 2\}, S_- = \{x \in S : 3 \leq d(x) < \frac{\Delta}{2} + 1\}, S_+ = \{x \in S : \frac{\Delta}{2} + 1 \leq d(x) < \Delta\}$ and $S_\Delta = \{x \in S : d(x) = \Delta\}$.

Claim 1. *If $S_2 \neq \emptyset$, then there are at least $\Delta - 1$ edges in the induced subgraph $G[T]$.*

Proof. Let $x \in S_2$ and $N(x) = \{y, z\}$. $G - xy$ is edge Δ-colorable since G is critical. Let ϕ be an edge coloring of $G - xy$ with color set $\{1, 2, \ldots, \Delta\}$ and $\phi(xz) = 1$. Let $k \in \{2, 3, \ldots, \Delta\}$, and $P_x(1,k)$ be the longest path colored with 1 and k with origin x. Then $P_x(1,k)$ ends at y and $P_x(1,k) + xy$ is an odd cycle. Since S is an independent set and $\{y,z\} \subseteq T$, so there is at least one edge of $P_x(1,k)$ colored k in $G[T]$. Since k is arbitrary, there are at least $\Delta - 1$ edges in $G[T]$. ■

By the same argument, it can be proved that if S contains a k-vertex, then $G[T]$ has at least $\Delta - k + 1$ edges; this can be used to improve the result of Theorem 1.5 slightly when $n_2 = 0$.

Define a charge function M_0 on V as follows:

$$M_0(x) = 0 \text{ for } x \in S, \quad M_0(y) = 3\Delta - 2 - \beta \text{ for } y \in T. \tag{4}$$

For each vertex $x \in S$ with $d(x) = k$, define:

$$f(x) = g(k) = \frac{2(\Delta - k)}{k}.$$

Clearly, g is a decreasing function of k.

For $y \in T$, let $d_T(y)$ denote the degree of y in the induced subgraph $G[T]$. Starting with the distribution M_0, we will redistribute charge according to the following steps.

Step 1. Each vertex in S receives charge 2 from each of its neighbors in T.

Step 2. If $n_2 \neq 0$, then each vertex $y \in T$ gives charge $2d_T(y)$ to S_2, distributed equally among all vertices in S_2.

Step 3. Each vertex $y \in T$ gives to each vertex $x \in N(y) \cap S_+$ charge $\frac{2}{3}$ if $\Delta = 4$, and charge $f(x)$ if $\Delta \geq 5$.

Step 4. Each vertex $y \in T$ distributes its remaining charge equally among all vertices in $N(y) \cap (S_2 \cup S_-)$ (if any).

Let the charge function after Step i be M_i, so that M_4 denotes the final charge distribution.

It is easy to see that $M_1(y) \geq \Delta - 2 - \beta + 2d_T(y)$ for each vertex $y \in T$, and $M_1(x) = 2d(x)$ for each vertex $x \in S$.

It follows that $M_2(y) \geq \Delta - 2 - \beta \geq \Delta - 3$ for each vertex $y \in T$, and $M_2(x) \geq 4 + \frac{4(\Delta-1)}{n_2}$ for each vertex $x \in S_2$, since $\sum_{y \in T} d_T(y) \geq 2(\Delta - 1)$ by Claim 1 if $n_2 \neq 0$. Also, $M_2(x) = M_1(x) = 2d(x)$ for each vertex $x \in S \setminus S_2$.

The proofs of Claims 2 and 3 are similar to those of Claims 1 and 2 in [10] respectively.

Claim 2. *$M_3(y) \geq 0$ for each vertex $y \in T$.*

Claim 3. *If $y \in T$ and $x_i, x_j \in N(y) \cap S$ and $3 \leq d(x_i) \leq d(x_j) \leq \Delta - 1$, then in Steps 3 and 4 y gives at least as much charge to x_i as it does to x_j.*

Note for future reference that if $\Delta \geq 5$ and $r \geq \frac{\Delta}{2} + 1$ and $r \in \mathbb{Z}$ then

$$\begin{aligned} r(\Delta - 3) - 2(r-1)(\Delta - r) &= 2r^2 - r\Delta - 5r + 2\Delta \\ &= (r-2)(2r - \Delta - 2) + r - 4 \geq 0, \end{aligned} \tag{5}$$

since r is an integer and so $r \geq 4$.

We will prove that $M_4(x) \geq 2\Delta - \frac{\gamma}{n_2}$ for each vertex $x \in S_2, M_4(x) \geq 2\Delta$ for each $x \in S \setminus S_2$, and $M_4(y) \geq 0$ for each $y \in T$. By the definition of M_0, this implies that $2\Delta|S| - \gamma \leq \sum_{v \in V} M_4(v) = \sum_{v \in V} M_0(v) = (3\Delta - 2 - \beta)|T|$, from which it immediately follows that $|S| \leq \frac{(3\Delta-2-\beta)n+\gamma}{5\Delta-2-\beta}$, as required.

(a) For $y \in T$, by Claim 2 and Step 4, it is easy to see that $M_4(y) \geq 0$.

(b) For $x \in S_+$, by Steps 1 and 3, it is easy to see that $M_4(x) \geq 2\Delta$.

(c) For $x \in S_2$ and $y \in N(x)$, every vertex in $N(y) \setminus \{x\}$ has degree Δ by Lemma 2.3(i), and so in Step 4 y gives x charge $M_3(y) = M_2(y) \geq \Delta - 2 - \beta$. Thus $M_4(x) \geq 4 + \frac{4(\Delta-1)}{n_2} + 2(\Delta - 2 - \beta) = 2\Delta - \frac{2\beta n_2 - 4(\Delta-1)}{n_2} \geq 2\Delta - \frac{\gamma}{n_2}$.

(d) If $x \in S_-$ and $d(x) = 3$, then $\Delta \geq 5$. Let $N(x) = \{y, z, w\}$. By VAL (Lemma 2.1), each neighbor of x has at most one $(< \Delta)$-neighbor other than x. By Lemma 2.4, at least two Δ-neighbors of x, say y and z, have no $(\leq \Delta - 2)$-neighbor except x; hence each gives charge at most $g(\Delta - 1) = \frac{2}{\Delta-1}$ in Step 3 and so gives x at least $\Delta - 3 - \frac{2}{\Delta-1}$ in Step 4. The remaining neighbor w may have an r-neighbor in $S_+ (r \geq \frac{\Delta}{2} + 1)$ and so gives x at least $\min\left\{\Delta - 3 - \frac{2(\Delta-r)}{r}, \frac{\Delta-3}{2}\right\} = \frac{\Delta-3}{2}$, since $\frac{\Delta-3}{2} \geq \frac{(r-1)(\Delta-r)}{r} > \frac{2(\Delta-r)}{r}$ by (5). It follows that $M_4(x) \geq 6 + 2\left(\Delta - 3 - \frac{2}{\Delta-1}\right) + \frac{\Delta-3}{2} = 2\Delta - \frac{4}{\Delta-1} + \frac{\Delta-3}{2} \geq 2\Delta$.

Now let $x \in S_-$ and $d(x) = k \geq 4$. Then $\Delta > 6$.

Define

$$h(k, l) = \frac{1}{k - l - 1}(\Delta - 3 - l \cdot g(\Delta - k + 2)) = \frac{1}{k - l - 1}\left(\Delta - 3 - l\frac{2(k-2)}{\Delta - k + 2}\right).$$

Claim 4. *If l is a nonnegative integer and y is a neighbor of x such that $\sigma(x, y) \geq \Delta - k + l + 1$, then y gives x at least $h(k, l)$ in Step 4.*

The proof of Claim 4 is similar to that of Claim 3 in [10].

Now define p as in Lemma 2.5(3). Let $N^+(x)$ be a set of $k - p - 1$ neighbors y of x for which $\sigma(x, y) \geq \Delta - p - 1$; the existence of such neighbors is proved in Lemma 2.5. Let $N^-(x) = N(x) \setminus N^+(x)$; then $N^-(x)$ contains $p + 1$ neighbors y of x, for which $\sigma(x, y) \geq \Delta - k + p + 1$ by the definition of p. Applying Claim 4 to the vertices in $N^-(x)$ with $l = p$, and to the vertices in $N^+(x)$ with $l = k - p - 2$, x receives charge of at least $\nu(k, p)$ in Step 4, where

$$\begin{aligned}\nu(k, p) &= (p + 1)h(k, p) + (k - p - 1)h(k, k - p - 2) \\ &= \frac{p+1}{k-p-1}\left(\Delta - 3 - p\frac{2(k-2)}{\Delta - k + 2}\right) \\ &\quad + \frac{k-p-1}{p+1}\left(\Delta - 3 - (k - p - 2)\frac{2(k-2)}{\Delta - k + 2}\right).\end{aligned}$$

Next we prove that this is at least $2(\Delta - d(x)) = 2(\Delta - k)$. Then this will imply that $M_4(x) \geq 2k + 2(\Delta - k) = 2\Delta$ by Steps 1 and 4.

Let $t = p + 1$, so that $1 \leq t \leq \frac{k}{2}$, since $0 \leq p \leq \frac{k}{2} - 1$ by Lemma 2.5(3). Setting

$$b = \frac{2(k-2)}{\Delta - k + 2} \quad \text{and} \quad a = \Delta - 3 + b,$$

we can write

$$\nu(k,p) = \frac{t(a-bt)}{k-t} + \frac{(k-t)(a-b(k-t))}{t}.$$

The derivative of this in respect to t is

$$\begin{aligned}\frac{d\nu(k,p)}{dt} &= \frac{ak-bk^2+b(k-t)^2}{(k-t)^2} - \frac{ak-bk^2+bt^2}{t^2} \\ &= \frac{ak-bk^2}{(k-t)^2} - \frac{ak-bk^2}{t^2} \\ &= k(a-bk)\cdot\frac{k(2t-k)}{t^2(k-t)^2}.\end{aligned}$$

Next we prove that $a - bk \geq 0$, which implies $\frac{d\nu(k,p)}{dt} \leq 0$ since $t \leq \frac{k}{2}$. In fact, $a - bk = \frac{(\Delta-3)(\Delta-k+2)-2(k-1)(k-2)}{\Delta-k+2} > \frac{(2k-5)k-2(k-1)(k-2)}{\Delta-k+2} = \frac{k-4}{\Delta-k+2} \geq 0$, since $4 \leq k < \frac{\Delta}{2}+1$. So $\nu(k,p)$ regarded as a function of p is decreasing in $[0, \frac{k}{2}-1]$.

To complete the proof, we need only to show that $\nu(k,0) \geq 2\Delta - k$, i.e. we show that

$$\frac{\Delta-3}{k-1} + (k-1)\left((\Delta-3) - \frac{2(k-2)^2}{\Delta-k+2}\right) \geq 2(\Delta-k). \tag{6}$$

Since $k < \frac{\Delta}{2}+1$, we can write $\Delta = 2k+s$, where $s \geq -1$. By (6), it suffices to show that $(\Delta-3) - \frac{2(k-2)^2}{\Delta-k+2} - \frac{2(\Delta-k)}{k-1} \geq 0$, i.e. we show that

$$2k+s-3-\frac{2(k+s)}{k-1} - \frac{2(k-2)^2}{k+s+2} \geq 0. \tag{7}$$

Since the left-hand-side of (7) is clearly an increasing function of s, it suffices to verify (4) for $s = -1$, where the left-hand-side becomes $2k-4-2-\frac{2(k-2)^2}{k+1} = \frac{4k-14}{k+1} > 0$ since $k \geq 4$.

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