

## CHROMATIC POLYNOMIALS OF MIXED HYPERCYCLES

JULIAN A. ALLAGAN AND DAVID SLUTZKY

*Department of Mathematics*  
*University of North Georgia*  
*Watkinsville, Georgia, USA*

**e-mail:** julian.allagan@ung.edu  
 david.slutzky@ung.edu

### Abstract

We color the vertices of each of the edges of a  $\mathcal{C}$ -hypergraph (or cohypergraph) in such a way that at least two vertices receive the same color and in every proper coloring of a  $\mathcal{B}$ -hypergraph (or bihypergraph), we forbid the cases when the vertices of any of its edges are colored with the same color (monochromatic) or when they are all colored with distinct colors (rainbow). In this paper, we determined explicit formulae for the chromatic polynomials of  $\mathcal{C}$ -hypercycles and  $\mathcal{B}$ -hypercycles.

**Keywords:** hypercycle, mixed hypergraph, chromatic polynomial.

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### 1. INTRODUCTION AND DEFINITIONS

For basic definitions and terminology we refer the reader to [2, 6, 18]. A *hypergraph*  $\mathcal{H}$  of order  $n$  is an ordered pair  $\mathcal{H}=(X, \mathcal{E})$ , where  $|X| = n$  is a finite nonempty set of *vertices* and  $\mathcal{E}$  is a collection of not necessarily distinct nonempty subsets of  $X$  called (*hyper*)*edges*.  $\mathcal{H}$  is said to be *k-uniform*, if the size of each of its edges is exactly  $k$ . A hypergraph is said to be *linear* if each pair of edges has at most one vertex in common. The *degree* of a vertex  $v$  is the number of edges containing  $v$ . A *hyperleaf* is a hyperedge which contains exactly one vertex of degree 2. In this paper all hypergraphs are assumed to be connected, linear and  $k$ -uniform unless stated otherwise. A linear *hypercycle* of length  $l$  is a hypergraph induced by a set of edges  $\{e_1, \dots, e_l\}$  ( $l \geq 3$ ), where

$$|e_i \cap e_j| = \begin{cases} 1 & \text{if } j = i + 1 \text{ or } \{i, j\} \in \{1, l\}, \\ 0 & \text{otherwise.} \end{cases}$$

We note that the term *elementary* hypercycle has also been used for *linear* hypercycle by Tomescu [15]. A (linear) hypercycle of length 2 induced by the set of edges  $\{e_1, e_2\}$  can be defined when  $|e_1 \cap e_2| = 2$ . In the case where  $l = 2$  and  $k = 2$  we allow for a loop, but our results are concerned with  $k > 2$  where the hypercycle of length 2 is a meaningful example (see Example 1.1).

An  $l$ -*unicyclic* hypergraph  $\mathcal{H} = (X, \mathcal{E})$  is a hypergraph in which there is exactly one set  $\{e_1, \dots, e_l\}$  which induces a hypercycle. A hypergraph which does not contain a hypercycle as a subhypergraph is called *acyclic*.

The concept of mixed-hypergraph coloring has been studied extensively by Voloshin *et al.* [9, 10, 18]. A *mixed hypergraph*  $\mathcal{H}$  with vertex set  $X$  is a triple  $(X, \mathcal{C}, \mathcal{D})$  such that  $\mathcal{C}$  and  $\mathcal{D}$  are subsets of  $X$ , called  $\mathcal{C}$ -(hyper)edges and  $\mathcal{D}$ -(hyper)edges, respectively. Elements of  $\mathcal{C} \cap \mathcal{D}$  are called  $\mathcal{B}$ -(hyper)edges (or *bi-edges*). A proper coloring of  $\mathcal{H}$  is a coloring of  $X$  such that each  $\mathcal{C}$ -edge has at least two vertices with a common color and each  $\mathcal{D}$ -edge has at least two vertices with distinct colors.

Given the mixed hypergraph  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ , when  $\mathcal{C} = \emptyset$ , we write  $\mathcal{H} = (X, \mathcal{D})$  and call it a  $\mathcal{D}$ -hypergraph (or *hypergraph*). In the case when  $\mathcal{D} = \emptyset$ , we write  $\mathcal{H} = (X, \mathcal{C})$  and call the mixed hypergraph a  $\mathcal{C}$ -hypergraph (or *cohypergraph*). In the case when  $\mathcal{C} = \mathcal{D}$ , we write  $\mathcal{H} = (X, \mathcal{B})$  and call it a  $\mathcal{B}$ -hypergraph (or *bihypergraph*). Several important results and open problems about mixed hypergraphs and bihypergraphs can be found in [7, 8, 11, 12, 13, 14].

**Example 1.1.** A hypercycle of length 2. Let  $\mathcal{H}_2^3 = (X, \mathcal{E})$ , where  $X = \{v_1, v_2, v_3, v_4\}$  and  $\mathcal{E} = \{e_1, e_2\}$  with  $e_1 = \{v_1, v_2, v_3\}$  and  $e_2 = \{v_1, v_2, v_4\}$ . Then Figure 1 is a representation of  $\mathcal{H}_2^3$ .

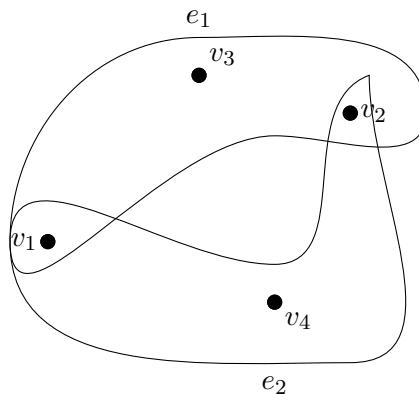


Figure 1. Linear 3-uniform hypercycle of length 2.

The *chromatic polynomial*  $P(\mathcal{H}, \lambda)$  of a mixed hypergraph  $\mathcal{H}$  is the function that counts the number of proper  $\lambda$ -colorings, which are mappings,  $f : X \rightarrow$

$\{1, 2, \dots, \lambda\}$  with the condition that every  $\mathcal{C}$ -edge has at least two vertices with a common color and every  $\mathcal{D}$ -edge has at least two vertices with distinct colors. We encourage the reader to refer to [9, 10, 18] for detailed information about chromatic polynomials, research, and applications of mixed hypergraph colorings.

For simplicity, throughout this paper, we will denote by  $\mathcal{H}_l^k = (X, \mathcal{E})$  a linear  $k$ -uniform hypergraph of length  $l$ , where  $|\mathcal{E}|=l$ . We also denote the falling factorial  $\lambda^{(t)} = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-t+1)$ . Further, we define the following parameters  $\gamma(i) = \gamma_k(\lambda, i) = (\lambda-i)^{(k-i)}$  and  $\zeta(i) = \zeta_k(\lambda, i) = \lambda^{k-i} - \gamma_k(i)$  with  $0 \leq i \leq k$ . The values of these parameters with  $i=1$  and  $i=2$  play a recurring role in the formulas of this paper, and we conjecture that the values for  $i > 2$  will play a role in the corresponding formulas for non-linear mixed hypergraphs. Note that  $\gamma(i)$  counts the number of rainbows formed using  $k-i$  vertices and  $\lambda-i$  colors and  $\zeta(i)$  counts the number of ways to color  $k-i$  vertices so that either at least two of the  $k-i$  vertices receive the same color from the  $\lambda-i$  colors, or at least one of the  $k-i$  vertices receives one of the other  $i$  specified colors.

## 2. THE CHROMATIC POLYNOMIALS OF SOME ACYCLIC MIXED HYPERGRAPHS

**Theorem 1.** *Let  $\Pi_l^k = (X, \mathcal{C})$  be a  $k$ -uniform linear connected acyclic  $\mathcal{C}$ -hypergraph of length  $l$ . Then  $P(\Pi_l^k, \lambda) = \lambda(\zeta(1))^l$ .*

**Proof.** We proceed by induction on  $l$ .

For  $l=1$ , consider the only edge  $e \in \mathcal{C}$ . There are  $\lambda$  ways to color each of its  $k$  vertices, giving  $\lambda^k$  colorings in total. But of these, exactly  $\lambda^{(k)}$  assign distinct colors to all  $k$  vertices. So the total number of proper  $\lambda$ -colorings is  $\lambda^k - \lambda^{(k)} = \lambda(\lambda^{k-1} - (\lambda-1)^{(k-1)})^1$ .

Assume the above formula is true for any  $k$ -uniform acyclic hypergraph with at most  $l-1$  edges, for some  $l \geq 2$ . Let  $\Pi_l^k = (X', \mathcal{E}')$  a  $k$ -uniform acyclic hypergraph with  $l$  edges. Since  $\Pi_l^k$  is acyclic, it contains a hyperleaf, say  $e_1$ . Let  $\Pi_{l-1}^k$  be the connected acyclic hypergraph  $\Pi_l^k - e_1$ . Thus,  $\Pi_{l-1}^k = (X'', \mathcal{E}'')$  is a  $k$ -uniform acyclic hypergraph with  $l-1$  edges, where  $\mathcal{E}'' = \mathcal{E}' - e_1$ . By the inductive hypothesis,  $P(\Pi_{l-1}^k, \lambda) = \lambda(\lambda^{k-1} - (\lambda-1)^{(k-1)})^{l-1}$ . Since  $\Pi_l^k$  is linear and  $e_1$  is a hyperleaf,  $X'' \cap e_1$  is a single vertex  $v$ .

For each coloring  $f$  of the  $P(\Pi_{l-1}^k, \lambda)$  proper colorings of  $\Pi_{l-1}^k$ , there exist  $\lambda^{k-1} - (\lambda-1)^{(k-1)}$  colorings of  $X(e_1) \setminus v$  in which not all vertices have colors distinct to  $f(v)$ . This produces all  $(\lambda^{k-1} - (\lambda-1)^{(k-1)})P(\Pi_{l-1}^k, \lambda) = \lambda(\lambda^{k-1} - (\lambda-1)^{(k-1)})^l = \lambda(\zeta(1))^l$  proper colorings of  $\Pi_l^k$ . ■

**Remark 2.** Most of the formulas in this paper are proven with an inductive argument similar to that of Theorem 1. We leave those inductive arguments to

the reader and will indicate the place for the argument by ending subsequent proofs with, "the result follows by induction on  $l$ ."

**Theorem 3.** *Let  $\Pi_l^k = (X, \mathcal{B})$  be a  $k$ -uniform linear connected acyclic  $\mathcal{B}$ -hypergraph of length  $l$ . Then  $P(\Pi_l^k, \lambda) = \lambda(\zeta(1) - 1)^l$ .*

**Proof.** Consider  $l = 1$ . There are  $\lambda^k$  ways to color each of its vertices while exactly  $\lambda$  assign the same color to all vertices and  $\lambda^{(k)}$  assign different colors to all  $k$  vertices of  $\Pi_l^k$ . Hence there are exactly  $\lambda^k - \lambda^{(k)} - \lambda = \lambda(\lambda^{k-1} - (\lambda - 1)^{(k-1)} - 1)$  ways to color the edge so that not all of its vertices are either colored with the same or with different colors. The result follows by induction on  $l$ . ■

**Theorem 4.** *Let  $\Pi_l^k = (X, \mathcal{D})$  be a  $k$ -uniform linear connected acyclic  $\mathcal{D}$ -hypergraph. Then  $P(\Pi_l^k, \lambda) = \lambda(\zeta(1) + \gamma(1) - 1)^l$ .*

**Proof.** Consider the case when  $l = 1$  and name the edge  $e$ . There are  $\lambda^k$  ways to color each of its vertices while exactly  $\lambda$  assign the same color to all vertices, bringing the number of proper  $\lambda$ -colorings to  $\lambda^k - \lambda = \lambda(\lambda^{k-1} - 1)^1$ . The result follows by induction on  $l$ . ■

**Corollary 5.** *Let  $\Pi_l^k = (X, \mathcal{C}, \mathcal{D})$  be a  $k$ -uniform linear connected acyclic mixed hypergraph. Then  $P(\Pi_l^k, \lambda) = \lambda(\gamma(1))^{p_1} (\zeta(1) - 1)^{p_2} (\zeta(1) + \gamma(1) - 1)^{p_3}$ , where  $|\mathcal{C} - \mathcal{D}| = p_1$ ,  $|\mathcal{B}| = |\mathcal{C} \cap \mathcal{D}| = p_2$  and  $|\mathcal{D} - \mathcal{C}| = p_3$ .*

**Proof.** The result follows from induction on  $l = p_1 + p_2 + p_3$ , by first considering the edges of  $\mathcal{C} - \mathcal{D}$ , then the edges of  $\mathcal{B}$ , and finally the edges of  $\mathcal{D} - \mathcal{C}$ . ■

### 3. THE CHROMATIC POLYNOMIALS OF SOME CYCLIC HYPERGRAPHS OF LENGTHS 2 AND 3

**Theorem 6.** *Let  $\mathcal{H}_2^k = (X, \mathcal{C})$  be a  $k$ -uniform  $\mathcal{C}$ -hypercycle of length 2. Then*

$$P(\mathcal{H}_2^k, \lambda) = \lambda^{n-1} + \lambda^{(2)}(\zeta(2))^2.$$

**Proof.** Let  $\mathcal{H}_2^k = (X, \mathcal{C})$  be a  $k$ -uniform linear hypercycle induced by the set of edges  $\{c_1, c_2\}$ . Consider their two vertices of degree 2, say,  $v_1$  and  $v_2$ . In each proper coloring of  $\mathcal{H}_2^k$ , one of the following is true.

*Case (i).*  $f(v_1) = f(v_2)$ . There are  $\lambda$  ways to color both vertices. Then the remaining  $k-2$  vertices of each edge can be properly colored in  $\lambda^{k-2}$  ways. Hence the number of colorings is

$$(1) \quad \lambda(\lambda^{k-2})^2.$$

*Case (ii).*  $f(v_1) \neq f(v_2)$ . There are  $\lambda(\lambda - 1)$  different ways to color both vertices. But there are  $(\lambda^{k-2} - (\lambda - 2)^{(k-2)})^2$  ways to color the remaining vertices of each edge, giving the number of colorings

$$(2) \quad \lambda(\lambda - 1) \left( \lambda^{k-2} - (\lambda - 2)^{(k-2)} \right)^2.$$

By combining (1) and (2) we obtain

$$(3) \quad P(\mathcal{H}_2^k, \lambda) = \lambda^{2k-3} + \lambda(\lambda - 1) \left( \lambda^{k-2} - (\lambda - 2)^{(k-2)} \right)^2$$

as desired. ■

**Theorem 7.** *Let  $\mathcal{H}_2^k = (X, \mathcal{B})$  be a  $k$ -uniform  $\mathcal{B}$ -hypercycle of length 2. Then*

$$P(\mathcal{H}_2^k, \lambda) = \lambda(\zeta(2) + \gamma(2) - 1)^2 + \lambda^{(2)}(\zeta(2))^2.$$

**Proof.** This proof is very similar to the one in Theorem 6. Let  $\mathcal{H}_2^k = (X, \mathcal{B})$  be a  $k$ -uniform linear hypergraph induced by the set of edges  $\{b_1, b_2\}$ . Consider their two vertices of degree 2, say,  $v_1$  and  $v_2$ . In each proper  $\lambda$ -coloring of  $\mathcal{H}_2^k$ , one of the following is true.

*Case (i).*  $f(v_1) = f(v_2)$ . There are  $\lambda$  ways to color both vertices. Then the remaining  $k - 2$  vertices of each edge can be properly colored in  $\lambda^{k-2} - 1$  ways. Hence the number of colorings is

$$(4) \quad \lambda(\lambda^{k-2} - 1)^2.$$

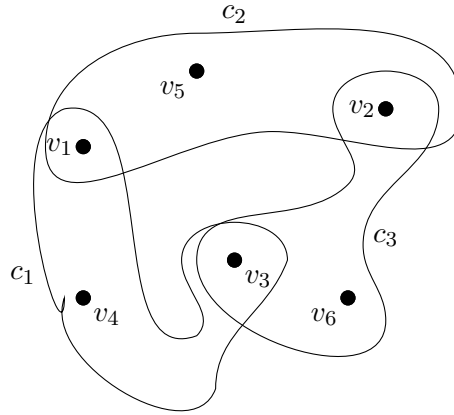
*Case (ii).*  $f(v_1) \neq f(v_2)$ . There are  $\lambda(\lambda - 1)$  different ways to color both vertices. But there are  $(\lambda^{k-2} - (\lambda - 2)^{(k-2)})^2$  ways to color the remaining vertices of each edge, giving the number of colorings

$$(5) \quad \lambda(\lambda - 1) \left( \lambda^{k-2} - (\lambda - 2)^{(k-2)} \right)^2.$$

By combining (4) and (5) we obtain that

$$(6) \quad P(\mathcal{H}_2^k, \lambda) = \lambda(\lambda^{k-2} - 1)^2 + \lambda(\lambda - 1) \left( \lambda^{k-2} - (\lambda - 2)^{(k-2)} \right)^2.$$

■

Figure 2. Linear 3-uniform  $\mathcal{C}$ -hypercycle of length 3.

**Example 2.1.** A proper 3-coloring of a linear 3-uniform  $\mathcal{C}$ -hypercycle of length 3. Let  $\mathcal{H}_3^3 = (X, \mathcal{C})$ , where  $X = \{v_1, v_2, v_3, v_4, v_5, v_6\}$  and  $\mathcal{C} = \{c_1, c_2, c_3\}$  with  $c_1 = \{v_1, v_3, v_4\}$ ,  $c_2 = \{v_1, v_2, v_5\}$  and  $c_3 = \{v_2, v_3, v_6\}$ . Figure 2 is a representation of  $\mathcal{H}_3^3$ , a linear 3-uniform hypercycle of length 3. Letting for instance  $f(v_1) = f(v_4) = 1$ ,  $f(v_2) = f(v_5) = 2$ , and  $f(v_3) = f(v_6) = 3$ , we have a proper 3-coloring of  $\mathcal{H}_3^3$ .

**Theorem 8.** Let  $\mathcal{H}_3^k = (X, \mathcal{C})$  be a  $k$ -uniform  $\mathcal{C}$ -hypercycle of length 3. Then  $P(\mathcal{H}_3^k, \lambda) = \lambda^{n-2} + 3\lambda^{(2)}(\lambda^{k-2})(\zeta(2))^2 + \lambda^{(3)}(\zeta(2))^3$ .

**Proof.** Let  $\mathcal{H}_3^k = (X, \mathcal{C})$  be a  $k$ -uniform linear hypergraph induced by the set of edges  $\{c_1, c_2, c_3\}$ . Consider the three intersecting vertices  $v_1, v_2, v_3$  such that  $c_1 \cap c_2 = \{v_1\}$ ,  $c_2 \cap c_3 = \{v_2\}$ ,  $c_3 \cap c_1 = \{v_3\}$ . In each proper  $\lambda$ -coloring of  $\mathcal{H}_3^k$ , one of the following is true.

*Case (i).* All three vertices  $v_1, v_2, v_3$  have the same color. There are  $\lambda$  ways to color the three vertices. Then the remaining  $k-2$  vertices of each edge can be properly colored in  $\lambda^{k-2}$  ways. Hence the number of colorings is

$$(7) \quad \lambda(\lambda^{k-2})^3.$$

*Case (ii).* Two colors are used to color these three vertices. Suppose  $f(v_1) \neq f(v_2) = f(v_3)$ . Then there are  $\lambda(\lambda-1)$  ways to color the three vertices. Now there are  $\lambda^{k-2}$  ways to color the remaining vertices of  $c_3$  (which does not contain  $v_1$ ) while there are  $(\lambda^{k-2} - (\lambda-2)^{(k-2)})^2$  ways to color the remaining vertices of  $c_1$  and  $c_2$  (which contain  $v_1$ ). Since there are three different ways of choosing the one vertex of different color, the number of colorings is

$$(8) \quad 3\lambda(\lambda-1)(\lambda^{k-2}) \left( \lambda^{k-2} - (\lambda-2)^{(k-2)} \right)^2.$$

*Case (iii).* All three vertices  $v_1, v_2, v_3$  have different colors. There are  $\lambda(\lambda-1)(\lambda-2)$  different ways to color the three vertices. There are  $(\lambda^{k-2} - (\lambda-2)^{(k-2)})^3$  ways to color the remaining vertices of each edge. The total number of colorings in this case is

$$(9) \quad \lambda(\lambda-1)(\lambda-2) \left( \lambda^{k-2} - (\lambda-2)^{(k-2)} \right)^3.$$

Now we combine (7), (8), and (9) to obtain the desired result.  $\blacksquare$

**Theorem 9.** *Let  $\mathcal{H}_3^k = (X, \mathcal{B})$  be a  $k$ -uniform  $\mathcal{B}$ -hypercycle of length 3. Then  $P(\mathcal{H}_3^k, \lambda) = \lambda(\zeta(2) + \gamma(2) - 1)^3 + 3\lambda^{(2)}(\zeta(2) + \gamma(2) - 1)(\zeta(2))^2 + \lambda^{(3)}(\zeta(2))^3$ .*

**Proof.** Using similar steps as in the proof of Theorem 8, we obtain that the total number of colorings  $P(\mathcal{H}_3^k, \lambda) = \lambda(\lambda^{k-2} - 1)^3 + 3\lambda(\lambda-1)(\lambda^{k-2} - 1)(\lambda^{k-2} - (\lambda-2)^{(k-2)})^2 + \lambda(\lambda-1)(\lambda-2)(\lambda^{k-2} - (\lambda-2)^{(k-2)})^3$ , giving the desired result.  $\blacksquare$

The chromatic polynomials of mixed hypergraphs are often computed using a recursive algorithm, commonly known as splitting-contraction [18]. To derive an explicit form for such formulas using the splitting-contraction algorithm is at least  $\sharp P$ -hard. However, using some combinatorial and recursive arguments, we obtained some (albeit not so simple) forms of these polynomials. These generalized formulas are presented in the next section and are built on the chromatic polynomials of the cyclic mixed hypergraphs already discussed in this section.

#### 4. CHROMATIC POLYNOMIALS OF CYCLIC HYPERGRAPHS OF ARBITRARY LENGTH

**Theorem 10.** *Let  $\mathcal{H}_l^k = (X, \mathcal{D})$  be a  $k$ -uniform  $\mathcal{D}$ -hypercycle. Then*

$$P(\mathcal{H}_l^k, \lambda) = (\lambda-1)^l \left( \sum_{i=0}^{k-2} \lambda^i \right)^l + (-1)^l(\lambda-1) \text{ for all } l \geq 2.$$

One of the authors proved this theorem in [1] and it has been established independently by Borowiecki and Łazuka in [4], as Walter pointed out in [19], simply because

$$\begin{aligned} (\lambda-1)^l \left( \sum_{i=0}^{k-2} \lambda^i \right)^l + (-1)^l(\lambda-1) &= \left( (\lambda-1) \sum_{i=0}^{k-2} \lambda^i \right)^l + (-1)^l(\lambda-1) \\ &= (\lambda^{k-1} - 1)^l + (-1)^l(\lambda-1). \end{aligned}$$

A considerable amount of literature has been written concerning the chromatic polynomials of certain families of  $\mathcal{D}$ -hypergraphs by Borowiecki *et al.* and Tomescu *et al.*, just to name a few researchers [3, 4, 5, 15]. However, very little is known about these formulas as they relate to mixed hypergraphs in general, particularly, the  $\mathcal{C}$ -hypergraphs and  $\mathcal{B}$ -hypergraphs. We present here some new results about these particular members of mixed hypergraphs.

**Theorem 11.** Let  $\mathcal{H}_l^k = (X, \mathcal{C})$  be a  $k$ -uniform  $\mathcal{C}$ -hypercycle of length  $l \geq 3$ . Then

$$(10) \quad P(\mathcal{H}_l^k, \lambda) = \zeta(2)P(\Pi_{l-1}^k, \lambda) + \gamma(2)P(\mathcal{H}_{l-1}^k, \lambda),$$

where  $\Pi_l^k$  is a  $k$ -uniform linear connected acyclic  $\mathcal{C}$ -hypergraph of length  $l \geq 3$ .

**Proof.** Let  $\mathcal{H}_l^k = (X, \mathcal{E})$  be any  $k$ -uniform  $\mathcal{C}$ -hypercycle of length  $l \geq 3$  induced by the set of edges  $\{c_1, \dots, c_l\}$ . Let  $u$  and  $v$  be the two vertices of degree 2 in  $c_l$ . In any proper coloring of the edge  $c_l$  using at most  $\lambda$  colors, either (i)  $u$  and  $v$  have the same color, or (ii)  $u$  and  $v$  have different colors. We therefore count the number of such colorings for each case in turn.

Case (i). There are  $\lambda^{k-2}$  ways to color the remaining  $k-2$  vertices in  $c_l \setminus \{u, v\}$  so that at least two vertices receive the same color, and there are  $P(\mathcal{H}_{l-1}^k, \lambda)$  ways to color the remaining vertices so that  $f(u) = f(v)$ . Hence, there are  $\lambda^{k-2}P(\mathcal{H}_{l-1}^k, \lambda)$  colorings.

Case (ii). Let  $\Pi_{l-1}^k$  be the hyperpath of length  $l-1$  induced by  $\{c_1, \dots, c_{l-1}\}$ . There are  $\lambda^{k-2} - (\lambda-2)^{(k-2)}$  colorings of the vertices in  $c_l \setminus \{u, v\}$ . For each such coloring, the number of colorings of the remaining vertices is

$$P(\Pi_{l-1}^k, \lambda) - P(\mathcal{H}_{l-1}^k, \lambda),$$

since the first term counts the number of colorings where  $u$  and  $v$  may have the same or different colors, and the second term counts the number of colors where  $u$  and  $v$  have the same color. So there are

$$\left(\lambda^{k-2} - (\lambda-2)^{(k-2)}\right) P(\Pi_{l-1}^k, \lambda) + (\lambda-2)^{(k-2)} P(\mathcal{H}_{l-1}^k, \lambda)$$

colorings altogether. ■

**Corollary 12.** Let  $\mathcal{H}_l^k = (X, \mathcal{C})$  be a  $k$ -uniform  $\mathcal{C}$ -hypercycle of length  $l \geq 3$ . Then

$$P(\mathcal{H}_l^k, \lambda) = (\gamma(2))^{l-2} \lambda^{2k-3} + \lambda \zeta(2) \sum_{j=1}^{l-2} (\gamma(2))^{j-1} (\zeta(1))^{l-j} + \lambda^{(2)} (\zeta(2))^2 (\gamma(2))^{l-2}.$$

**Proof.** When  $l = 2$ , the middle term is set to zero to yield  $P(\mathcal{H}_2^k, \lambda) = \lambda^{2k-3} + \lambda^{(2)} (\zeta(2))^2$ , which becomes the basis of the recursive argument for the proof. When  $l = 3$ , the formula in Theorem 8 can be expanded (although messy) to support this result. Now, for  $l \geq 3$ , we obtain from (10) that

$$P(\mathcal{H}_l^k, \lambda) = (\gamma(2))^{l-2} P(\mathcal{H}_2^k, \lambda) + \zeta(2) \sum_{j=1}^{l-2} (\gamma(2))^{j-1} P(\Pi_{l-j}^k, \lambda).$$

Using Theorems 1 and 6 we obtain the result after substitution. ■



**Theorem 13.** Let  $\mathcal{H}_l^k = (X, \mathcal{B})$  be a  $k$ -uniform  $\mathcal{B}$ -hypercycle of length  $l \geq 3$ . Then

$$(11) \quad P(\mathcal{H}_l^k, \lambda) = \zeta(2)P(\Pi_{l-1}^k, \lambda) + (\gamma(2) - 1)P(\mathcal{H}_{l-1}^k, \lambda),$$

where  $\Pi_{l-1}^k$  is a  $k$ -uniform linear connected acyclic  $\mathcal{B}$ -hypergraph.

**Proof.** Let  $\mathcal{H}_l^k = (X, \mathcal{B})$  be any  $k$ -uniform  $\mathcal{B}$ -hypercycle of length  $l$  induced by the set of edges  $\{b_1, \dots, b_l\}$  ( $l \geq 3$ ). Let  $u$  and  $v$  be the 2 vertices of degree 2 in  $b_l$ . In any proper coloring of the edge  $b_l$  using  $\lambda$ -colors, either (i)  $u$  and  $v$  have the same color, or (ii)  $u$  and  $v$  have different colors. We therefore count the number of such colorings for each case in turn.

*Case (i).* There are  $\lambda^{k-2} - 1$  ways to color the remaining  $k - 2$  vertices in  $b_l \setminus \{u, v\}$  so that at least two vertices (of the remaining  $k - 2$  vertices) receive different colors, and there are  $P(\mathcal{H}_{l-1}^k, \lambda)$  ways to color the remaining vertices so that  $f(u) = f(v)$ . Hence, there are  $(\lambda^{k-2} - 1)P(\mathcal{H}_{l-1}^k, \lambda)$  colorings.

*Case (ii).* Let  $\Pi_{l-1}^k$  be the hyperpath of length  $l - 1$  induced by  $\{b_1, \dots, b_{l-1}\}$ . There are  $\lambda^{k-2} - (\lambda - 2)^{(k-2)}$  colorings of the vertices in  $b_l \setminus \{u, v\}$ . For each such coloring the number of colorings of the remaining vertices is

$$P(\Pi_{l-1}^k, \lambda) - P(\mathcal{H}_{l-1}^k, \lambda),$$

since the first term counts the number of colorings where  $u$  and  $v$  may have the same or different colors, and the second term counts the number of colors where  $u$  and  $v$  have the same color. So there are

$$\left(\lambda^{k-2} - (\lambda - 2)^{(k-2)}\right)P(\Pi_{l-1}^k, \lambda) + \left((\lambda - 2)^{(k-2)} - 1\right)P(\mathcal{H}_{l-1}^k, \lambda)$$

colorings altogether. ■

**Corollary 14.** Let  $\mathcal{H}_l^k = (X, \mathcal{B})$  be a  $k$ -uniform  $\mathcal{B}$ -hypercycle of length  $l \geq 3$ . Then

$$\begin{aligned} P(\mathcal{H}_l^k, \lambda) &= \lambda(\zeta(2) + \gamma(2) - 1)^2(\gamma(2) - 1)^{l-2} \\ &\quad + \lambda\zeta(2) \sum_{j=1}^{l-2} (\gamma(2) - 1)^{j-1} (\zeta(1) - 1)^{l-j} + \lambda^{(2)}(\zeta(2))^2(\gamma(2) - 1)^{l-2}. \end{aligned}$$

**Proof.** When  $l = 2$ , the middle term is set to zero to yield  $P(\mathcal{H}_2^k, \lambda) = \lambda(\zeta(2) + \gamma(2) - 1)^2 + \lambda^{(2)}(\zeta(2))^2$ , which becomes the basis of the recursive argument for the proof just as in the previous corollary. For  $l \geq 3$ , we obtain from (11) that  $P(\mathcal{H}_l^k, \lambda) = (\gamma(2) - 1)^{l-2}P(\mathcal{H}_2^k, \lambda) + \zeta(2) \sum_{j=1}^{l-2} (\gamma(2) - 1)^{j-1}P(\Pi_{l-j}^k, \lambda)$ . Using Theorems 3 and 7 we obtain the desired formula. ■

These results obtained in this section can easily be rewritten to obtain the chromatic polynomials of several other families of linear connected uniform hypergraphs. In particular the chromatic polynomials of unicyclic mixed hypergraphs and mixed hypercacti [9] can be written and are left as exercises for the reader. As it is, rewriting these formulas in terms of the standard basis is doable but messy. Further work could look for simpler forms for these expressions or address the remaining open problems of interpreting the coefficients of these polynomials and finding their roots.

Furthermore, by using  $\gamma$  and  $\zeta$  as functions of  $|e|$  (i.e., of any value other than just  $k$ ), it is reasonable to extend the formulas discussed in this paper to non-uniform mixed hypergraphs (see Corollary 16). Recently, Walter [19] has found the formulas for some non-uniform  $\mathcal{D}$ -hypergraphs. As a step in this direction, we close this paper with a more general result concerning non-uniform acyclic mixed hypergraphs.

It is easy to verify that the chromatic polynomials of an isolated hyperedge, cohyperedge and bihyperedge are as follows.

**Proposition 15.** *Let  $e$  be an isolated hyperedge. Then the chromatic polynomials of  $e$  when viewed as a  $\mathcal{D}$ -hyperedge,  $\mathcal{C}$ -hyperedge, or  $\mathcal{B}$ -hyperedge are*

$$(12) \quad \begin{aligned} P_{\mathcal{D}}(e) &= \lambda(\lambda^{|e|-1} - 1), \\ P_{\mathcal{C}}(e) &= \lambda(\lambda^{|e|-1} - (\lambda - 1)^{|e|-1}) = \lambda\zeta_{|e|}(1), \\ P_{\mathcal{B}}(e) &= \lambda(\lambda^{|e|-1} - (\lambda - 1)^{|e|-1} - 1) = \lambda(\zeta_{|e|}(1) - 1), \end{aligned}$$

respectively.

For instance, the case when  $e \in \mathcal{D}$ , there are  $\lambda^{|e|} - \lambda = \lambda(\lambda^{|e|-1} - 1)$  ways to properly color each hyperedge.

From (12), we can extend Corollary 5 (following the argument used in Theorem 1) to obtain the following.

**Corollary 16.** *Let  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  be an acyclic mixed hypergraph. Then the chromatic polynomial of any (non-uniform) acyclic mixed hypergraph is given by*

$$P(\mathcal{H}) = \lambda \prod_{\substack{e_1 \in \mathcal{D}, e_2 \in \mathcal{C} \\ e_3 \in \mathcal{B}}} (\lambda^{|e_1|-1} - 1) \zeta_{|e_2|}(1) (\zeta_{|e_3|}(1) - 1).$$

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