Discussiones Mathematicae
Graph Theory 34 (2014) 559–565
doi:10.7151/dmgt.1749

CHARACTERIZATION OF CUBIC GRAPHS G WITH $ir_t(G) = IR_t(G) = 2$

Changiz Eslahchi, Shahab Haghi

Department of Mathematics
Shahid Beheshti University, G. C.
Evin, Tehran, Iran

e-mail: eslahchi.ch@gmail.com
shahab_ha1365@yahoo.com

and

Nader Jafari Rad

Department of Mathematics
Shahrood University of Technology
Shahrood, Iran

e-mail: n.jafarirad@gmail.com

Abstract

A subset S of vertices in a graph G is called a *total irredundant set* if, for each vertex v in G, v or one of its neighbors has no neighbor in $S - \{v\}$. The *total irredundance number*, $ir(G)$, is the minimum cardinality of a maximal total irredundant set of G, while the upper total irredundance number, $IR(G)$, is the maximum cardinality of a such set. In this paper we characterize all cubic graphs G with $ir_t(G) = IR_t(G) = 2$.

Keywords: total domination, total irredundance, cubic.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. A set of vertices S in G is a *total dominating set*, (or just

TDS), if $N(S) = V(G)$. The *total domination number* $\gamma_t(G)$, of G, is the minimum cardinality of a total dominating set of G. For graph theory notation and terminology in general we follow [3].

Total irredundance in graphs was introduced by Hedetniemi *et al.* in [4], and further studied for example in [1, 2, 5]. A set S of vertices in a graph G is called a *total irredundant set* (or just TIS) if, for each vertex v in G, v or one of its neighbors has no neighbor in $S - \{v\}$. The *total irredundance number*, $ir_t(G)$, is the minimum cardinality of a maximal TIS of G, while the upper total irredundance number, $IR_t(G)$, is the maximum cardinality of a such set.

Favaron *et al.* in [1] proved that for every cubic graph $G \neq K_4$, $ir_t(G) \geq 2$. In this paper we characterize all cubic graphs G of order at least six with $ir_t(G) = IR_t(G) = 2$.

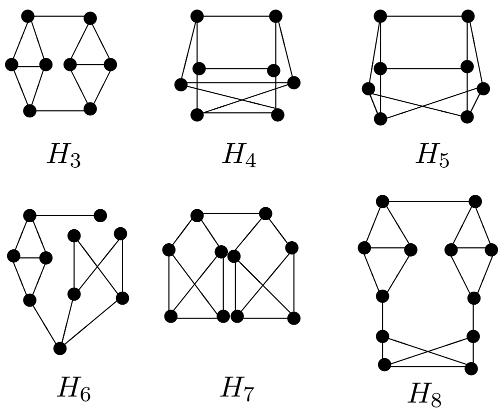

Figure 1. Graphs H_3,...,H_8.

2. MAIN RESULT

It is well known that there are only two cubic graphs of order 6. Let H_1 and H_2 be the two cubic graphs of order 6, and Q_3 be the (3-dimensional) hypercube. Let $H_3, H_4, \ldots, H_8$ be the graphs shown in Figure 1. We prove the following.

Theorem 1. *For a connected cubic graph G of order at least six, $ir_t(G) = IR_t(G) = 2$ if and only if $G = Q_3, H_1, H_2, \ldots,$ or H_8.*

Proof. First it is a routine matter to see that $ir_t(Q_3) = IR_t(Q_3) = ir_t(H_i) = IR_t(H_i) = 2$ for $i = 1, 2, \ldots, 8$. Let G be a connected cubic graph of order $n \geq 6$ with $ir_t(G) = IR_t(G) = 2$. Since there is no cubic graph of order 6 different from H_1, H_2, we assume that $n \geq 8$. Since $IR_t(G) = 2$, any minimum maximal TIS of G is also a maximum maximal TIS of G.

Lemma 2. *There is a minimum maximal TIS S of G such that the two vertices of S are adjacent.*

Proof. Suppose to the contrary that there is no minimum maximal TIS S containing two adjacent vertices. Let x and y be two adjacent vertices of G. By assumption $S = \{x, y\}$ is not a minimum maximal TIS. Since $IR_t(G) = ir_t(G) = 2$, $IR_t(G)$ is maximum among all maximal total irredundant sets, and $ir_t(G)$ is minimum among all maximal total irredundant sets, we deduce that S is not a TIS. This implies that there is a vertex v such that $N[v]-N[S-\{v\}] = \emptyset$. We consider the following cases.

Case 1. $v \in S$. Without loss of generality assume that $v = x$. Let $N(x) = \{y, x_1, x_2\}$. Then $N[y] = \{x, x_1, x_2\}$. Since $n \geq 8$, $x_1 \notin N(x_2)$. If $N(x_1) \cap N(x_2) - \{x, y\} \neq \emptyset$, then $\{w, w_1\}$ is a TIS of G, where $w \in N(x_1) \cap N(x_2) - \{x, y\}$, and $w_1 \in N(w) - \{x_1, x_2\}$. This contradiction implies that $N(x_1) \cap N(x_2) - \{x, y\} = \emptyset$. Now $\{x_1, w\}$ is a TIS of G, where $w \in N(x_1) - \{x, y\}$, a contradiction.

Case 2. $v \notin S$. If $N(v) \cap \{x, y\} = \emptyset$ then $v \in N[v] - N[S - \{v\}]$, a contradiction. Thus without loss of generality assume that $v \in N(x)$. We show that $v \notin N(y)$. Suppose to the contrary that $v \in N(y)$. Let $v_1 \in N(v) - \{x, y\}$. Then $v_1 \in N(x) \cup N(y)$, since $N[v] - N[S - \{v\}] = \emptyset$. Since $n \geq 8$, we obtain that $\{x, y\} \not\subseteq N(v_1)$. If $v_1 \in N(x)$, then we consider a vertex $w \in N(v_1) - \{x, v\}$. If $w \notin N(y)$, then $\{v_1, w\}$ is a TIS of G, a contradiction. Thus $w \in N(y)$. Now $\{w, w_1\}$ is a TIS of G, where $w_1 \in N(w) - \{v_1, y\}$, a contradiction. We deduce that $v_1 \notin N(x)$, and so $v_1 \in N(y)$. Let $w \in N(v_1) - \{v, y\}$. If $w \notin N(x)$, then $\{w, v_1\}$ is a TIS of G, a contradiction. Thus $w \in N(x)$. Now $\{w, w_1\}$ is a TIS of G, where $w_1 \in N(w) - \{x, v_1\}$, a contradiction. We conclude that $v \notin N(y)$.

Let $N(v) = \{x, v_1, v_2\}$. We next show that $\{v_1, v_2\} \not\subseteq N(y)$. Assume to the contrary that $\{v_1, v_2\} \subseteq N(y)$. If $x \notin N(v_1) \cup N(v_2)$, then $\{x, x_1\}$ is a TIS of G, where $x_1 \in N(x) - \{v, y\}$. This is a contradiction. So without loss of generality we may assume that $x \in N(v_1)$. Let $w \in N(v_2) - \{v, y\}$. Then $\{v_2, w\}$ is a TIS of G, a contradiction. We conclude that $\{v_1, v_2\} \not\subseteq N(y)$. Without loss of generality, assume that $v_1 \in N(x)$ and $v_2 \in N(y)$. Since $n \geq 8$, we may assume without loss of generality that $N(v_1) - \{v, v_2, x\} \neq \emptyset$. Let $w \in N(v_1) - \{v, v_2, x\}$. Then $\{v_1, w\}$ is a TIS of G, a contradiction. □

Let $S = \{u, v\}$ be a minimum maximal TIS of G such that u is adjacent to v. Since $IR(G) = 2$, S is a maximum maximal TIS of G. Since S is a TIS of G, we obtain that $|N(u) \cap N(v)| \leq 1$. We proceed with Lemma 3.

Lemma 3. $|N(u) \cap N(v)| = 0$.

Proof. Assume to the contrary that $|N(u) \cap N(v)| = 1$. Let $N(u) \cap N(v) = \{x\}$, $N[u]-N[v] = \{y\}$ and $N[v]-N[u] = \{z\}$. If $x \in N(y) \cup N(z)$, then $N[x]-N[S-$

$\{x\}] = \emptyset$, a contradiction. Thus $x \notin N(y) \cup N(z)$. Let $N(x) - \{u, v\} = \{w\}$. Since $\{u, v, w\}$ is not a TIS of G, there is a vertex k such that $N[k] - N[\{u, v, w\} - \{k\}] = \emptyset$. Clearly $k \in \{y, w, z\}$. We show that $k \neq w$. Assume that $k = w$. Then $N[w] \subseteq N[\{u, v, x\}]$. This implies that $w \in N(y) \cap N(z)$. Since $n \geq 8$, we find that $y \notin N(z)$. Let $t_1 \in N(y) - \{u, w\}$ and $t_2 \in N(z) - \{v, w\}$. If $t_1 = t_2$, then $\{w, z, t_1\}$ is a TIS of G, a contradiction. Thus $t_1 \neq t_2$. Since $\{w, z, t_2\}$ is not a TIS of G, we obtain that $N[t_1] \subseteq N[\{w, z, t_2\}]$. In particular, $t_1 \in N(t_2)$, and $N(t_1) - \{y, t_2\} = N(t_2) - \{z, t_1\}$. Let $N(t_1) - \{y, t_2\} = \{t_3\}$. Then $\{t_1, t_2, t_3\}$ is a TIS of G, a contradiction. We deduce that $k \neq w$. Without loss of generality assume that $k = y$. Then $N[y] \subseteq N[\{u, v, x\}]$. In particular, $y \in N(w) \cap N(z)$. Since $n \geq 8$, we find that $z \notin N(w)$. Let $N(z) - \{y, v\} = \{t\}$. Since $\{y, z, t\}$ is not a TIS of G, there is a vertex k_1 such that $N[k_1] \subseteq N[\{y, z, t\} - \{k_1\}]$. Clearly $k_1 \notin \{y, z, t, u, v, w\}$. We deduce that k_1 is a vertex with $N(k_1) \subseteq N(w) \cup N(t)$. Let $N(k_1) = \{w, t, t_1\}$, where $t_1 \in N(t) - \{z, k_1\}$. Then $\{t, k_1, t_1\}$ is a TIS of G, a contradiction. □

Thus $|N(u) \cap N(v)| = 0$. Let $N(u) - N(v) = \{u_1, u_2\}$ and $N(v) - N(u) = \{v_1, v_2\}$. We consider the following cases depending on adjacency among u_1 and u_2.

Case 1. $u_1 \in N(u_2)$. Assume that $v_1 \in N(v_2)$. Since $\{u, u_1, u_2\}$ and $\{v, v_1, v_2\}$ are not maximal TIS of G, we obtain that $N(u_1) \cap N(u_2) - \{u, v_1, v_2\} \neq \emptyset$ and $N(v_1) \cap N(v_2) - \{v, u_1, u_2\} \neq \emptyset$. Let $N(u_1) \cap N(u_2) - \{u, v_1, v_2\} = \{s_1\}$ and $N(v_1) \cap N(v_2) - \{v, u_1, u_2\} = \{s_2\}$. If $s_1 \in N(s_2)$, then $G = H_3$. Thus assume that $s_1 \notin N(s_2)$. If $N(s_1) \cap N(s_2) \neq \emptyset$, then $\{s_1, s_2, w\}$ is a TIS of G, where $w \in N(s_1) \cap N(s_2)$. This contradiction implies that $N(s_1) \cap N(s_2) = \emptyset$. Let $w_1 \in N(s_1) - \{u_1, u_2\}$ and $w_2 \in N(s_2) - \{v_1, v_2\}$. Suppose that w_1 is adjacent to w_2. If $N(w_1) \cap N(w_2) = \emptyset$, then $\{s_1, s_2, w_1\}$ is a maximal TIS of G, a contradiction. So we assume that $N(w_1) \cap N(w_2) \neq \emptyset$. Let $w_3 \in N(w_1) \cap N(w_2)$. Then $\{w_1, w_2, w_3\}$ is a TIS of G, a contradiction. We deduce that $w_1 \notin N(w_2)$. If $N(w_1) \cap N(w_2) = \emptyset$, then $\{w_1, s_1, s_2\}$ is a TIS of G, a contradiction. Thus $N(w_1) \cap N(w_2) \neq \emptyset$. Let $w_3 \in N(w_1) \cap N(w_2)$. If $N(w_1) - \{s_1\} \neq N(w_2) - \{s_2\}$, then $\{v, v_1, w_3\}$ is a TIS of G, a contradiction. Thus $N(w_1) - \{s_1\} = N(w_2) - \{s_2\}$. Let $N(w_1) - \{s_1\} = \{w_3, w_4\}$. If $w_3 \notin N(w_4)$, then $\{u, v, w_5\}$ is a TIS of G, where $w_5 \in N(w_4) - N(w_3)$. This contradiction implies that $w_3 \in N(w_4)$. Consequently, $G = H_8$. Thus we assume that $v_1 \notin N(v_2)$. If $N(u_1) \cap N(u_2) - \{u\} = \emptyset$, then $\{u, u_1, u_2\}$ is a TIS of G, a contradiction. Thus $N(u_1) \cap N(u_2) - \{u\} \neq \emptyset$. Let $N(u_1) \cap N(u_2) - \{u\} = \{w\}$. We show that $w \notin N(v_1) \cup N(v_2)$. Without loss of generality assume that $w \in N(v_1)$. Since $\{u, v, v_2\}$ is not a TIS for G, there is a vertex k such that $N[k] - N[\{u, v, v_2\} - \{k\}] = \emptyset$. It is obvious that $k \notin \{u, v, v_1, v_2, u_1, u_2\}$. Thus $k \in N(v_1) \cap N(v_2) \cap N(k_1)$, where $k_1 \in N(v_2) - \{k, v\}$. Then $\{k, k_1, v_2\}$ is a TIS of G, a contradiction. We deduce that $w \notin N(v_1) \cup N(v_2)$. But there is a vertex k such that $N[k] - N[\{u, v, v_2\} - \{k\}] = \emptyset$, since $\{u, v, v_2\}$

is not a TIS of G. It is obvious that $k \notin \{u, v, v_2, u_1, u_2\}$. Assume that $k \neq v_1$. Thus $k \in N(v_2)$. Since $N[k] \subseteq N[\{u, v, v_2\} - \{k\}]$, we obtain that $k \in N(v_1)$, and $N(k) \cap N(v_2) \neq \emptyset$. Let $t_1 \in N(k) \cap N(v_2)$. If $t_1 \notin N(v_1)$, then $\{t_1, k, w\}$ is a TIS of G, a contradiction. Thus $t_1 \in N(v_1)$. Let $w_1 \in N(w) - \{u_1, u_2\}$. Then $\{w, w_1, v_2\}$ is a TIS of G, a contradiction. Thus $k = v_1$. It follows that $N(v_1) = N(v_2)$. Let $N(v_1) - \{v\} = \{t_1, t_2\}$. Since $\{w, t_1, t_2\}$ is not a TIS of G, there is a vertex k_1 such that $N[k_1] - N[\{w, t_1, t_2\} - \{k_1\}] = \emptyset$. It is obvious that $k_1 \notin \{u_1, u_2, v_1, v_2\}$. Assume that $\{w, t_1, t_2\}$ is not independent. If $w \in N(t_1)$, then $\{v_1, w, t_3\}$ is a TIS of G, where $t_3 \in N(t_2) - \{v_1, v_2\}$. This contradiction implies that $w \notin N(t_1)$. Thus $t_2 \in N(t_1)$. Now $\{w, w_1, v_2\}$ is a TIS of G, where $w_1 \in N(w) - \{u_1, u_2\}$, a contradiction. Thus $\{w, t_1, t_2\}$ is an independent set. Then $k_1 \notin \{w, t_1, t_2\}$. This implies that $k_1 \in N(w) \cap N(t_1) \cap N(t_2)$, and thus $G = H_6$.

Case 2. $u_1 \notin N(u_2)$. According to Case 1, we may assume that $v_1 \notin N(v_2)$. We consider the following subcases depending on adjacency among u_2 and v_1.

Subcase 2.1. $u_2 \in N(v_1)$. Assume that $u_1 \in N(v_2)$. We first show that $N(u_2) \cap N(v_1) = \emptyset$. Assume to the contrary that $N(u_2) \cap N(v_1) \neq \emptyset$. Let $w \in N(u_2) \cap N(v_1)$. Since $\{u_2, v_1, w\}$ is not a TIS of G, there is a vertex k such that $N[k] - N[\{u_2, v_1, w\} - \{k\}] = \emptyset$. It is obvious that $k \notin \{u_2, v, w, u_1, v_2\}$. Thus $k \in \{u, v\}$. Without loss of generality assume that $k = u$. Then $u_1 \in N(w)$. Let $t \in N(v_2) - \{u_1, v\}$. Then $\{v, v_2, t\}$ is a TIS of G, a contradiction. We conclude that $N(u_2) \cap N(v_1) = \emptyset$. Let $w_1 \in N(u_2) - \{u, v_1\}$ and $w_2 \in N(v_1) - \{u_2, v\}$. Since $\{u, u_2, v_1\}$ is not a TIS of G, there is a vertex k such that $N[k] - N[\{u, u_2, v_1\} - \{k\}] = \emptyset$. It is easy to see that $k \notin \{u, u_2, v_1, v, u_1\}$. Thus $k \in \{w_1, w_2\}$. Assume that $k = w_1$. Then, $w_1 \in N(u_1) \cap N(w_2)$. Since $\{u_2, v, v_1\}$ is not a TIS of G, there is a vertex k_1 such that $N[k_1] \subseteq N[\{u_2, v, v_1\}]$. It is easy to see that $k_1 \notin \{u, v, u_1, u_2, v_1, v_2, w_1\}$. Thus $k_1 = w_2$. This implies that $w_2 \in N(v_2)$. Consequently, $G = Q_3$. Next assume that $k = w_2$. Then, $w_2 \in N(w_1) \cap N(u_1)$. Since $\{v, v_1, u_2\}$ is not a TIS of G, there is a vertex k_2 such that $N[k_2] \subseteq N[\{u_2, v_1, v\}]$. It is easy to see that $k_2 \notin \{u, v, u_1, u_2, v_1, v_2, w_2\}$. Thus $k_2 = w_1$. This implies that $w_1 \in N(v_2)$. Consequently, $G = H_4$. Thus we may assume that $u_1 \notin N(v_2)$. We first show that $N(u_2) \cap N(v_1) = \emptyset$. Assume to the contrary that $N(u_2) \cap N(v_1) \neq \emptyset$. Let $t \in N(u_2) \cap N(v_1)$. Since $\{u_2, v_1, t\}$ is not a TIS of G, there is a vertex k such that $N[k] \subseteq N[\{u_2, v_1, t\}]$, and we observe that $k \notin \{u_2, v_1, t, u_1, v_2\}$. It follows that $k \in \{u, v\}$. Without loss of generality assume that $k = u$. Then $u_1 \in N(t)$ and $\{v, v_1, v_2\}$ is a TIS of G, a contradiction. Thus $N(u_2) \cap N(v_1) = \emptyset$. Let $t_1 \in N(u_2) - \{u\}$ and $t_2 \in N(v_1) - \{v\}$. Since $\{u, v, u_2\}$ is not a TIS of G, there is a vertex k_1 such that $N[k_1] \subseteq N[\{u, v, u_2\}]$, and we can see that $k_1 \notin \{u, v, u_2, v_1, v_2\}$. This implies that $k = t_1$, and thus $t_1 \in N(u_1) \cap N(v_2)$. Similarly, since $\{u, v, v_1\}$ is not a TIS of G, there is a vertex k_2 such that $N[k_2] \subseteq N[\{u, v, v_1\}]$, and we can see that $k_2 = t_2$ which implies that $t_2 \in N(u_1) \cap N(v_2)$. Consequently, $G = H_5$.

Subcase 2.2. $u_2 \notin N(v_1)$. Since $\{u, v, u_2\}$ is not a TIS of G, there is a vertex k such that $N[k] \subseteq N[\{u, v, u_2\}]$, and clearly $k \notin \{u, v, u_2\}$. If $k = v_1$, then there are two vertices t_1 and t_2 such that $\{u_2, v_1\} \subseteq N(t_1) \cap N(t_2)$. If $t_1 \in N(t_2)$, then $\{u_1, u_2, u\}$ is a TIS of G, a contradiction. So $t_1 \notin N(t_2)$. If $u_1 \in N(t_1) \cap N(t_2)$, then $\{v, v_1, v_2\}$ is a TIS of G, and if $u_1 \notin N(t_1)$ or $u_1 \notin N(t_2)$, then $\{u_1, u_2, u\}$ is a TIS of G, a contradiction. We conclude that $k \neq v_1$. Similarly $k \neq v_2$. Thus $k \in \{u_1, t\}$, where $t \in N(u_2) - \{u\}$. We continue according the two possibilities of k.

Subcase 2.2.1. $k = u_1$. Then $N(u_1) = N(u_2)$. Let $N(u_1) = \{u, t_1, t_2\}$. Since $\{u, v, v_1\}$ is not a TIS of G, there is a vertex k' such that $N[k'] \subseteq N[\{u, v, v_1\}]$, and clearly $k' \notin \{u, v, v_1\}$. Suppose that $k' \neq v_2$. We assume that $t_3 \notin N(v_2)$, since the case $t_3 \in N(v_2)$ has been checked earlier. Then $\{t_3, k', u_2\}$ is a TIS of G, a contradiction. Thus $k' = v_2$. Then $N(v_1) = N(v_2)$. Let $N(v_1) = \{t_3, t_4\}$. We show that $t_1 \in N(t_2)$ and $t_3 \in N(t_4)$. Assume without loss of generality that $t_3 \notin N(t_4)$. We show that $t_1 \notin N(t_2)$. Assume to the contrary that $t_1 \in N(t_2)$. If $N(t_3) \cap N(t_4) \neq \emptyset$, and $w \in N(t_4) \cap N(t_3)$, then $\{u, v, w_1\}$ is a TIS of G, where $w_1 \in N(w) - \{t_3, t_4\}$, a contradiction. Thus $N(t_3) \cap N(t_4) = \emptyset$. Let $w_1 \in N(t_3) - \{v_1, v_2\}$ and $w_2 \in N(t_4) - \{v_1, v_2\}$. If $w_1 \in N(w_2)$, then $\{w_1, w_2, u\}$ is a TIS of G, a contradiction. Thus $w_1 \notin N(w_2)$. Let $w_3 \in N(w_1) - \{t_3\}$. Since $\{u, v, w_3\}$ is not a TIS of G, there is a vertex k_1 such that $N[k_1] \subseteq N[\{u, v, w_3\}]$, and it can be easily seen that $k_1 \in N(w_1) \cap N(w_3)$. Furthermore, $|N(k_1) \cap N(w_3)| = 2$. Let $w_4 \in N(k_1) \cap N(w_3) - \{w_1\}$. Then $\{u, v, w_4\}$ is a TIS of G, a contradiction. We deduce that $t_1 \notin N(t_2)$. If $t_1 \in N(t_4)$ and $t_2 \in N(t_3)$, then $IR_t(G) = 3$, a contradiction. Thus without loss of generality assume that $t_1 \notin N(t_4)$. We next show that $t_2 \notin N(t_3)$. Assume to the contrary that $t_2 \in N(t_3)$. Let $t_5 \in N(t_4) - \{v_1, v_2\}$, and let $t_6 \in N(t_5) - \{t_1, t_4\}$. Since $\{t_2, t_3, t_6\}$ is not a TIS of G, there is a vertex k_2 such that $N[k_2] \subseteq N[\{t_2, t_3, t_6\}]$. It is obvious that $k_2 \notin \{t_2, t_3, t_5, t_6\}$. So $k_2 = t_1$ or $k_2 \in N(t_5) \cap N(t_6) \cap N(t_7)$, where $t_7 \in N(t_6) - \{t_1, t_5\}$. If $k_2 = t_1$, then $t_1 \in N(t_6)$ and $\{u, v, a\}$ is a TIS of G, where $a \in N(t_6) - \{t_1, t_5\}$, a contradiction. So $k_2 \in N(t_5) \cap N(t_6) \cap N(t_7)$. Since $\{u, v, t_7\}$ is not a TIS of G, there is a vertex k_3 such that$N[k_3] - N[\{u, v, t_7\} - \{k_3\}] = \emptyset$, and we can see that $k_3 = t_1$. Now $t_1 \in N(t_7)$, and $\{v, t_3, t_7\}$ is a TIS of G, a contradiction. Thus $t_2 \notin N(t_3)$. Since $\{t_1, t_2, t_3\}$ is not a TIS of G, there is a vertex k_3 such that $N[k_3] \subseteq N[\{t_1, t_2, t_3\}]$. It is obvious that $k_3 \notin \{t_1, t_2, t_3, u_1, u_2, v_1, v_2\}$. Thus $k_3 \in N(t_1) \cap N(t_2) \cap N(t_3)$. Now $\{t_2, t_3, t_4\}$ is a TIS of G, a contradiction. Thus $t_1 \in N(t_2)$ and $t_3 \in N(t_4)$. Consequently, $G = H_7$.

Subcase 2.2.2. $k = t$, where $t \in N(u_2) - \{u\}$. We show that $t \notin N(u_1) \cap N(v_1)$. Assume to the contrary that $t \in N(u_1) \cap N(v_1)$. Since $\{u, u_1, u_2\}$ is not a TIS of G, there is a vertex k such that $N[k] \subseteq N[\{u, u_1, u_2\} - \{k\}]$, and we can see that

$k \notin \{u, u_2, v, t, v_1, v_2\}$. Then $k = u_1$ which implies that $N(u_1) \cap N(u_2) - \{t\} \neq \emptyset$. Let $t_1 \in N(u_1) \cap N(u_2) - \{t\}$. Since $\{v, v_1, v_2\}$ is not a TIS of G, there is a vertex a such that $N[a] \subseteq N[\{v, v_1, v_2\} - \{a\}]$, and we observe that a is a vertex adjacent to both v_1 and v_2. Let $a_1 \in N(v_2) - \{a, v\}$. Then $a \in N(a_1)$, and $\{v_2, a_1, a\}$ is a TIS of G, a contradiction. Thus $t \notin N(u_1) \cap N(v_1)$. Similarly, $t \notin N(u_1) \cap N(v_2)$. If $t \in N(v_1)$, then we let $t_1 \in N(u_2) - \{t, u\}$. It follows that $t \in N(t_1)$. If $t_1 \in N(v_2)$ then $\{u, u_1, u_2\}$ is a TIS of G, and if $t_1 \notin N(v_2)$ then $\{u_2, t_1, t_2\}$ is a TIS of G, both are contradictions. We deduce that $t \notin N(v_1)$, and similarly $t \notin N(v_2)$. Thus $t \in N(u_1)$. Let $t_1 \in N(u_2) - \{t, u\}$. Then $t_1 \in N(t)$. We show that $t_1 \notin N(v_1) \cup N(v_2)$. Assume to the contrary that $t_1 \in N(v_1) \cup N(v_2)$. Without loss of generality assume that $t_1 \in N(v_1)$. Since $\{t_1, t_2, v_2\}$ is not a TIS of G, there are two vertices t_2 and t_3 such that $\{t_2, t_3\} \subseteq N(v_2)$, $t_2 \in N(v_1)$ and $t_3 \in N(t_2)$. Then $\{v_2, t_2, t_3\}$ is a TIS of G, a contradiction. Thus $t_1 \notin N(v_1) \cup N(v_2)$. Since $\{u, v, u_2\}$ is not a TIS of G, we find that $t_1 \in N(u_1)$. Since $\{u, v, v_1\}$ is not a TIS of G, there is a vertex b such that $N[b] \subseteq N[\{u, v, v_1\}]$, and we observe that $b \notin \{u, v, u_1, u_2, v_1\}$. If $b = v_2$, then there are two vertices $t_1^*, t_2^* \in N(v_1) \cap N(v_2) - \{v\}$. But this is an earlier possibility in the Subcase 2.2.1, which has been discussed. Thus $b \in N(v_1) - \{v\}$. Let $t_4 \in N(v_1) - \{v, b\}$. Then $b \in N(t_4) \cap N(v_2)$. If $t_4 \notin N(v_2)$, then $\{u_2, t_4, b\}$ is a TIS of G, a contradiction. Thus $t_4 \in N(v_2)$. Consequently, $G = H_7$. ■

References

[1] O. Favaron, T.W. Haynes, S.T. Hedetniemi, M.A. Henning and D.J. Knisley, *Total irredundance in graphs*, Discrete Math. **256** (2002) 115–127. doi:10.1016/S0012-365X(00)00459-3

[2] T.W. Haynes, S.T. Hedetniemi, M.A. Henning and D.J. Knisley, *Stable and unstable graphs with total irredundance number zero*, Ars Combin. **61** (2001) 34–46.

[3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1998).

[4] S.M. Hedetniemi, S.T. Hedetniemi and D.P. Jacobs, *Total irredundance in graphs: theory and algorithms*, Ars Combin. **35** (1993) 271–284.

[5] Q.X. Tu and Z.Q. Hu, *Structures of regular graphs with total irredundance number zero*, Math. Appl. (Wuhan) **18** (2005) 41–44 (in Chinese).

Received 8 January 2013
Revised 3 June 2013
Accepted 27 June 2013