# 5-STARS OF LOW WEIGHT IN NORMAL PLANE MAPS WITH MINIMUM DEGREE 5 

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#### Abstract

It is known that there are normal plane maps $M_{5}$ with minimum degree 5 such that the minimum degree-sum $w\left(S_{5}\right)$ of 5 -stars at 5 -vertices is arbitrarily large. In 1940, Lebesgue showed that if an $M_{5}$ has no 4 -stars of cyclic type $(5,6,6,5)$ centered at 5 -vertices, then $w\left(S_{5}\right) \leq 68$. We improve this bound of 68 to 55 and give a construction of a $(5,6,6,5)$-free $M_{5}$ with $w\left(S_{5}\right)=48$.


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## 1. Introduction

A normal plane map (NPM for short) is a plane pseudograph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three. The degree of a vertex $v$ is denoted by $d(v)$. A $k$-vertex is a vertex $v$ with $d(v)=k$. A $k^{+}$-vertex $\left(k^{-}\right.$-vertex) is one of degree at least $k$ (at most $k$ ). An NPM with minimum degree $\delta$ at least 5 is denoted by $M_{5}$. The weight of a subgraph of an NPM is the sum of degrees of its vertices. A $k$-star $S_{k}(v)$ is minor if its center $v$ has degree (in the NPM) at most 5 . All stars considered in this note are minor. By $w\left(S_{k}\right)$ we denote the minimum weight of minor $k$-stars in a given NPM.

In 1904, Wernicke [15] proved that every $M_{5}$ has a 5 -vertex adjacent to a $6^{-}$-vertex. This result was strengthened by Franklin [8] in 1922 to the existence of a 5 -vertex with two $6^{-}$-neighbors. In 1940, Lebesgue [14, p. 36] gave an approximate description of the neighborhoods of 5 -vertices in $M_{5}$ s. In particular, this description implies the results in $[15,8]$ and shows that there is a 5 -vertex with three $8^{-}$-neighbors.

For $M_{5}$ s, the bounds $w\left(S_{1}\right) \leq 11$ (Wernicke [15]) and $w\left(S_{2}\right) \leq 17$ (Franklin [8]) are tight. It was proved by Lebesgue [14, p. 36] that $w\left(S_{3}\right) \leq 24$, which was improved in 1996 by Jendrol' and Madaras [11] to the tight bound $w\left(S_{3}\right) \leq 23$. Furthermore, Jendrol' and Madaras [11] gave a precise description of minor 3stars in $M_{5} \mathrm{~s}$.

For arbitrary NPMs, the following results concerning $(d-2)$-stars at $d$ vertices, $d \leq 5$, are known. Van den Heuvel and McGuinness [10] proved (in particular) that there is a vertex $v$ such that either $w\left(S_{1}(v)\right) \leq 14$ with $d(v)=3$, or $w\left(S_{2}(v)\right) \leq 22$ with $d(v)=4$, or $w\left(S_{3}(v)\right) \leq 29$ with $d(v)=5$. Balogh et al. [1] proved that there is a $5^{-}$-vertex adjacent to at most two $11^{+}$-vertices. Harant and Jendrol' [9] strengthened these results by proving (in particular) that either $w\left(S_{1}(v)\right) \leq 13$ with $d(v)=3$, or $w\left(S_{2}(v)\right) \leq 19$ with $d(v)=4$, or $w\left(S_{3}(v)\right) \leq 23$ with $d(v)=5$. Recently, we obtained a precise description of $(d-2)$-stars in NPMs [6].

For $M_{5} \mathrm{~s}$, Lebesgue [14, p. 36] proved $w\left(S_{4}\right) \leq 31$, which was improved by Borodin and Woodall [3] to the tight bound $w\left(S_{4}\right) \leq 30$. Note that $w\left(S_{3}\right) \leq 23$ easily implies $w\left(S_{2}\right) \leq 17$ and immediately follows from $w\left(S_{4}\right) \leq 30$ (in both cases, it suffices to delete a vertex of maximum degree from a minor star of the minimum weight). Recently, we obtained a precise description of 4 -stars in $M_{5} \mathrm{~s}$ [7].

For arbitrary NPMs, the problem of describing $(d-1)$-stars at $d$-vertices, $d \leq 5$, called pre-complete stars, appears difficult. As follows from the double $n$-pyramid, the minimum weight $w\left(S_{d-1}\right)$ of pre-complete stars in NPMs can be arbitrarily large. Even when $w\left(S_{d-1}\right)$ is restricted by appropriate conditions,
the tight upper bounds on it are unknown. Borodin et al. [4, 5] proved (in particular) that if a planar graph with $\delta \geq 3$ has no edge joining two $4^{-}$-vertices, then there is a star $S_{d-1}(v)$ with $w\left(S_{d-1}(v)\right) \leq 38+d(v)$, where $d(v) \leq 5$ (see [5, Theorem 2.A]). Jendrol' and Madaras [12] proved that if the weight of every edge in a planar graph with $\delta \geq 3$ is at least 9 , then there is a pre-complete star in which every vertex has degree at most 20 , where the bound of 20 is best possible.

The more general problem of describing $d$-stars at $d$-vertices, $d \leq 5$, called complete stars, at the moment seems intractable for arbitrary NPMs and difficult even for $M_{5} \mathrm{~s}$. In this note we make a modest contribution for the case of $M_{5} \mathrm{~s}$.

The following well-known construction shows that $w\left(S_{5}\right)$ is unbounded in $M_{5} \mathrm{~s}$. Take three concentric $n$-cycles $C^{i}=v_{1}^{i} \ldots v_{n}^{i}$, where $n$ is large and $1 \leq i \leq 3$, and join $C^{2}$ with $C^{1}$ by edges $v_{j}^{2} v_{j}^{1}$ and $v_{j}^{2} v_{j+1}^{1}$ whenever $1 \leq j \leq n$ (addition modulo $n$ ). The same is done with $C^{2}$ and $C^{3}$. Finally, join all vertices of $C^{1}$ to a new $n$-vertex, and do the same with $C^{3}$.

Definition. A 5 -vertex $v$ surrounded by vertices $v_{1}, \ldots, v_{5}$ in a cyclic order is a $(5,6,6,5)$-vertex if there is a $k, k \leq 5$, such that $d\left(v_{k+1}\right)=d\left(v_{k+4}\right)=5$, $d\left(v_{k+2}\right) \leq 6$, and $d\left(v_{k+3}\right) \leq 6$ (addition modulo 5 ).

Clearly, each 5 -vertex in the $M_{5}$ constructed above is a ( $5,6,6,5$ )-vertex and, moreover, it has two 5 -neighbors and two 6 -neighbors. Lebesgue [14, p. 36] proved that if an $M_{5}$ has no ( $5,6,6,5$ )-vertices, then $w\left(S_{5}\right) \leq 68$.

The purpose of this note is to improve the bound of 68 to 55 (Theorem 1) and give a construction of a $(5,6,6,5)$-free $M_{5}$ with $w\left(S_{5}\right)=48$ (see Figure 1). We first truncate all vertices of the dodecahedron, and then join the mid-points of the edges of each triangle and put a 2 -vertex on each edge not incident with a triangle. Finally, we insert a 20 -wheel inside every 20 -face as shown in Figure 1. As a result, every 5 -vertex has neighbors of degrees $5,6,7,5$, and 20 in this order.

Theorem 1. If a normal plane map $M_{5}$ with minimum degree 5 does not contain (5, 6, 6, 5)-vertices, then $M_{5}$ contains a 5 -star of weight at most 55 with a 5 -vertex as its center.

Problem 2. Find best possible version of Theorem 1.

## 2. Proof of Theorem 1

It suffices to prove the theorem for triangulations, since adding a diagonal edge into a non-triangular face of a normal plane map with $\delta=5$ cannot create a new minor 5 -star, nor can it reduce the weight of any existing minor 5 -star. So


Figure 1. An $M_{5}$ without (5, 6, 6, 5)-vertices such that $w\left(S_{5}\right)=48$.
suppose that a triangulation $T$, with its sets of vertices, edges, and faces denoted by $V, E$, and $F$, respectively, is a counterexample to Theorem 1. Euler's formula $|V|-|E|+|F|=2$ for $T$ implies

$$
\begin{equation*}
\sum_{v \in V}(d(v)-6)=-12 \tag{1}
\end{equation*}
$$

Assign an initial charge $\mu(v)=d(v)-6$ to each $v \in V$, so that only 5 -vertices have negative initial charge. Also, put $\mu(f)=0$ for each $f \in F$. Using the properties of our $T$ as a counterexample, we define a local redistribution of charges, preserving their sum, such that the final charge $\mu_{3}(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to -12 . The technique of discharging is often used in solving structural and coloring problems on plane graphs.

Definition. For integer $n, n \geq 6$, put $\xi(n)=\frac{n-6}{n}$. For any $6^{+}$-vertex $v$, put $\psi(v)=\xi(d(v))$.

In what follows, it is convenient to use the upwards convexity of the increasing function $\xi(n)$ for integer $n \geq 6$, which is checked easily.

Lemma 3. For any integers $p$ and $q$, where $6 \leq p<q$, we have

$$
\xi(p)+\xi(q) \leq \xi(p+1)+\xi(q-1)
$$

The final charge $\mu_{3}(v)$ of vertex $v$ is defined by applying the rules R1-R3 as follows.

R1. Each $6^{+}$-vertex $v$ sends $\psi(v)$ to each incident 3 -face.
R2. Let $f=x y z$ be a face such that $d(x)=5$ and $d(z) \geq 6$. Then $x$ receives from $f$ the following charge:
(a) $\frac{\psi(z)}{2}$ if $d(y)=5$, or
(b) $\psi(y)+\psi(z)$ if $d(y) \geq 6$.

The charge of $x$, where $x \in V \cup F$, after applying R1 and R2 is denoted by $\mu_{2}(x)$.
Definition. A 5 -vertex $v$ surrounded by vertices $v_{1}, \ldots, v_{5}$ in a cyclic order is bad if $d\left(v_{1}\right)=d\left(v_{2}\right)=d\left(v_{4}\right)=5$ and $d\left(v_{3}\right)=7$.

Note that each bad 5 -vertex $v$ satisfies $d\left(v_{5}\right) \geq 29$ since $w\left(S_{5}\right) \geq 56$ by assumption.

R3. Suppose $v$ is bad, and let the neighbors of $v_{2}$ and $v_{1}$ in the cyclic order be $v, v_{3}, x, y, v_{1}$ and $v, v_{2}, y, z, v_{5}$, respectively (see Figure 2 ).
(a) If $d(x)=5$ (which means that $v_{2}$ is also bad), then $v_{1}$ gives $\frac{1}{14}$ to each of $v$ and $v_{2}$.
(b) If $d(x) \geq 6$ and $v_{1}$ is also bad (that is $d(y)=7$, and $d(z)=5$ ), then $v_{2}$ gives $\frac{1}{14}$ to each of $v$ and $v_{1}$.
(c) If $d(x) \geq 6$ but $v_{1}$ is not bad, then $v_{2}$ gives $\frac{1}{14}$ only to $v$.

Clearly, $\mu_{3}(v)=\mu_{2}(v)=0$ for each $6^{+}$-vertex $v$, and $\mu_{3}(f)=\mu_{2}(f) \geq 0$ for each face $f$. It remains to show that every 5 -vertex receives at least 1 in total by R1-R3. If $v$ is a bad 5 -vertex, then $\mu_{2}(v) \geq 5-6+\xi(7)+\xi(28)=-\frac{1}{14}$, which implies that $\mu_{3}(v) \geq 0$.

Our next goal is to show that every non-bad 5 -vertex $v$ satisfies $\mu_{2}(v) \geq 0$, and then we will complete the proof of Theorem 1 by checking that in fact $\mu_{3}(v) \geq 0$.

Remark 4. Suppose a 5 -vertex $v$ is adjacent to a $6^{+}$-vertex $w$; then $v$ receives from $w$ by R1 and R2 one of the charges $\psi(w), \frac{3 \psi(w)}{2}$, or $2 \psi(w)$ depending on the number of 5 -vertices in the two 3 -faces incident with edge $v w$ : three, two, or one, respectively.

Definition. The type of a 5 -vertex $v$ is the vector $\left(d_{1}, \ldots, d_{5}\right)$ of the degrees of the neighbors of $v$ in the non-decreasing order. (So, $d_{1}$ is the smallest degree among the neighbors of $v, d_{2}$ is the second smallest, and so on.)

Lemma 5. If $v$ is a non-bad 5 -vertex, then $\mu_{2}(v) \geq 0$.


Figure 2. Rule R3.

Proof. Clearly, $\mu_{2}(v)=-1+\mu_{1,2}^{+}(v)$, where $\mu_{1,2}^{+}(v)$ denotes the total donation to $v$ from its neighbors by R1 and R2. Thus it suffices to prove that $\mu_{1,2}^{+}(v) \geq 1$. It will be helpful to note that

$$
\begin{equation*}
\frac{3}{2} \xi(27)=\frac{3}{2} \cdot \frac{21}{27}=\frac{7}{6}>\frac{8}{7}>1 \tag{2}
\end{equation*}
$$

Let $v$ be of type $\left(d_{1}, \ldots, d_{5}\right)$; then $d_{1}+\cdots+d_{5} \geq 56-5=51$ since $T$ is a counterexample to Theorem 1.

Now our proof splits into four cases.
Case 1. $6 \leq d_{1}$. By Remark 4, $\mu_{1,2}^{+}(v)=2\left(\psi\left(d_{1}\right)+\cdots+\psi\left(d_{5}\right)\right)$. This is smallest when $d_{1}=\cdots=d_{4}=6$ and $d_{5}=27$, as otherwise it can be made smaller by using Lemma 3 , or just by reducing $d_{5}$ if $d_{5}>27$. Thus $\mu_{1,2}^{+}(v) \geq 2 \xi(27)>1$ by (2).

Case 2. $5=d_{1}<d_{2}$. By Remark 4, $\mu_{1,2}^{+}(v) \geq \frac{3}{2}\left(\xi\left(d_{2}\right)+\cdots+\xi\left(d_{5}\right)\right)$, which is smallest when $d_{2}=\cdots=d_{4}=6$ and $d_{5}=28$. Thus $\mu_{1,2}^{+}(v) \geq \frac{3}{2} \xi(28)>1$ by (2).

Case 3. $5=d_{1}=d_{2}<d_{3}$. If the two 5 -neighbors of $v$ form a 3 -face with $v$, then $\mu_{1,2}^{+}(v) \geq \frac{3}{2}\left(\xi\left(d_{3}\right)+\xi\left(d_{4}\right)+\xi\left(d_{5}\right)\right)$, which is smallest when $d_{3}=d_{4}=6$ and $d_{5}=29$. Thus $\mu_{1,2}^{+}(v) \geq \frac{3}{2} \xi(29)>1$ by (2).

So assume that the two 5 -neighbors of $v$ do not form a 3 -face with $v$. By Remark $4, \mu_{1,2}^{+}(v) \geq \frac{3}{2}(\xi(a)+\xi(b))+\xi(c)$, where $a, b, c$ is a permutation of $d_{3}, d_{4}, d_{5}$ such that the $a$-vertex and $b$-vertex form a 3 -face with $v$. Recall that $v$ is not a $(5,6,6,5)$-vertex by assumption. So if $d_{3}=d_{4}=6$ then $c=6$ and $\max \{a, b\} \geq 29$; thus $\mu_{1,2}^{+}(v) \geq \frac{3}{2} \xi(29)>1$ by (2). Otherwise, $d_{4} \geq 7$ and $\mu_{1,2}^{+}(v) \geq \frac{3}{2} \xi(7)+\xi(28)=$ $\frac{3}{2} \cdot \frac{1}{7}+\frac{22}{28}=1$.

Case 4. $5=d_{1}=d_{2}=d_{3}<d_{4}$. Then $d_{4}+d_{5} \geq 36$. If $6 \leq d_{4} \leq 7$, then the $d_{4}$-vertex and the $d_{5}$-vertex form a 3 -face with $v$, since $v$ is neither bad nor a ( $5,6,6,5$ )-vertex by assumption; thus $\mu_{1,2}^{+}(v) \geq \frac{3}{2} \xi\left(d_{5}\right) \geq \frac{3}{2} \xi(29)>1$ by (2). If $d_{4} \geq 8$, then $\mu_{1,2}^{+}(v) \geq \xi(8)+\xi(28)=\frac{29}{28}>1$.

Lemma 6. If $v$ is a non-bad 5 -vertex, then $\mu_{3}(v) \geq 0$.
Proof. If $v$ gives nothing to bad 5 -vertices by R3, then $\mu_{3}(v)=\mu_{2}(v) \geq 0$ by Lemma 5. So we may assume that $v$ gives either $\frac{1}{14}$ or $\frac{1}{7}$ in total to bad 5 -vertices by R3. It suffices to prove that $\mu_{1,2}^{+}(v) \geq \frac{8}{7}$.

We may assume that $v$ is either the vertex $v_{1}$ in $\mathrm{R} 3(\mathrm{a})$ or the vertex $v_{2}$ in R3(b) or R3(c). In both cases, $d(x)+d(y) \geq 34$ since $w\left(S_{5}\left(v_{2}\right)\right) \geq 56$. In the first case, $d(x)=5$ and $d(y) \geq 29$ (see Figure 2), and so $\mu_{1,2}^{+}(v) \geq 2 \xi(29)>\frac{8}{7}$ by (2). In the second case, $\mu_{1,2}^{+}(v) \geq \frac{3}{2}(\xi(6)+\xi(28))=\frac{3}{2} \xi(28)>\frac{8}{7}$ by (2).

Thus we have proved $\mu_{3}(x) \geq 0$ for every $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 1 .

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