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A REDUCTION OF THE GRAPH RECONSTRUCTION CONJECTURE

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Abstract

A graph is said to be *reconstructible* if it is determined up to isomorphism from the collection of all its one-vertex deleted unlabeled subgraphs. Reconstruction Conjecture (RC) asserts that all graphs on at least three vertices are reconstructible. In this paper, we prove that interval-regular graphs and some new classes of graphs are reconstructible and show that RC is true if and only if all non-geodetic and non-interval-regular blocks G with $diam(G) = 2$ or $diam(G) = diam(\overline{G}) = 3$ are reconstructible.

Keywords: reconstruction, diameter, geodetic graph, interval-regular graph.

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1. Introduction

All graphs in this paper are finite, simple and undirected. We use the terminology of Harary [10]. The degree of a vertex v of a graph G is denoted by $deg_G(v)$. By $d_G(u, v)$, we mean the distance between two vertices u and v in G. The *eccentricity* $e_G(v)$ of a vertex in a connected graph G is maximum of $d_G(u, v)$ for all u. The *radius* and *diameter* of a graph G, denoted by $rad(G)$ and $diam(G)$ respectively, are the minimum and maximum of the vertex eccentricities respectively. A connected graph is called *separable* if it has cut vertices and is called a *block* otherwise. The set of all neighbours of a vertex u of G is denoted by $N(u)$ (or $N_G(u)$). For any two vertices u and v in G, the set $I(u, v) = \{w \in V(G) : w$

lies on a shortest $u - v$ path$\}$ is the interval in G between u and v. A connected graph G is called *interval-regular* if $|I(u,v) \cap N(u)| = d_G(u,v)$ for all $u, v \in V(G)$. Interval-regular graphs have been introduced and studied in [17]. Graph in which every pair of vertices have unique shortest path is a *geodetic graph.*

A *vertex-deleted subgraph* (or *card*) $G - v$ of a graph G is the unlabeled subgraph obtained from G by deleting v and all edges incident with v. The collection of all cards of G is called the *deck* of G. A graph H is called a *reconstruction* of G if H has the same deck as G. A graph is said to be *reconstructible* if it is isomorphic to all its reconstructions. A family $\mathscr{F}$ of graphs is *recognizable* if, for each $G \in \mathscr{F}$, every reconstruction of G is also in $\mathscr{F}$, and *weakly reconstructible* if, for each graph $G \in \mathscr{F}$, all reconstructions of G that are in $\mathscr{F}$ are isomorphic to G. A family $\mathscr{F}$ of graphs is *reconstructible* if, for any graph $G \in \mathscr{F}$, G is reconstructible (i.e. if $\mathscr{F}$ is both recognizable and weakly reconstructible). A parameter p defined on graphs is reconstructible if, for any graph G, it takes the same value on every reconstruction of G. Ulam's Conjecture, also called Reconstruction Conjecture (RC) [9] asserts that all graphs on at least three vertices are reconstructible. [1, 2, 4, 13, 14], and [18] are surveys of workdone on RC and related problems.

Yang Yongzhi [22] settled the problem listed in survey [1] when he proved the following.

Theorem 1. *RC is true if and only if all* 2*-connected graphs are reconstructible.*

Definition 2 [8]**.** $pav(G,i)$ $(pv(G,i))$ (where $i \in [0, n-2]$) is the number of adjacent (non-adjacent) pairs of vertices of G such that, for each pair, there are exactly i paths of length two between the two vertices.

Using the parameters $pv(G,i)$, Gupta *et al.* [8] have proved the following reduction of RC.

Theorem 3. *RC is true if and only if all graphs G with $diam(G) = 2$ or $diam(G) = diam(\overline{G}) = 3$ are weakly reconstructible.*

If a connected graph G has $pv(G,\lambda) = pav(G,\lambda) > 0$ and $pv(G,i) = pav(G,i) = 0$ for all $i \neq \lambda$, then every pair of vertices of G has exactly λ common neighbours or none at all, and vice-versa. Mulder [16] defined such a connected graph as $(0,\lambda)$-graph and proved that it is regular (graph in which all vertices have equal degree).

In their book [3], Brouwer *et al.* have discussed many classes of regular graphs. In particular, they studied regular graphs in which any two non-adjacent vertices have precisely μ common neighbours, and they called such graphs are *co-edge-regular* graphs. Co-edge-regular graphs with $\mu > 0$ are clearly connected and have diameter at most two. Here we prove, using the parameters $pv(G,i)$, that connected graphs G, in which every pair of non-adjacent vertices has precisely μ common neighbours or none at all, are reconstructible. For $\mu = 1$, graph G is geodetic and for $\mu = 2$, graph G is interval-regular. We also show that all graphs are reconstructible if and only if all non-geodetic and non-interval-regular blocks G with $diam(G) = 2$ or $diam(G) = diam(\overline{G}) = 3$ are reconstructible.

2. Geodetic Graphs and Interval-regular Graphs

The following three lemmas are proved by Gupta *et al.* [8].

Lemma 4. *For a graph G of order n and $i \in [0, n-2]$, the parameters $pv(G, i)$ are reconstructible.*

Lemma 5. *Graphs G with $diam(G) = 2$ are recognizable.*

Definition 6. For a graph G of order n and $\mu \in [0, n-2]$, $S_\mu(G)$ is the set of all vertices u such that there exists a vertex v non-adjacent to u such that u and v have exactly μ common neighbours in G.

Theorem 7. *Connected graphs G in which every pair of non-adjacent vertices has exactly μ common neighbours or none at all are reconstructible, where μ is a constant greater than one.*

Proof. *Recognition.* The class of connected graphs G under consideration is recognizable if $pv(G, \mu) > 0$ and $pv(G, i) = 0$ for $i \in [1, n-2]$ and $i \neq \mu$.

Weak reconstruction. For $v \in V(G)$, we have $d_G(x, y) \leq 2$ for all $x, y \in N_G(v)$. Depending on this fact, we consider two cases as follows.

Case 1. There is a vertex v in G such that for each vertex $x \in N_G(v)$, there exists a vertex $y \in N_G(v)$ such that $d_G(x, y) = 2$. This case is recognizable if there exists a card $G - v$ such that $|S_{\mu-1}(G - v)| = deg_G(v)$. Since $pv(G, \mu - 1) = 0$, it follows that the set $S_{\mu-1}(G - v)$ in the card $G - v$ must be the neighbourhood of v in G. Now all graphs obtained from $G - v$ by adding a new vertex and joining it with all the vertices in $S_{\mu-1}(G - v)$ are isomorphic to G. Hence G is reconstructible.

Case 2. For each vertex $v \in V(G)$, there exists a vertex $x \in N_G(v)$ such that $d_G(x, y) = 1$ for all $y \in N_G(v)$. This case occurs if there exists no card $G - v$ satisfying the equation $|S_{\mu-1}(G - v)| = deg_G(v)$. Now consider a card $G - x$ such that $deg_G(x) = \Delta(G)$ (the maximum degree). Let (A, B) be the bipartition of $N_G(x)$ in G such that $A = \{y \in N_G(x) : y$ is non-adjacent to at least one vertex in $N_G(x)\}$ and $B = N_G(x) - A$. Since $|S_{\mu-1}(G - v)| \neq deg_G(v)$ for every card $G - v$ of G, it follows that $|S_{\mu-1}(G - x)| < deg_G(x)$ and hence B is non-empty (by our hypothesis in Case 2).

In the card $G-x$, the set A is identifiable uniquely as the set $S_{\mu-1}(G-x)$. But the set B can be identified up to automorphism in $G-x$ (as in (i) below). Now all graphs obtained by adding a new vertex to $G-x$ and joining it with all the vertices in $A \cup B$ are isomorphic to G. Hence G is reconstructible.

(i) The set B consists of neighbours of x that are adjacent to all other neighbours of x in G. Therefore, in the card $G-x$, the set B is identifiable as a vertex set of a complete subgraph, say C, of size $\Delta(G)-|A|$ such that each vertex of C is adjacent to all the vertices in the identifiable set A and is non-adjacent to any vertex in $V(G-x)-(A\cup V(C))$. If there exists more than one such complete subgraph, say C_1 and C_2 in $G-x$, then the mapping $\alpha : V(G-x) \longrightarrow V(G-x)$, defined by $\alpha(V(C_1)) = V(C_2)$ and fixing all other vertices, is an automorphism of $G-x$. ■

Theorem 8. *Graphs G in which every pair of non-adjacent vertices has exactly μ common neighbours are reconstructible, where μ is a non-zero constant.*

Proof. The graph G is clearly connected and $diam(G) = 2$ and hence $pv(G,0)=0$.

Recognition. The class of connected graphs G under consideration is recognizable if $pv(G,\mu)>0$ and $pv(G,i)=0$ for $i\in[0,n-2]$ and $i\neq\mu$.

Weak reconstruction. If $\mu\geq 2$, then G is reconstructible by Theorem 7. So assume that $\mu=1$. Then there exists a card $G-v$ (obtained by deleting the unique neighbour of a non-adjacent pair of vertices) with $pv(G-v,0)>0$. Since $pv(G,0)=0$, graph G can be obtained uniquely (up to isomorphism) from the card $G-v$ by adding a new vertex w and joining it with all the vertices in $S_0(G-v)$. ■

For $\mu=1$, Theorem 8 implies

Corollary 9. *Geodetic graphs of diameter two are reconstructible.*

Since any two non-adjacent vertices of distance two in an interval-regular graph have exactly two common neighbours, Theorem 7 for $\mu=2$ gives

Corollary 10. *Interval-regular graphs are reconstructible.*

Theorem 11. *Let G be a connected graph of diameter two with k pairs of non-adjacent vertices; let $\mu_1,\ \mu_2,\ldots,\mu_k$ be the number of common neighbours of the k pairs. If $|\mu_r-\mu_s|\geq 2$ for all $r\neq s$, then G is reconstructible.*

Proof. *Recognition.* Follows from Lemmas 4 and 5.

Weak reconstruction. We shall assume that $pv(G,0)=0$ (as otherwise, $diam(G)\neq 2$). Then since $diam(G)=2$, it follows that $pv(G,i)>0$ for some $i\in[1,n-2]$. If $pv(G,j)=0$ for all $j\in[1,n-2]$ and $j\neq i$, then G is reconstructible by Theorem 8 with $\mu=i$. So, we shall assume that $pv(G,j)>0$ for some $j \in [1, n-2]$ and $j \neq i$. But from our hypothesis, $pv(G, i-1) = pv(G, i+1) = pv(G, j-1) = pv(G, j+1) = 0$.

If there exists a card $G - v$ such that $|S_{i-1}(G-v) \cup S_{j-1}(G-v)| = deg_G(v)$, then the set $S_{i-1}(G-v) \cup S_{j-1}(G-v)$ in the card $G-v$ must be the neighbourhood of v in G (because $pv(G, i-1) = pv(G, j-1) = 0$) and hence G is reconstructible. So, we shall assume that there exists no card $G - v$ satisfying the equation $|S_{i-1}(G-v) \cup S_{j-1}(G-v)| = deg_G(v)$.

Now consider a card $G - x$ such that $deg_G(x) = \Delta(G)$. Let (A, B) be the bipartition of $N_G(x)$ in G such that $A = \{y \in N_G(x) : y$ is non-adjacent to at least one vertex in $N_G(x)\}$ and $B = N_G(x) - A$. Since $|S_{i-1}(G-v) \cup S_{j-1}(G-v)| \neq deg_G(v)$ for every card $G - v$ of G, it follows that $|S_{i-1}(G-v) \cup S_{j-1}(G-v)| < deg_G(v)$ and for every vertex v in G, there exists a neighbour w of v such that w is adjacent to all other neighbours of v in G. Therefore B is non-empty.

In the card $G - x$, the set A is identifiable uniquely as the set $S_{i-1}(G-x) \cup S_{j-1}(G-x)$. But the set B can be identified (up to automorphism) in $G - x$ by using similar arguments we employed for identifying B in Case 2 of Theorem 7. Hence G is reconstructible. ■

3. Geodetic Blocks of Diameter Three

Parthasarathy and Srinivasan [20] have obtained some structural properties of geodetic blocks of diameter three and proved that geodetic blocks of diameter three are self-centered (graphs H with $diam(H) = rad(H)$).

Theorem 12 [20]**.** *Every geodetic block of diameter three is self-centered.*

Using Theorem 12, we first prove that if a geodetic block has diameter three, then its complement has diameter two. It will be used while proving Theorem 16. The next theorem is well known.

Theorem 13 [21]**.** *If* $diam(G) \geq 3$*, then* $diam(\overline{G}) \leq 3$.

When U and W are disjoint subsets of the vertex set $V(G)$ of a graph G, $U \sim W$ means that every vertex in U is adjacent to every vertex in W, and when $u \notin W$, $u \sim W$ means that u is adjacent to every vertex in W.

Theorem 14. *If G is a geodetic block of diameter three, then $\overline{G}$ has diameter two.*

Proof. Since G is a block, $diam(\overline{G}) \neq 1$ and hence $diam(\overline{G}) = 2$ or 3, by Theorem 13. We assume, to the contrary, that $diam(\overline{G}) = 3$. Then since G is a geodetic block of diameter three, $e_G(v) = 3$ for all vertices v in G, by Theorem 12. Consider a vertex v in G. Let $N_i(v)$ $(i = 1, 2, 3)$ be the set of all vertices which are at distance i from v in G. Since $e_G(v) = 3$, $N_i(v)$ is non-empty for $i = 1, 2, 3$.

Now in $\overline{G}$, $v \sim N_2(v) \cup N_3(v)$ and $N_1(v) \sim N_3(v)$. Therefore, $d_{\overline{G}}(u, w) \leq 2$ for every pair of vertices u and w except for the case when one is in $N_1(v)$ and the other is in $N_2(v)$. Since $diam(\overline{G}) = 3$, it follows that $d_{\overline{G}}(x, y) = 3$ for some $x \in N_1(v)$ and $y \in N_2(v)$ see (Figure 1); in Figure 1, a thick line denotes the existence of all possible edges, a dashed line denotes non-adjacency. Since $N_1(v) \sim N_3(v)$ in $\overline{G}$, it follows that the vertex y is not adjacent to any vertex in $N_3(v)$ in $\overline{G}$ (as otherwise, $d_{\overline{G}}(x, y) = 2$). Consequently, we get $y \sim N_3(v)$ in G.

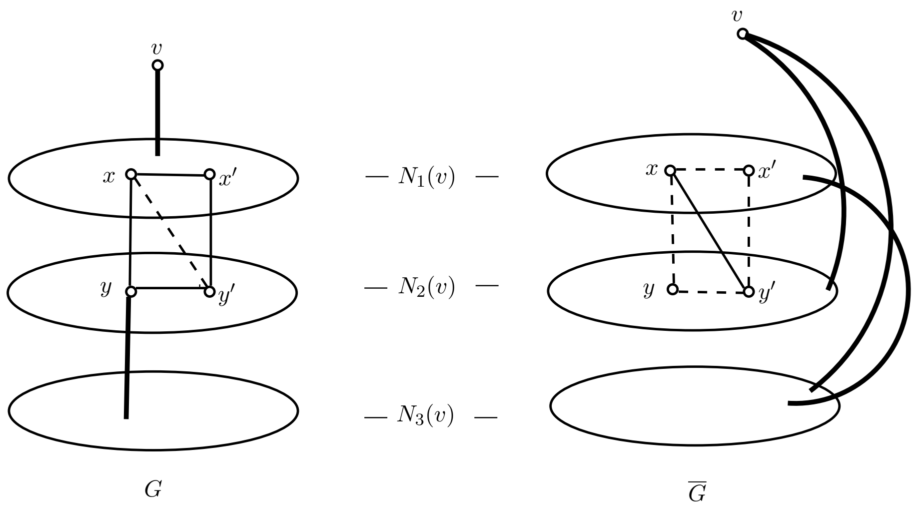

Figure 1

Now in G, the vertex y is adjacent to the vertex x (because $d_{\overline{G}}(x, y) = 3$) and so y is not adjacent to any other vertex in $N_1(v)$ (because G is geodetic). Consequently, the vertex y is adjacent to all the vertices in $N_1(v) \backslash \{x\}$ in $\overline{G}$. This, together with $d_{\overline{G}}(x, y) = 3$, implies that the vertex x is not adjacent to any vertex in $N_1(v)$ in $\overline{G}$. Consequently, we get $x \sim N_1(v)$ in G. Also, we have $x \sim N_2(v)$ (as in (i) below).

Thus in G, we have proved that $x \sim N_1(v)$, $y \sim N_3(v)$, and $x \sim N_2(v)$. Therefore $d_G(x, w) \leq 2$ for every vertex w of G and hence $e_G(x) = 2$, giving a contradiction to G being a self-centered graph of diameter 3.

(i) Suppose there exists a vertex y' in $N_2(v)$ such that y' were not adjacent to x in G. Since G is geodetic, y' would have a unique neighbour say x' in $N_1(v)$. Now in $\overline{G}$, no vertex in $N_2(v)$ is adjacent to both x and y (because $d_{\overline{G}}(x, y) = 3$) and hence the vertex y' would be adjacent to exactly one of the vertices of x and y. Consequently, we would have $yy' \in E(G)$. Hence in G, the non-adjacent vertices x and y' would have two common neighbours (namely x' and y), giving a contradiction to G being geodetic. ■

The next theorem is well known.

Theorem 15 [1]. *A graph G is reconstructible if and only if $\overline{G}$ is reconstructible.*

Theorem 16. *All 2-connected graphs are reconstructible if and only if all non-geodetic and non-interval-regular blocks G such that $diam(G) = 2$ or $diam(G) = diam(\overline{G}) = 3$ are weakly reconstructible.*

Proof. The necessity is obvious.

Sufficiency: Let G be a 2-connected graph. Then, by Theorems 13 and 15, it is enough to reconstruct only 2-connected graphs G with $diam(G) \leq 3$. If $diam(G) = 1$, then G is a complete graph and hence G is reconstructible. If $diam(G) = 2$, then G is reconstructible by Corollary 9 or 10, or our hypothesis. So we assume that $diam(G) = 3$.

Now $diam(\overline{G}) \leq 3$. If $diam(\overline{G}) = 1$, then $\overline{G}$ is reconstructible. If $diam(\overline{G}) = 2$ and $\overline{G}$ is 2-connected, then $\overline{G}$ is reconstructible by Corollary 9 or 10, or our hypothesis. If $diam(\overline{G}) = 2$ and $\overline{G}$ is separable, then all blocks of $\overline{G}$ are end-blocks and hence $\overline{G}$ has only one cutvertex, say v. Since $diam(\overline{G}) = 2$, the vertex v must be adjacent to all other vertices of $\overline{G}$ and hence $\overline{G}$ is reconstructible. So we shall assume that $diam(\overline{G}) = 3$. Then $\overline{G}$ is reconstructible by either Theorem 14 and Corollary 10 or our hypothesis. ■

Theorem 17. *All graphs on at least three vertices are reconstructible if and only if all non-geodetic and non-interval-regular blocks G such that $diam(G) = 2$ or $diam(G) = diam(\overline{G}) = 3$ are weakly reconstructible.*

Proof. Follows from Theorems 1 and 16. ■

4. Conclusion

Some subclasses of 2-connected graphs are already proved to be reconstructible (see [5, 6, 7, 11, 12, 15, 19]). Several other classes of graphs already proved to be reconstructible contain 2-connected graphs. Attempts to prove RC using them may lead to the reconstruction of more classes of graphs and further narrow down the classes of graphs to be reconstructed to prove RC. Using the parameters $pv(G, i)$ and $pav(G, i)$, short of proving the reconstructibility of these narrowed down classes, one can divide it further into some subclasses and can try to prove the reconstructibility of these subclasses. If at all RC is false, then these narrowed down classes must contain a pair of non-isomorphic graphs having the same deck.

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