Discussiones Mathematicae
Graph Theory 34 (2014) 497–507
doi:10.7151/dmgt.1745

TWO GRAPHS WITH A COMMON EDGE

Lidia Badura

Mathematics and Computer Science Department
University of Opole
Oleska 48
45-052 Opole, Poland

e-mail: badura@math.uni.opole.pl

Abstract

Let $G = G_1 \cup G_2$ be the sum of two simple graphs G_1, G_2 having a common edge or $G = G_1 \cup e_1 \cup e_2 \cup G_2$ be the sum of two simple disjoint graphs G_1, G_2 connected by two edges e_1 and e_2 which form a cycle C_4 inside G. We give a method of computing the determinant $\det A(G)$ of the adjacency matrix of G by reducing the calculation of the determinant to certain subgraphs of G_1 and G_2. To show the scope and effectiveness of our method we give some examples.

Keywords: graph, adjacency matrix, determinant of graph, path, cycle.

2010 Mathematics Subject Classification: 05C50.

1. Introduction

We say that a graph is *singular* if the determinant of its adjacency matrix is equal to 0. For a large n, computations of the determinant of an $n \times n$ matrix are generally difficult. There are known certain reduction procedures for calculating the determinant of the adjacency matrix of some graphs, presented by F. Harary [2], H.M. Rara [4], L. Huang and W. Yan [3] and A. Abdollahi [1]. In S. Arwon and P. Wojtylak [5], some reduction procedures are used in case of paths of cycles and cycles of cycles. This paper was an inspiration for the present work in which we consider sums of two graphs with a common edge and two separate graphs connected by two edges.

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a non-empty set of *vertices* and $E(G)$ is a set of unordered pairs of vertices, called *edges*. Let $[x, y]$ stand for the edge with the vertices x and y. If $[x, y] \in E(G)$, we also say that y

is a *neighbour* of x. We denote the set of all neighbours of x by $N_G(x)$. The cardinality of the set $N_G(x)$, denoted by $deg_G(x)$, is the degree of the vertex x. If all vertices of a graph have the same degree we call it a *regular* graph.

The graph P_n, where $n \geq 1$, with the vertex set $V(P_n) = \{1, 2, \ldots, n\}$ and the edge set $E(P_n) = \{\{x, y\} \subseteq V(P_n) : |x - y| = 1\}$ is called a *path of order n*. And the graph C_n, where $n \geq 3$, with the vertex set $V(C_n) = \{1, 2, \ldots, n\}$ and the edge set $E(C_n) = \{\{x, y\} \subseteq V(C_n) : |x - y| \equiv 1 (\text{mod } n)\}$ is called a *cycle of order n*.

Let G and H be graphs. Then H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $x \in V(G)$ then $G \setminus x$ means the subgraph of G with the vertex set $V(G) \setminus \{x\}$ and the edge set $E(G) \setminus \{[x, y] : y \in N_G(x)\}$. By $G \setminus [x, y]$ we mean the subgraph G with the vertex set $V(G)$ and the edge set $E(G) \setminus [x, y]$. A subgraph H of G is called a *spanning subgraph* if $V(G) = V(H)$. The *union* $G \cup H$ of two graphs G and H is a graph defined by two equations $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. *Sesquivalent graphs* are (disjoint) unions of regular graphs of degree 1 or 2. In other words, a graph is sesquivalent if it consists of single edges and cycles.

Let $v_1, v_3, \ldots, v_n$ be the vertices of a graph G and assume that $v_i \neq v_j$ if $i \neq j$. The *adjacency matrix* of G is the matrix $A(G) = [a_{ij}]_{n \times n}$ where

$$a_{ij} = \begin{cases} 1 \text{ if } v_i \text{ and } v_j \text{ are adjacent in } G, \\ 0 \text{ otherwise.} \end{cases}$$

Lemma 1. *If G and H are (vertex) disjoint graphs, then*

$$\det A(G \cup H) = \det A(G) \cdot \det A(H).$$

By the following theorem, see Harary [2], we can compute the determinant of the adjacency matrix of any graph:

Theorem 2. *For each graph G*

$$\det A(G) = \sum (-1)^{r(\Gamma)} \cdot 2^{s(\Gamma)}$$

where the summation is over all sesquivalent spanning subgraphs Γ of G where $c(\Gamma)$ is the number of components of the graph Γ, and $r(\Gamma) = |V(\Gamma)| - c(\Gamma)$ and $s(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + c(\Gamma)$.

In particular, we obtain the following results.

Corollary 1. *For each $n \geq 1$,*

$$\det A(P_n) = \begin{cases} 1 & if \;\; n \equiv 0 \,(\text{mod } 4), \\ -1 & if \;\; n \equiv 2 \,(\text{mod } 4), \\ 0 & otherwise. \end{cases}$$

Corollary 2. *For each $n \geq 1$,*

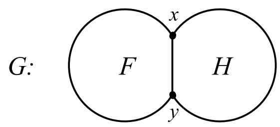
$$\det A(C_n) = \begin{cases} 0 & if \;\; n \equiv 0 \,(\mathrm{mod}\ 4), \\ -4 & if \;\; n \equiv 2 \,(\mathrm{mod}\ 4), \\ 2 & otherwise. \end{cases}$$

2. Sums of Two Graphs with a Common Edge

Theorem 3. *Let G, F and H be simple graphs such that $G = F \cup H$, and $V(F) \cap V(H) = \{x, y\}$ and $E(F) \cap E(H) = [x, y]$. Then*

$$\begin{aligned} \det A(G) \;=\; & \det A(G \setminus [x,y]) + \det A(F) \cdot \det A(H \setminus \{x,y\}) \\ & + \det A(F \setminus \{x,y\}) \cdot \det A(H \setminus [x,y]) \\ & + \det A(F \setminus \{x,y\}) \cdot \det A(H \setminus \{x,y\}). \end{aligned}$$

Proof. Let Γ be a sesquivalent spanning subgraph of $G = F \cup H$.



If x and y belong to a cycle in Γ, then Γ must be included in one (and only one) of the graphs $G \setminus [x, y]$ or $F \cup H \setminus \{x, y\}$ or $F \setminus \{x, y\} \cup H$.

If the edge $[x, y]$ belongs to Γ, then Γ must be included in $F \cup H \setminus \{x, y\}$ or in $F \setminus \{x, y\} \cup H$, though it may be included in both. However, if it happens, then Γ is included in $F \setminus \{x, y\} \cup H \setminus \{x, y\} \cup [x, y]$.

If $[x, y] \notin E(\Gamma)$ (and x, y do not belong to a cycle in Γ), then there are $t, z \in V(G)$ such that $[x, t], [y, z] \in E(\Gamma)$. If one of the elements $\{t, z\}$ is in F and the other in H, then Γ is a sesquivalent spanning subgraph of $G \setminus [x, y]$. If $\{t, z\} \subseteq V(H)$, then Γ is a sesquivalent spanning subgraph of $G \setminus [x, y]$ and $F \setminus \{x, y\} \cup H \setminus [x, y]$. Similarly if $\{t, z\} \subseteq V(H)$, then Γ is a sesquivalent spanning subgraph of $G \setminus [x, y]$ and $F \setminus [x, y] \cup H \setminus \{x, y\}$.

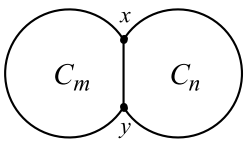
Summing up, a sesquivalent spanning subgraph Γ of G is a subgraph of $G \setminus [x, y]$, or $F \cup H \setminus \{x, y\}$ or $F \setminus \{x, y\} \cup H$. No sesquivalent spanning subgraph of G can be a subgraph of all three graphs $G \setminus [x, y]$, and $F \cup H \setminus \{x, y\}$ and $F \setminus \{x, y\} \cup H$. It may happen, however, that Γ is included in two of them. But it takes place if and only if Γ is a sesquivalent spanning subgraph in one (and only one) of the graphs: $F \setminus \{x, y\} \cup H \setminus \{x, y\} \cup [x, y]$, $F \setminus [x, y] \cup H \setminus \{x, y\}$ or $F \setminus \{x, y\} \cup H \setminus [x, y]$. Therefore, using Theorem 2 and Lemma 1 we obtain

$$\begin{aligned}\det A(G) &= \det A(G \setminus [x,y]) + \det A(F \cup H \setminus \{x,y\}) + \det A(F \setminus \{x,y\} \cup H)\\ &- \det A(F \setminus \{x,y\} \cup H \setminus [x,y]) - \det A(F \setminus \{x,y\} \cup H \setminus [x,y])\\ &- \det A(F \setminus \{x,y\} \cup H \setminus \{x,y\} \cup [x,y]) = \det A(G \setminus [x,y])\\ &+ \det A(F) \cdot \det A(H \setminus \{x,y\}) + \det A(F \setminus \{x,y\}) \cdot \det A(H)\\ &- \det A(F \setminus [x,y]) \cdot \det A(H \setminus \{x,y\}) - \det A(F \setminus \{x,y\})\\ &\cdot \det A(H \setminus [x,y]) + \det A(F \setminus \{x,y\}) \cdot \det A(H \setminus \{x,y\}).\end{aligned}$$

■

3. Two Cycles with a Common Edge

In this part we will consider the graph $G = C_m \cup C_n$, where C_m and C_n, $m, n > 2$, are two cycles with a common edge such that $V(C_m) \cap V(C_n) = \{x, y\}$ and $E(C_m) \cap E(C_n) = [x, y]$.

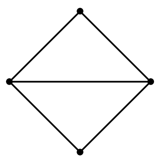
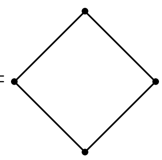


According to Theorem 5,

Corollary 3. *If m and n are odd, then*

$$\det A(C_m \cup C_n) = \det A(C_{m+n-2}) = \begin{cases} 4 & \text{if } m+n \equiv 0 \pmod 4,\\ 0 & \text{if } m+n \equiv 2 \pmod 4.\end{cases}$$

For example,

$\det\ A(C_3 \cup C_3) =$  $=$  $= 0.$

Now, let m be odd and n even. Then we have

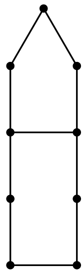
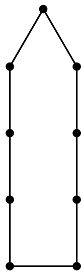
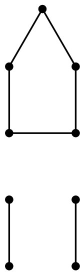
$$\det A(G) = \det A(C_m \cup C_n) = \det A(C_{m+n-2}) + \det A(C_m) \cdot \det A(P_{n-2}).$$

Therefore, we obtain

Corollary 4. *If m is odd, then*

$$\det A(C_m \cup C_n) = \begin{cases} 4 & \text{if } n \equiv 2 \pmod 4,\\ 0 & \text{if } n \equiv 0 \pmod 4.\end{cases}$$

For example,

$\det\, A(C_5 \cup C_6) =$  $=$  $+$  $= 2 + 2 \cdot (-1) \cdot (-1) = 4.$

It remains for us to consider the case when m and n are even numbers. Then

$$\begin{aligned}
\det A(G) \;&=\; \det A(C_m \cup C_n) = \det A(C_{m+n-2}) + \det A(C_m) \cdot \det A(P_{n-2}) \\
&+ \det A(P_{m-2}) \cdot \det A(C_n) - \det A(P_m) \cdot \det A(P_{n-2}) \\
&- \det A(P_{m-2}) \cdot \det A(P_n) + \det A(P_{m-2}) \cdot \det A(P_{n-2}) \\
&= \det A(C_{m+n-2}) + \det A(C_m) \cdot (-1)^{\frac{n-2}{2}} + (-1)^{\frac{m-2}{2}} \cdot \det A(C_n) \\
&- (-1)^{\frac{m}{2}} \cdot (-1)^{\frac{n-2}{2}} - (-1)^{\frac{m-2}{2}} \cdot (-1)^{\frac{n}{2}} + (-1)^{\frac{m-2}{2}} \cdot (-1)^{\frac{n-2}{2}} \\
&= \det A(C_{m+n-2}) + (-1)^{\frac{n-2}{2}} \cdot \det A(C_m) + (-1)^{\frac{m-2}{2}} \cdot \det A(C_n) \\
&+ 3 \cdot (-1)^{\frac{m+n}{2}}.
\end{aligned}$$

We have three possibilities:

1. If $m, n \equiv 0 \,(\mathrm{mod}\ 4)$, then $\det A(G) = -4 + (-1) \cdot 0 + (-1) \cdot 0 + 3 \cdot 1 = -1$.

2. If $m, n \equiv 2 \,(\mathrm{mod}\ 4)$, then $\det A(G) = -4 + 1 \cdot (-4) + 1 \cdot (-4) + 3 \cdot 1 = -9$.

3. If $m \equiv 0 \,(\mathrm{mod}\ 4), n \equiv 2 \,(\mathrm{mod}\ 4)$, then $\det A(G) = 0 + 1 \cdot 0 + (-1) \cdot (-4)$ $+3 \cdot (-1) = 1$.

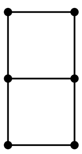
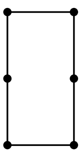
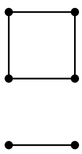
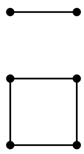
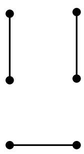
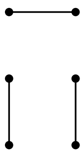
Hence,

Corollary 5. *If m and n are even, then*

$$\det A(G) = \begin{cases} -9 & \text{if } m, n \equiv 2 \,(\mathrm{mod}\ 4), \\ -1 & \text{if } m, n \equiv 0 \,(\mathrm{mod}\ 4), \\ 1 & \text{if } m \equiv 0 \,(\mathrm{mod}\ 4),\ n \equiv 2 \,(\mathrm{mod}\ 4). \end{cases}$$

For example,

$\det\, A(C_4 \cup C_4) =$

$=$  $=$  $+$  $+$  $-$  $-$  $+$  $=$

$= -4 + 0 + 0 - (-1) - (-1) + 1 = -1.$

To recapitulate,

Theorem 4. *Let $G = C_m \cup C_n$, $m, n > 2$ be the sum of two cycles C_m and C_n with common vertices $V(C_m) \cap V(C_n) = \{x, y\}$ and common edges $E(C_m) \cap E(C_n) = [x, y]$. The determinant of adjacency matrix of the graph is given by*

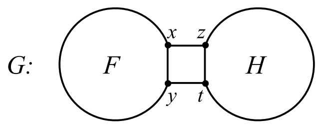
$$\det A(G) = \begin{cases} -9 & if \quad m \equiv 2 \,(\mathrm{mod}\ 4), n \equiv 2\,(\mathrm{mod}\ 4), \\ -4 & if \quad m + n \equiv 0\,(\mathrm{mod}\ 4), m, n \text{ are odd}, \\ -1 & if \quad m \equiv 0\,(\mathrm{mod}\ 4), n \equiv 0\,(\mathrm{mod}\ 4), \\ 0 & if \quad m + n \equiv 2\,(\mathrm{mod}\ 4), m, n \text{ are odd or} \\ & \qquad m \equiv 0\,(\mathrm{mod}\ 4), n \text{ is odd}, \\ 1 & if \quad m \equiv 0\,(\mathrm{mod}\ 4), n \equiv 2\,(\mathrm{mod}\ 4), \\ 4 & if \quad m \equiv 2\,(\mathrm{mod}\ 4), n \text{ is odd}. \end{cases}$$

4. Sums of Two Graphs Connected by Two Edges

Theorem 5. *Let F and H be simple disjoint graphs and G be graph such that $G = F \cup H \cup [x, z] \cup [y, t]$, where $\{x, y\} \subseteq V(F)$, $\{z, t\} \subseteq V(H)$ and $[x, y] \in E(F)$, $[z, t] \in E(H)$. Then*

$$\begin{aligned} \det A(G) = \ & \det A(G \setminus \{[x, y], [z, t]\}) + \det A(F) \cdot \det A(H) \\ & + \det A(F \cup \{z, t\} \setminus [x, y]) \cdot \det A(H \setminus \{z, t\}) \\ & + \det A(F \setminus \{x, y\}) \cdot \det A(H \cup \{x, y\} \setminus [z, t]) \\ & - \det A(F \setminus [x, y]) \cdot \det A(H \setminus [z, t]) \\ & - 4 \cdot \det A(F \setminus \{x, y\}) \cdot \det A(H \setminus \{z, t\}) \\ & + \det A(F \setminus [x, y]) \cdot \det A(H \setminus \{z, t\}) \\ & + \det A(F \setminus \{x, y\}) \cdot \det A(H \setminus [z, t]). \end{aligned}$$

Proof. Let Γ be a sesquivalent spanning subgraph of $G = F \cup H \cup [x, z] \cup [y, t]$.



If x, y, z and t belong to a cycle in Γ, then Γ must be included in one (and only one) of the graphs $G \setminus \{[x, y], [z, t]\}$ or $F \cup H$ or $(F \cup \{z, t\} \setminus [x, y]) \cup H \setminus \{z, t\}$ or $F \setminus \{x, y\} \cup (H \cup \{x, y\} \setminus [z, t])$ or $F \setminus \{x, y\} \cup C_4 \cup H \setminus \{x, y\}$, where C_4 is a cycle such that $\{x, y, z, t\} \in V(C_4)$.

Notice that, if $[x, z]$ belongs to Γ, then Γ must be included in $G \setminus \{[x, y], [z, t]\}$ or $(F \cup \{z, t\} \setminus [x, y]) \cup H \setminus \{z, t\}$ or $F \setminus \{x, y\} \cup (H \cup \{x, y\} \setminus [z, t])$ or $F \setminus \{x, y\} \cup C_4 \cup H \setminus \{x, y\}$, though it may be included in each of them. However, if it happens, then Γ is included in $F \setminus \{x, y\} \cup [x, z] \cup [y, t] \cup H \setminus \{z, t\}$.

If the edge $[x, y]$ and $[z, t]$ belongs to Γ, then Γ must be included in $F \cup H$ or $F \setminus \{x, y\} \cup C_4 \cup H \setminus \{x, y\}$, though it may be included in both. However, if it happens, then Γ is included in $F \setminus \{x, y\} \cup [x, y] \cup [z, t] \cup H \setminus \{z, t\}$.

If the edge $[x,y]$ belongs to Γ and $[z,t]$ does not belong to Γ, then Γ must be included in $F\cup H$ or $F\setminus[x,y]\cup(H\cup\{x,y\}\setminus[z,t])$, though it may be included in both. However, if it happens, then Γ is included in $F\setminus\{x,y\}\cup[x,y]\cup H\setminus[z,t]$.

Similarly, if the edge $[x,y]$ does not belong to Γ and $[z,t]$ belongs to Γ, then Γ must be included in $F\cup H$ or $(F\cup\{z,t\}\setminus[x,y])\cup H\setminus[z,t]$, though it may be included in both. However, if it happens, then Γ is included in $F\setminus[x,y]\cup[z,t]\cup(H\cup\{z,t\})$.

If $[x,z]$ and $[x,y]$ do not belong to Γ, then x,y,z do not belong to a cycle in Γ and there are $\{k,p\}\in V(G)$ such that $[x,k]$, $[z,p]\in E(\Gamma)$ and $[x,k]\in F\setminus[x,y]$ and $[z,p]\in H\setminus[z,t]$. So Γ is a sesquivalent spanning subgraph of $G\setminus\{[x,y],[z,t]\}$ or $F\cup G$, though it may be included in both. However, if it happens, then Γ is included in $F\setminus[x,y]\cup H\setminus[z,t]$.

Summing up, a sesquivalent spanning subgraph Γ of G is a subgraph of $G\setminus\{[x,y],[z,t]\}$ or $F\cup H$ or $(F\cup\{z,t\}\setminus[x,y])\cup H\setminus\{z,t\}$ or $F\setminus\{x,y\}\cup(H\cup\{x,y\}\setminus[z,t])$ or $F\setminus\{x,y\}\cup C_4\cup H\setminus\{x,y\}$.

Let us notice that:

- the sesquivalent spanning subgraph Γ of $F\setminus[x,y]\cup H\setminus[z,t]$ can be a subgraph of two graphs $G\setminus\{[x,y],[z,t]\}$ and $F\cup H$,
- the sesquivalent spanning subgraph Γ of $F\setminus\{x,y\}\cup[x,z]\cup[u,t]\cup H\setminus\{z,t\}$ can be a subgraph of four graphs $G\setminus\{[x,y],[z,t]\}$ and $(F\cup\{z,t\}\setminus[x,y])\cup H\setminus\{z,t\}$ and $F\setminus\{x,y\}\cup(H\cup\{x,y\}\setminus[z,t])$ and $F\setminus\{x,y\}\cup C_4\cup H\setminus\{x,y\}$,
- the sesquivalent spanning subgraph Γ of $F\setminus[x,y]\cup[z,t]\cup H\setminus\{z,t\}$ can be a subgraph of two graphs $F\cup H$ and $(F\cup\{z,t\}\setminus[x,y])\cup H\setminus\{z,t\}$,
- the sesquivalent spanning subgraph Γ of $F\setminus\{x,y\}\cup[x,y]\cup H\setminus[z,t]$ can be a subgraph of two graphs $F\cup H$ and $F\setminus\{x,y\}\cup(H\cup\{x,y\}\setminus[z,t])$,
- the sesquivalent spanning subgraph Γ of $F\setminus\{x,y\}\cup[x,y]\cup[z,t]\cup H\setminus\{z,t\}$ can be a subgraph of two graphs $F\cup H$ and $F\setminus\{x,y\}\cup C_4\cup H\setminus\{x,y\}$.

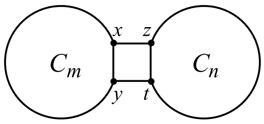
Therefore, using Theorem 2 and Lemma 1 we obtain

$$
\begin{aligned}
\det A(G) &= \det A(G\setminus\{[x,y],[z,t]\})+\det A(F\cup H)\\
&+\det A((F\cup\{z,t\}\setminus[x,y])\cup H\setminus\{z,t\})\\
&+\det A(F\setminus\{x,y\}\cup(H\cup\{x,y\}\setminus[z,t]))\\
&+\det A(F\setminus\{x,y\}\cup C_4\cup H\setminus\{z,t\})\\
&-\det A(F\setminus[x,y]\cup H\setminus[z,t])\\
&-3\cdot\det A(F\setminus\{x,y\}\cup[x,z]\cup[y,t]\cup H\setminus\{z,t\})\\
&-\det A(F\setminus[x,y]\cup[z,t]\cup H\setminus\{z,t\})\\
&-\det A(F\setminus\{x,y\}\cup[x,y]\cup H\setminus[z,t])\\
&-\det A(F\setminus\{x,y\}\cup[x,y]\cup[z,t]\cup H\setminus\{z,t\})\\
&=\det A(G\setminus\{[x,y],[z,t]\})+\det A(F)\cdot\det A(H)\\
&+\det A(F\cup\{z,t\}\setminus[x,y])\cdot\det A(H\setminus\{z,t\})\\
&+\det A(F\setminus\{x,y\})\cdot\det A(H\cup\{x,y\}\setminus[z,t])\\
&-\det A(F\setminus[x,y])\cdot\det A(H\setminus[z,t])
\end{aligned}
$$

$$
\begin{aligned}
&-4 \cdot \det A(F \setminus \{x,y\}) \cdot \det A(H \setminus \{z,t\}) \\
&+ \det A(F \setminus [x,y]) \cdot \det A(H \setminus \{z,t\}) \\
&+ \det A(F \setminus \{x,y\}) \cdot \det A(H \setminus [z,t]). \qquad \blacksquare
\end{aligned}
$$

5. Two Cycles Connected by Two Edges

In this part we will consider the graph $G = C_m \cup [x,z] \cup [y,t] \cup C_n$, where C_m and C_n, $m,n > 2$ are two cycles such that $\{x,y\} \in V(C_m)$, $\{z,t\} \in V(C_n)$ and $[x,y] \in E(C_m)$, $[z,t] \in E(C_n)$.



According to Theorem 5 we get

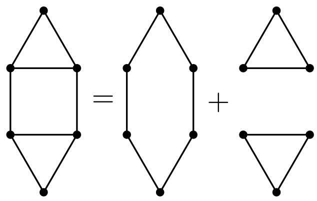
$\det A(G) = \det A(C_{m+n}) + \det A(C_m) \cdot \det A(C_n) + \det A(C_{m+2}) \cdot \det A(P_{n-2}) + \det A(P_{m-2}) \cdot \det A(C_{n+2}) - \det A(P_m) \cdot \det A(P_n) - 4 \cdot \det A(P_{m-2}) \cdot \det A(P_{n-2}) + \det A(P_m) \cdot \det A(P_{n-2}) + \det A(P_{m-2}) \cdot \det A(P_n)$.

It follows that

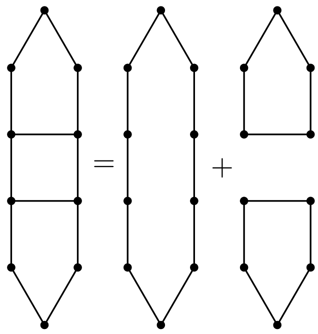
Corollary 6. *If m and n are odd, then*

$$\det A(G) = \det A(C_{m+n}) + 4 = \begin{cases} 0 & \text{if } m+n \equiv 2 \pmod 4, \\ 4 & \text{if } m+n \equiv 0 \pmod 4. \end{cases}$$

For example,



$= -4 + 2 \cdot 2 = 0.$



$= 0 + 2 \cdot 2 = 4.$

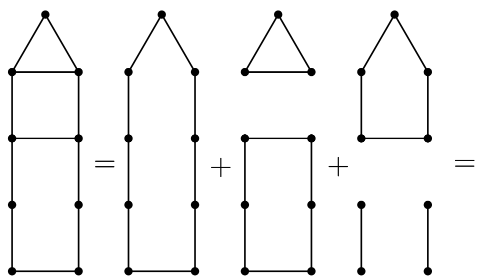
Now, let m be odd and n even. Then we have $\det A(G) = 2 + 2 \cdot \det A(C_n) + 2 \cdot \det A(P_{n-2})$.

Therefore, we get

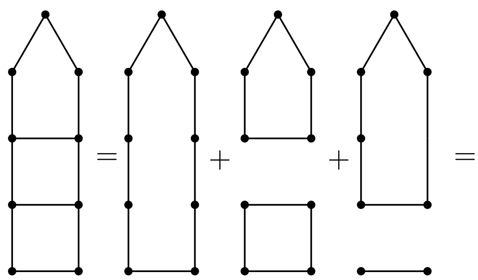
Corollary 7. *If m is odd, then*

$$\det A(G) = \begin{cases} -4 & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

For example,



$$= 2 + 2 \cdot (-4) + 2 \cdot (-1) \cdot (-1) = -4.$$



$$= 2 + 2 \cdot 0 + 2 \cdot (-1) = 0.$$

It remains for us to consider the case when m and n are even numbers. Then

$$\begin{aligned} \det A(G) &= \det A(C_{m+n}) + \det A(C_m) \cdot \det A(C_n) \\ &+ \det A(C_{m+2}) \cdot \det A(P_{n-2}) + \det A(P_{m-2}) \cdot \det A(C_{n+2}) \\ &- \det A(P_m) \cdot \det A(P_n) - 4 \cdot \det A(P_{m-2}) \cdot \det A(P_{n-2}) \\ &+ \det A(P_m) \cdot \det A(P_{n-2}) + \det A(P_{m-2}) \cdot \det A(P_n) \\ &= \det A(C_{m+n}) + \det A(C_m) \cdot \det A(C_n) + \det A(C_{m+2}) \cdot (-1)^{\frac{n-2}{2}} \\ &+ (-1)^{\frac{m-2}{2}} \cdot \det A(C_{n+2}) - (-1)^{\frac{m}{2}} \cdot (-1)^{\frac{n}{2}} \\ &- 4 \cdot (-1)^{\frac{m-2}{2}} \cdot (-1)^{\frac{n-2}{2}} + (-1)^{\frac{m}{2}} \cdot (-1)^{\frac{n-2}{2}} \\ &+ (-1)^{\frac{m-2}{2}} \cdot (-1)^{\frac{n}{2}} = \det A(C_{m+n}) + \det A(C_m) \cdot \det A(C_n) \\ &- (-1)^{\frac{n}{2}} \cdot \det A(C_{m+2}) - (-1)^{\frac{m}{2}} \cdot \det A(C_{n+2}) - 7 \cdot (-1)^{\frac{m+n}{2}}. \end{aligned}$$

We have three possibilities:

1. If $m, n \equiv 0 \,(\mathrm{mod}\ 4)$, then $\det A(G) = 0 + 0 \cdot 0 - 1 \cdot (-4) - 1 \cdot (-4) - 7 = 1$.

2. If $m, n \equiv 2 \,(\mathrm{mod}\ 4)$, then $\det A(G) = 0 + (-4)^2 - (-1) \cdot 0 - (-1) \cdot 0 - 7 = 9$.

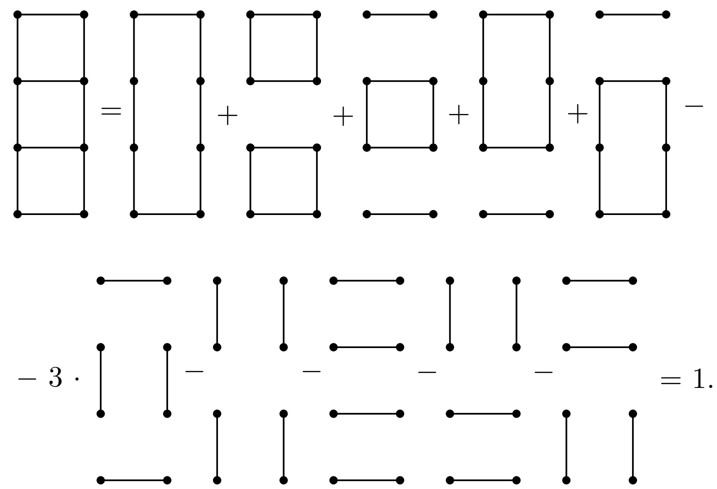
3. If $m \equiv 0 \,(\mathrm{mod}\ 4), n \equiv 2 \,(\mathrm{mod}\ 4)$, then $\det A(G) = -4 + 0 \cdot (-4) - (-1) \cdot (-4) - 1 \cdot 0 - 7 = -1$.

Hence

Corollary 8. *If m and n are even then*

$$\det A(G) = \begin{cases} 9 & \text{if} \quad m, n \equiv 2 \,(\mathrm{mod}\ 4), \\ 1 & \text{if} \quad m, n \equiv 0 \,(\mathrm{mod}\ 4), \\ -1 & \text{if}\ m \equiv 0 \,(\mathrm{mod}\ 4),\ n \equiv 2 \,(\mathrm{mod}\ 4). \end{cases}$$

For example



To recapitulate,

Theorem 6. *Let $G = C_m \cup [x, z] \cup [y, t] \cup C_n$, where C_m and C_n, $m, n > 2$ are two cycles such that $\{x, y\} \subseteq V(C_m)$, $\{z, t\} \subseteq V(C_n)$ and $[x, y] \in E(C_m)$, $[z, t] \in E(C_n)$. The determinant of adjacency matrix of the graph is given by,*

$$\det A(G) = \begin{cases} 9 & if \quad m \equiv 2 \,(\mathrm{mod}\ 4), n \equiv 2 \,(\mathrm{mod}\ 4), \\ 4 & if \quad m + n \equiv 0 \,(\mathrm{mod}\ 4)), m, n \text{ are odd}, \\ 1 & if \quad m \equiv 0 \,(\mathrm{mod}\ 4), n \equiv 0 \,(\mathrm{mod}\ 4), \\ 0 & if \quad m + n \equiv 2 \,(\mathrm{mod}\ 4)), m, n \text{ are odd} \\ & \qquad \text{or } n \equiv 0 \,(\mathrm{mod}\ 4), m \text{ is odd}, \\ -1 & if \quad m \equiv 0 \,(\mathrm{mod}\ 4), n \equiv 2 \,(\mathrm{mod}\ 4), \\ -4 & if \quad n \equiv 2 \,(\mathrm{mod}\ 4), m \text{ is odd}. \end{cases}$$

Comparing the above Theorem 6 with Theorem 4, we easily notice that we get the same formula (for computing the determinant) with the opposite sign. Thus

Observation 9. *The determinant of the adjacency matrix of two cycles with a common edge and the determinant of the adjacency matrix of two cycles connected by two edges differ only with the sign* ± 1.

References

[1] A. Abdollahi, *Determinants of adjacency matrices of graphs*, Trans. Combin. **1(4)** (2012) 9–16.

[2] F. Harary, *The Determinant of the adjacency matrix of a graph*, SIAM Rev. **4** (1961) 202–210.
doi:10.1137/1004057

[3] L. Huang and W. Yan, *On the determinant of the adjacency matrix of a type of plane bipartite graphs*, MATCH Commun. Math. Comput. Chem. **68** (2012) 931–938.

[4] H.M. Rara, *Reduction procedures for calculating the determinant of the adjacency matrix of some graphs and the singularity of square planar grids*, Discrete Math. **151** (1996) 213–219.
doi:10.1016/0012-365X(94)00098-4

[5] P. Wojtylak and S. Arworn, *Paths of cycles and cycles of cycles*, (2010) preprint.

Received 30 October 2012
Revised 8 May 2013
Accepted 8 May 2013