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ON EULERIAN IRREGULARITY IN GRAPHS

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Abstract

A closed walk in a connected graph G that contains every edge of G exactly once is an Eulerian circuit. A graph is Eulerian if it contains an Eulerian circuit. It is well known that a connected graph G is Eulerian if and only if every vertex of G is even. An Eulerian walk in a connected graph G is a closed walk that contains every edge of G at least once, while an irregular Eulerian walk in G is an Eulerian walk that encounters no two edges of G the same number of times. The minimum length of an irregular Eulerian walk in G is called the Eulerian irregularity of G and is denoted by $EI(G)$. It is known that if G is a nontrivial connected graph of size m, then $\binom{m+1}{2} \leq EI(G) \leq 2\binom{m+1}{2}$. A necessary and sufficient condition has been established for all pairs k, m of positive integers for which there is a nontrivial connected graph G of size m with $EI(G) = k$. A subgraph F in a graph G is an even subgraph of G if every vertex of F is even. We present a formula for the Eulerian irregularity of a graph in terms of the size of certain even subgraph of the graph. Furthermore, Eulerian irregularities are determined for all graphs of cycle rank 2 and all complete bipartite graphs.

Keywords: Eulerian walks, Eulerian irregularity.

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1. Introduction

A closed walk in a nontrivial connected graph or multigraph G that contains every edge of G exactly once is an *Eulerian circuit*. A graph or multigraph is

Eulerian if it contains an Eulerian circuit. It is well known [3] that a connected graph (or multigraph) G is Eulerian if and only if every vertex of G is even. In [1], an *Eulerian walk* in a connected graph G is defined as a closed walk that contains every edge of G at least once. If every edge of a nontrivial connected graph G is replaced by two parallel edges, then the resulting multigraph is Eulerian, which implies that G contains a closed walk in which every edge of G appears exactly twice. Hence if G is not Eulerian, then the minimum length of an Eulerian walk in G is more than m (the size of G) but not more than $2m$ and every edge appears once or twice in such an Eulerian walk in G. This problem is directly related to a well-known problem called the Chinese Postman Problem named by Alan Goldman for the Chinese mathematician Meigu Guan (often known as Mei-Ko Kwan) who introduced this problem in 1960 [4].

The Chinese Postman Problem. *Suppose that a postman starts from the post office and has mail to deliver to the houses along each street on his mail route. Once he has completed delivering the mail, he returns to the post office. Determine the minimum length of a round trip that accomplishes this.*

While the Chinese Postman Problem asks for the minimum length of a closed walk in a connected graph G such that every edge of G appears on the walk once or twice, another problem of interest is that of determining the minimum length of a closed walk in G in which no two edges of G appear the same number of times. Such walks in a graph G distinguish the edges of G by their occurrences on the walk. This gives rise to the concept of irregular Eulerian walks in graphs, which were introduced and studied in [1].

An *irregular Eulerian walk* in a nontrivial connected graph G is an Eulerian walk that encounters no two edges of G the same number of times. The minimum length of an irregular Eulerian walk in G is defined as the *Eulerian irregularity* of G and is denoted by $EI(G)$. If the size of G is m, then the length of an irregular Eulerian walk in G is at least $1 + 2 + \cdots + m = \binom{m+1}{2}$. Furthermore, if $E(G) = \{e_1, e_2, \ldots, e_m\}$ and each edge e_i $(1 \le i \le m)$ of G is replaced by $2i$ parallel edges, then the resulting multigraph M is Eulerian and each Eulerian circuit in M gives rise to an irregular Eulerian walk in which each edge e_i of G appears exactly $2i$ times in the walk. Thus G contains an irregular Eulerian walk of length $2 + 4 + 6 + \cdots + 2m = 2\binom{m+1}{2}$. The length of a walk W is denoted by $L(W)$. If W is an irregular Eulerian walk of minimum length in a connected graph G of size m, then $\binom{m+1}{2} \le L(W) \le 2\binom{m+1}{2}$. A problem of interest here is that of determining the minimum length of an irregular Eulerian walk in G, which is defined in [1] as the *Eulerian irregularity* of G and is denoted by $EI(G)$. Therefore, if G is a connected graph of size m, then

$$\binom{m+1}{2} \le EI(G) \le 2\binom{m+1}{2}. \tag{1}$$

Both upper and lower bounds in (1) are sharp. In fact, all nontrivial connected graphs of size m having Eulerian irregularity $\binom{m+1}{2}$ and $2\binom{m+1}{2}$ have been characterized in [1]. A subgraph F in a graph G is an *even subgraph* of G if every vertex of F is even.

Theorem 1.1 [1]**.** *If G is a nontrivial connected graph of size m, then*
(i) $EI(G) = 2\binom{m+1}{2}$ *if and only if G is a tree.*
(ii) $EI(G) = \binom{m+1}{2}$ *if and only if G contains an even subgraph of size $\lceil m/2 \rceil$.*

In this work, we continue the study of irregular walks in graphs. In Section 2, we provide a necessary and sufficient condition for all pairs k, m of positive integers with $\binom{m+1}{2} \le k \le 2\binom{m+1}{2}$ for which there is a nontrivial connected graph G of size m with $EI(G) = k$ and then establish a formula for the Eulerian irregularity of a connected graph G in terms of the size of G and the size of certain even subgraphs in G. In Sections 3 and 4, we determine the Eulerian irregularities of graphs of cycle rank 2 and complete bipartite graphs. We refer to the book [2] for graph-theoretical notation and terminology not described in this paper.

2. A Realization Result on Eulerian Irregularity

We have seen in (1) that if G is a nontrivial connected graph of size m, then $\binom{m+1}{2} \le EI(G) \le 2\binom{m+1}{2}$. This gives rise to the following question: For given positive integers k and m with $\binom{m+1}{2} \le k \le 2\binom{m+1}{2}$, is there a connected graph G of size m such that $EI(G) = k$? In this section, we present a necessary and sufficient condition for a pair k, m of positive integers such that there is a nontrivial connected graph G of size m with $EI(G) = k$. In order to do this, we first present some preliminary results.

A *weighted graph* is a graph in which each edge e is assigned a positive integer called the *weight* of the edge and denoted by $w(e)$. The *degree of a vertex v* in a weighted graph H is the sum of the weights of the edges incident with v and is denoted by $\deg_H v$ (or $\deg v$ if the weighted graph H under consideration is clear). A weighted graph H is *Eulerian* if H is connected and every vertex has even degree. For an Eulerian walk W of a connected graph G, let G_W be the weighted graph obtained from G by assigning to each edge uv of G the number of times uv is encountered on W. In this case, G_W is said to be *induced* by W. Consequently, the vertex set of G_W is $V(G)$ and every vertex in G_W has even degree. Thus, the weighted graph G_W induced by an Eulerian walk W in G is Eulerian. Furthermore, for an Eulerian walk W of a connected graph G, let M be the multigraph obtained from G by replacing each edge uv of G by the number of parallel edges equal to the number of times uv is encountered on W. In this case, M is said to be *induced* by W. Consequently, M is an Eulerian multigraph whose vertex set is $V(G)$.

For a connected graph G of size m with edge set $\{e_1, e_2, \ldots, e_m\}$ and an Eulerian walk W, let a_i be the number of times that e_i is encountered in W for $1 \leq i \leq m$. If W is an Eulerian walk of minimum length, then $a_i \in \{1, 2\}$, while if W is an irregular Eulerian walk of minimum length, then $a_i \in \{1, 2, \ldots, 2m\}$ and $a_i \neq a_j$ for all i, j with $1 \leq i \neq j \leq m$. In general, a multiset $S = \{a_1, a_2, \ldots, a_m\}$ of positive integers is *Eulerian realizable* if there is a connected graph G of size m, an ordering $e_1, e_2, \ldots, e_m$ of the edges of G and an Eulerian walk W in G such that e_i is encountered exactly a_i times in W for $1 \leq i \leq m$. We now present a necessary and sufficient conditions for a multiset S of $m \geq 3$ positive integers to be Eulerian realizable.

Theorem 2.1. *For an integer $m \geq 3$, a multiset $S = \{a_1, a_2, \ldots, a_m\}$ of positive integers is Eulerian realizable if and only if either*

(i) *no element in S is odd or*

(ii) *at least three elements in S are odd.*

Proof. First, suppose that exactly one or exactly two elements of S are odd. For any connected graph F of size m and any ordering $f_1, f_2, \ldots, f_m$ of the edges of F, let H be the weighted graph obtained from F by assigning the weight a_i to f_i for $1 \leq i \leq m$. Since either exactly one edge of F is assigned an odd weight or exactly two edges of F are assigned odd weights, it follows that H must have at least two vertices of odd degree. Hence F cannot have a closed walk in which f_i is encountered a_i times for $i = 1, 2, \ldots, m$.

To verify the converse, first suppose that no element in S is odd. Let G be any connected graph of size m with $E(G) = \{e_1, e_2, \ldots, e_m\}$ and let H be the weighted graph obtained by assigning the weight a_i to e_i for $1 \leq i \leq m$. Since every element in S is even, each vertex of H has even degree and so G has a closed walk in which e_i is encountered a_i times for $i = 1, 2, \ldots, m$. Next, suppose that exactly $k \geq 3$ elements in S are odd. If $k = m$, then let $G = C_m$; while if $k < m$, then let G be the graph obtained from the k-cycle C_k of order k and the path P_{m-k} of order $m - k$ by joining an end-vertex of P_{m-k} to a vertex of C_k. Then the size of G is m. Let H be the weighted graph obtained by assigning the k odd weights to the k edges of C_k and the $m - k$ even weights to the remaining $m - k$ edges of G. Then each vertex of H is even and so there is an ordering $e_1, e_2, \ldots, e_m$ of edges of G and a closed walk W in G such that e_i is encountered a_i times in W for $i = 1, 2, \ldots, m$. ■

In the problem of finding an Eulerian walk W of minimum length in G, we minimize the number of edges that are encountered exactly twice in W. In the problem of finding an irregular Eulerian walk W of minimum length in G, we have a different situation. For an Eulerian walk W in G, let $m_1 = m_1(W)$ be the number of edges that are encountered exactly once in W and $m_2 = m_2(W)$ the number of edges that are encountered exactly twice in W, where then $m = m_1 + m_2$. Let $e_1, e_2, \ldots, e_{m_1}$ be those edges occurring exactly once on W and let $f_1, f_2, \ldots, f_{m_2}$ be those edges occurring exactly twice on W. We construct an Eulerian multigraph M by replacing each edge e_i $(1 \le i \le m_1)$ by $2i - 1$ parallel edges and replacing each edge f_j $(1 \le j \le m_2)$ by $2j$ parallel edges. An Eulerian circuit in M gives rise to an irregular Eulerian walk W^* in G such that e_i $(1 \le i \le m_1)$ appears exactly $2i-1$ times in W^* and f_j $(1 \le j \le m_2)$ appears exactly $2j$ times in W^*. Thus, the length of W^* is $[1 + 3 + \cdots + (2m_1 - 1)] + [2 + 4 + \cdots + 2m_2] = m_1^2 + m_2(m_2 + 1)$ where $m = m_1 + m_2$. Therefore, in the problem of finding an irregular Eulerian walk of minimum length in G, we investigate those connected graphs G that minimize $|m_1(W) - m_2(W)|$ over all Eulerian walks W in G. In the view of this observation, we present the following lemma.

Lemma 2.2. *Let G be a nontrivial connected graph of size m. If G contains an even subgraph F of size x, then there is an irregular Eulerian walk of length $x^2 + (m - x)(m - x + 1)$ in G and so $EI(G) \le x^2 + (m - x)(m - x + 1)$.*

Proof. Let F be an even subgraph of size x in G and let

$$E(G) = \{e_1, e_2, \ldots, e_x\} \cup \{f_1, f_2, \ldots, f_{m-x}\},$$

where $E(F) = \{e_1, e_2, \ldots, e_x\}$.

We construct an Eulerian multigraph M by replacing each edge e_i where $1 \le i \le x$ by $2i-1$ parallel edges and replacing each edge f_j where $1 \le j \le m-x$ by $2j$ parallel edges. An Eulerian circuit in M gives rise to an irregular Eulerian walk W in G such that each edge e_i of G appears exactly $2i - 1$ times in W where $1 \le i \le x$ and each edge f_j of G appears exactly $2j$ times in W where $1 \le j \le m - x$. Then the length of W is $x^2 + (m - x)(m - x + 1)$ and so $EI(G) \le L(W) = x^2 + (m - x)(m - x + 1)$. ∎

With the aid of Lemma 2.2, we determine the Eulerian irregularity of a special class of connected graphs. A graph G is *unicyclic* if G is connected and contains exactly one cycle. The next result provides the Eulerian irregularity of a unicyclic graph in terms of its size and the size of its unique cycle.

Proposition 2.3. *If G is a unicyclic graph of size $m \ge 3$ and the unique cycle in G is a k-cycle for some integer $k \ge 3$, then*

$$EI(G) = k^2 + (m - k)(m - k + 1).$$

In particular, if $G = C_n$, then $EI(C_n) = n^2$.

Proof. Since G contains an even subgraph of size k, namely C_k, it follows by Lemma 2.2 that $EI(G) \leq k^2+(m-k)(m-k+1)$. Now, let $C_k = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1)$ be the unique cycle in G and let $E(G) - E(C_k) = \{f_1, f_2, \ldots, f_{m-k}\}$. Let W be an irregular Eulerian walk of minimum length in G. Since each edge $f_j \in E(G) - E(C_k)$ is a bridge in G for $1 \leq j \leq m-k$, it follows that f_j must be encountered an even number of times on W. Furthermore, either every edge on C_k is encountered an odd number of times on W or every edge on C_k is encountered an even number of times on W. Thus

$$\begin{aligned} EI(G) &= L(W) \geq [1+3+\cdots+(2k-1)] + [2+4+\cdots+2(m-k)] \\ &= k^2 + (m-k)(m-k+1), \end{aligned}$$

giving the desired result. In particular, if $G = C_n$, then an irregular Eulerian walk of minimum length encounters each edge of C_n an odd number of times and so $EI(C_n) = 1+3+\cdots+(2n-1) = n^2$. ■

If W is an irregular Eulerian walk of minimum length in a nontrivial connected graph G, then the set of occurrences of edges of G in W satisfies certain conditions, which are described in the next result.

Lemma 2.4. *Let G be a nontrivial connected graph G of size m and let W be an irregular Eulerian walk of minimum length in G. If there are x edges of G that are encountered an odd number of times in W and there are $m-x$ edges of G that are encountered an even number of times in W, then the numbers of times of the edges of G encountered in W are* $1, 3, \ldots, 2x-1, 2, 4, \ldots, 2(m-x)$ *and so* $EI(G) = x^2 + (m-x)(m-x+1)$.

Proof. Let W be an irregular Eulerian walk of minimum length in G, where then $L(W) = EI(G)$. For each edge e of G, let $w(e)$ be the number of times that e is encountered in W. Let $\{e_1, e_2, \ldots, e_x\}$ be the set of edges of G that are encountered an odd number of times in W and $\{f_1, f_2, \ldots, f_y\}$ the set of edges that are encountered an even number of times in W, where $y = m-x$. We may assume that $w(e_1) < w(e_2) < \cdots < w(e_x)$ and $w(f_1) < w(f_2) < \cdots < w(f_y)$. Thus $w(e_i) \geq 2i-1$ for $1 \leq i \leq x$ and $w(f_j) \geq 2j$ for $1 \leq j \leq y$, which implies that $L(W) \geq x^2 + y(y+1)$. Now consider the Eulerian multigraph M obtained from G by replacing each edge e_i $(1 \leq i \leq x)$ by $2i-1$ parallel edges and each edge f_j $(1 \leq j \leq y)$ by $2j$ parallel edges. An Eulerian circuit in M gives rise to an irregular Eulerian walk W^* in G such that e_i $(1 \leq i \leq x)$ appears exactly $2i-1$ times in W^* and f_j $(1 \leq j \leq y)$ appears exactly $2j$ times in W^*. Thus, the length of $W^* = x^2 + y(y+1)$. Since W is an irregular Eulerian walk of minimum length, $L(W) \leq L(W^*) = x^2 + y(y+1)$. Therefore, $L(W) = x^2 + y(y+1)$ and so $w(e_i) = 2i-1$ for $1 \leq i \leq x$ and $w(f_j) = 2j$ for $1 \leq j \leq y$. Therefore, the numbers of times of the edges of G encountered in W are $1, 3, \ldots, 2x-1, 2, 4, \ldots, 2(m-x)$ and so $EI(G) = x^2 + (m-x)(m-x+1)$. ■

We are now prepared to present the following realization result.

Theorem 2.5. *Let k and m be positive integers with $\binom{m+1}{2} \leq k \leq 2\binom{m+1}{2}$. Then there exists a nontrivial connected graph G of size m with $EI(G) = k$ if and only if there exists integer x with $0 \leq x \leq m$ and $x \neq 1, 2$ such that $x^2 + (m-x)(m-x+1) = k$.*

Proof. First, suppose that G is a nontrivial connected graph of size m such that $EI(G) = k$. Let W be an irregular Eulerian walk of length $EI(G)$ in G. Suppose that there are $x \geq 0$ edges of G that are encountered an odd number of times in W and $m - x$ edges that are encountered an even number of times in W. It then follows by Lemma 2.4 that $L(W) = x^2 + (m-x)(m-x+1)$. Furthermore, $x \neq 1, 2$ by Theorem 2.1.

For the converse, let k and m be positive integers with $\binom{m+1}{2} \leq k \leq 2\binom{m+1}{2}$ and let x be an integer such that $0 \leq x \leq m$, $x \neq 1, 2$, and $x^2+(m-x)(m-x+1) = k$. By Theorem 1.1, we may assume that $\binom{m+1}{2} < k < 2\binom{m+1}{2}$. Thus $x > 0$ and so $x \geq 3$. Let G be a unicyclic graph of size m that contains the cycle C_x of order x. It then follows by Proposition 2.3 that $EI(G) = x^2 + (m-x)(m-x+1) = k$. ■

By Theorem 2.5, a pair k, m of positive integers with $\binom{m+1}{2} \leq k \leq 2\binom{m+1}{2}$ can be realized as the Eulerian irregularity and the size of some nontrivial connected graph if and only if there exists an integer x with $0 \leq x \leq m$ and $x \neq 1, 2$ such that $x^2 + (m-x)(m-x+1) = k$. To determine the possible values of such integers x, we consider the real-valued function

$$L(x) = x^2 + (m-x)(m-x+1) = 2x^2 - (2m+1)x + m^2 + m. \tag{2}$$

Since $L(x)$ is a concave-up parabola which has the minimum value at $x_0 = \frac{2m+1}{4}$, it follows that the closer x is to x_0, the closer $L(x)$ is to $L(x_0)$. For a positive integer m, let $[0..m]$ be the set of all integers x with $0 \leq x \leq m$. We list the elements of $[0..m]$ as an ordered sequence s of length $m + 1$ where

$$s = (x_1, x_2, \ldots, x_{m+1}) \tag{3}$$

such that

$$L(x_1) \leq L(x_2) \leq \cdots \leq L(x_{m+1}), \tag{4}$$

where then $L(x_1) = \binom{m+1}{2}$, $L(x_2) = \binom{m+1}{2}+1$, $L(x_3) = \binom{m+1}{2}+3, \ldots, L(x_{m+1}) = 2\binom{m+1}{2}$. The sequence s in (3) that satisfies (2) and (4) is referred to as the *Eulerian irregular sequence of* m. We now state an useful observation on Eulerian irregular sequences.

Observation 2.6. *Let m be a positive integer.*

(i) *If m is even, then the Eulerian irregular sequence of m is*

$$\begin{aligned}(5)\qquad &\Big(\Big\lceil\frac{m}{2}\Big\rceil, \Big\lceil\frac{m}{2}\Big\rceil+1, \Big\lceil\frac{m}{2}\Big\rceil-1, \Big\lceil\frac{m}{2}\Big\rceil+2, \Big\lceil\frac{m}{2}\Big\rceil-2, \ldots,\\ &\Big\lceil\frac{m}{2}\Big\rceil+\Big(\Big\lfloor\frac{m}{2}\Big\rfloor-1\Big), \Big\lceil\frac{m}{2}\Big\rceil-\Big(\Big\lfloor\frac{m}{2}\Big\rfloor-1\Big), m, 0\Big);\end{aligned}$$

(ii) *If m is odd, then the Eulerian irregular sequence of m is*

$$\begin{aligned}(6)\qquad &\Big(\Big\lceil\frac{m}{2}\Big\rceil, \Big\lceil\frac{m}{2}\Big\rceil-1, \Big\lceil\frac{m}{2}\Big\rceil+1, \Big\lceil\frac{m}{2}\Big\rceil-2, \Big\lceil\frac{m}{2}\Big\rceil+2, \ldots,\\ &\Big\lceil\frac{m}{2}\Big\rceil-\Big\lfloor\frac{m}{2}\Big\rfloor, \Big\lceil\frac{m}{2}\Big\rceil+\Big\lfloor\frac{m}{2}\Big\rfloor = m, 0\Big).\end{aligned}$$

The following theorem [4] will be useful to us in the proof of the next result.

Kwan's Theorem. *Let G be a connected graph and let W be a closed walk of minimum length containing every edge of G at least once. Then W encounters no edge of G more than twice and no more than half of the edges in any cycle appear twice.*

We next present a formula for the Eulerian irregularity $EI(G)$ of a graph G in terms of the size of G and the size of a certain even subgraph of G.

Theorem 2.7. *Let G be a nontrivial connected graph of size m and $(x_1, x_2, \ldots, x_{m+1})$ the Eulerian irregular sequence of m. If*

$$\alpha = \min\{i : G \text{ contains an even subgraph } F \text{ of size } x_i,\ 1 \le i \le m+1\},$$

then $EI(G) = x_\alpha^2 + (m - x_\alpha)(m - x_\alpha + 1)$.

Proof. By Theorem 1.1, we may assume that G is not a tree. Since G contains an even subgraph of size x_α, it follows by Lemma 2.2 that $EI(G) \le x_\alpha^2 + (m - x_\alpha)(m - x_\alpha + 1)$. Let W be an irregular Eulerian walk of length $EI(G)$ in G. Let E' be the set of edges of G that are encountered an odd number of times in W and let E'' be the set of edges of G that are encountered an even number of times in W. Since G is not a tree, it follows by Kwan's Theorem that $E' \ne \emptyset$. Let F' be the subgraph induced by E' and F'' the subgraph induced by E''. We claim that every vertex of F' is even. Let M be the weighted graph obtained by assigning the weight $w(e)$ to each edge e of G, where $w(e)$ is the number of times that e is encountered in W. Let H' be the weighted subgraph of M induced by the edges of F' and let H'' be the weighted subgraph of M induced by the edges of F''. Since G has an Eulerian walk in which each edge e appears exactly $w(e)$ times, every vertex of M has even degree. Since $\deg_M v = \deg_{H'} v + \deg_{H''} v$ for every vertex v of G and $\deg_M v$ and $\deg_{H''}$ are both even, it follows that $\deg_{H'} v$ is even.

Suppose that $\deg_{F'} v = k$. Then v is incident with k edges in G, each of odd weight. Since $\deg_{H'} v$ is even, k is even and so v is an even vertex in F'. Therefore, F' is an even subgraph. Suppose that the size of F' is x, where then $1 \leq x \leq m$. It then follows by Lemma 2.4 that $EI(G) = L(W) = x^2 + (m - x)(m - x + 1)$. By the defining property of x_α and Observation 2.6, it follows that $x = x_\alpha$ and so $EI(G) = x_\alpha^2 + (m - x_\alpha)(m - x_\alpha + 1)$. ■

3. Eulerian Irregularities of Graphs of Cycle Rank 2

For a connected graph G of order n and size m, the number of edges that must be deleted from G to obtain a spanning tree of G is $m - n + 1$. The number $m - n + 1$ is called the *cycle rank* of G. Thus the cycle rank of a tree is 0 and the cycle rank of a *unicyclic* graph is 1. The cycle rank of a connected graph of order n and size $m = n + 1$ is therefore 2. In this section, we study the Eulerian irregularity of graphs of cycle rank 2.

Let G be a connected graph of order $n \geq 5$ and cycle rank 2. Then G contains one of the following three graphs of Figure 1 as a subgraph. If G contains two edge-disjoint cycles, then we say that G *is of type I*; otherwise, G is *is of type II*. If G is of type I, then G contains a subgraph H_1 obtained from two edge-disjoint cycles C_{k_1} and C_{k_2} by either identifying a vertex of C_{k_1} with a vertex of C_{k_2} or by connecting a vertex of C_{k_1} and a vertex of C_{k_2} by a path as shown in Figure 1(a) and (b). In this case, H_1 is called a (k_1, k_2)-*subgraph* of G. If G is of type II, then G contains a subgraph H_2 obtained from three internally disjoint $u - v$ paths $P_{k_1+1}, P_{k_2+1}, P_{k_3+1}$ of lengths k_1, k_2, k_3, respectively, as shown in Figure 1(c). In this case, H_2 is called a (k_1, k_2, k_3)-*subgraph* of G.

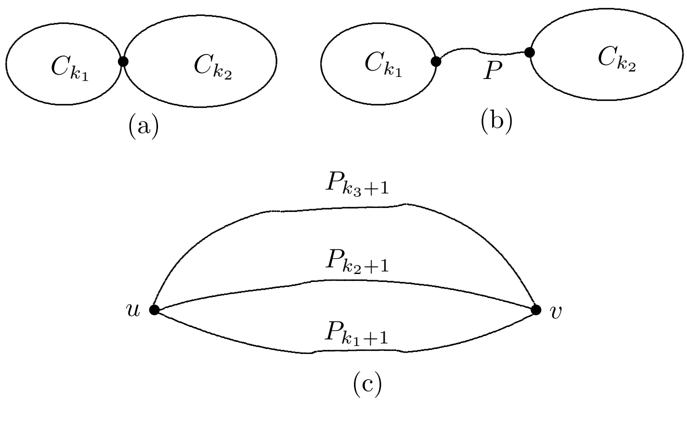

Figure 1. Subgraphs in a graph of cycle rank 2.

Theorem 3.1. *Let G be a graph of order $n \geq 5$, size m and cycle rank 2. If G*

is of type I and contains a (k_1, k_2)-subgraph, where $3 \le k_1 \le k_2$, then

$$EI(G) = \begin{cases} (k_1+k_2)^2 + 2\binom{m-k_1-k_2+1}{2} & \text{if } m-(k_1+k_2) \ge k_2, \\ k_2^2 + 2\binom{m-k_2+1}{2} & \text{if } m-(k_1+k_2) < k_2. \end{cases}$$

Proof. Since G is of cycle rank 2, $m = n+1$. First, we make an observation. Since each bridge of G is encountered an even number of times in an irregular Eulerian walk W, it follows that either all edges on C_{k_1} are encountered an odd number of times in W or all edges on C_{k_1} are encountered an even number of times in W. Similarly, this is the case for all edges on C_{k_2}. Divide the edge set $E(G)$ into three sets E_1, E_2 and E_3, where $E_i = E(C_{k_i})$ for $i = 1, 2$ and $E_3 = E(G) - (E_1 \cup E_2)$ is the set of all bridges of G. Thus $\{E_1, E_2, E_3\}$ is a partition of $E(G)$ if $E_3 \ne \emptyset$. Let W be an irregular Eulerian walk of minimum length in G. Now let $E'(W)$ be the set of edges of G that are encountered an odd number of times in W and $E''(W)$ the set of edges that are encountered an even number of times in W. As we indicated above, $E_3 \subseteq E''(W)$. Since W is an irregular Eulerian walk of minimum length in G, we may assume that $E'(W) = \{e_1, e_2, \ldots, e_a\}$ and $E''(W) = \{f_1, f_2, \ldots, f_b\}$ for some nonnegative integers a and b such that e_i $(1 \le i \le a)$ appears exactly $2i-1$ times in W and f_j $(1 \le j \le b)$ appears exactly $2j$ times in W. Therefore,

$$\begin{aligned} L(W) &= [1+3+\cdots+(2a-1)] + (2+4+\cdots+2b) \\ &= a^2 + b(b+1). \end{aligned}$$

Let $p = m - (k_1+k_2)$. We consider two cases.

Case 1. $p \ge k_2$. There are four possibilities for W, according to the sets $E'(W)$ and $E''(W)$.

- If $E'(W) = E_1 \cup E_2$ and $E''(W) = E_3$, then $L(W) = (k_1+k_2)^2 + p(p+1)$.
- If $E'(W) = E_1$ and $E''(W) = E_2 \cup E_3$, then, since $p \ge k_2 \ge k_1$,
$$\begin{aligned} L(W) &= k_1^2 + (k_2+p)(k_2+p+1) = k_1^2 + k_2^2 + 2pk_2 + p^2 + k_2 + p \\ &\ge k_1^2 + 2k_1k_2 + k_2^2 + p(p+1) = (k_1+k_2)^2 + p(p+1). \end{aligned}$$
- If $E'(W) = E_2$ and $E''(W) = E_1 \cup E_3$, then, since $p \ge k_2$,
$$\begin{aligned} L(W) &= k_2^2 + (k_1+p)(k_1+p+1) = k_1^2 + k_2^2 + 2pk_1 + p^2 + k_1 + p \\ &\ge k_1^2 + 2k_1k_2 + k_2^2 + p(p+1) = (k_1+k_2)^2 + p(p+1). \end{aligned}$$
- If $E''(W) = E_1 \cup E_2 \cup E_3$, then
$$L(W) = (k_1+k_2+p)(k_1+k_2+p+1) \ge (k_1+k_2)^2 + p(p+1).$$

Thus $L(W) = (k_1 + k_2)^2 + p(p + 1)$ is minimum when $E'(W) = E_1 \cup E_2$ and $E''(W) = E_3$, in which case, the difference between $|E'(W)|$ and $|E''(W)|$ in absolute value is the minimum. Therefore, $EI(G) = (k_1 + k_2)^2 + p(p + 1)$.

Case 2. $p < k_2$. Again, there are four possibilities for W, according to the sets $E'(W)$ and $E''(W)$.

- If $E'(W) = E_2$ and $E''(W) = E_1 \cup E_3$, then $L(W) = k_2^2 + (k_1 + p)(k_1 + p + 1)$.
- If $E'(W) = E_1$ and $E''(W) = E_2 \cup E_3$, then, since $k_2 \geq k_1$,
$$\begin{aligned} L(W) &= k_1^2 + (k_2 + p)(k_2 + p + 1) = k_1^2 + k_2^2 + 2pk_2 + p^2 + k_2 + p \\ &\geq k_2^2 + k_1^2 + 2pk_1 + p^2 + k_1 + p \\ &= k_2^2 + (k_1 + p)^2 + k_1 + p = k_2^2 + (k_1 + p)(k_1 + p + 1). \end{aligned}$$
- If $E'(W) = E_1 \cup E_2$ and $E''(W) = E_3$, then, since $k_1 \geq 3$ and so $2k_1 > k_1$,
$$\begin{aligned} L(W) &= (k_1 + k_2)^2 + p(p + 1) = k_1^2 + 2k_1k_2 + k_2^2 + p^2 + p \\ &\geq k_2^2 + k_1^2 + 2k_1(p + 1) + p^2 + p \quad (\text{since } p + 1 \leq k_2) \\ &= k_2^2 + (k_1 + p)^2 + 2k_1 + p \\ &> k_2^2 + (k_1 + p)^2 + k_1 + p = k_2^2 + (k_1 + p)(k_1 + p + 1). \end{aligned}$$
- If $E''(W) = E_1 \cup E_2 \cup E_3$, then
$$\begin{aligned} L(W) &= (k_1 + k_2 + p)(k_1 + k_2 + p + 1) \\ &= (k_1 + k_2 + p)^2 + (k_1 + k_2 + p) \\ &= (k_1 + k_2)^2 + 2(k_1 + k_2)p + p^2 + k_1 + k_2 + p \\ &\geq (k_1 + k_2)^2 + k_1 + p(p + 1) > k_2^2 + (k_1 + p)^2 + k_1 + p \\ &= k_2^2 + (k_1 + p)(k_1 + p + 1) \end{aligned}$$

Thus $L(W) = k_2^2 + (k_1 + p)(k_1 + p + 1)$ is minimum when $E'(W) = E_2$ and $E''(W) = E_1 \cup E_3$. Therefore, $EI(G) = k_2^2 + (k_1 + p)(k_1 + p + 1)$. ■

Theorem 3.2. *Let G be a graph of order $n \geq 4$, size m and cycle rank 2. Suppose that G is of type II and contains a (k_1, k_2, k_3)-subgraph, where $1 \leq k_1 \leq k_2 \leq k_3$. Let*

$$M = \min \left\{ \left| (k_i + k_j) - \left\lceil \frac{m}{2} \right\rceil \right| : i, j \in \{1, 2, 3\}, i \neq j \right\}. \tag{7}$$

(1) *If $M = 0$, then $EI(G) = \binom{m+1}{2}$;*

(2) *For $M \geq 1$,*

- *if there exist at least two distinct pairs (i, j) where $i, j \in \{1, 2, 3\}$ such that $\left|(k_i + k_j) - \lceil \frac{m}{2} \rceil\right| = M$ and $(k_i + k_j) - \lceil \frac{m}{2} \rceil$ are different in signs; that is, there are at lest two pairs (r, s) and (ℓ, t), where $r, s, \ell, t \in \{1, 2, 3\}$ and $(r, s) \neq (\ell, t)$, such that*
$$(k_r + k_s) - \lceil \tfrac{m}{2} \rceil = M \text{ and } \lceil \tfrac{m}{2} \rceil - (k_\ell + k_t) = M,$$

then

$$(8) \quad EI(G) = \begin{cases} (k_r+k_s)^2+(m-k_r-k_s)(m-k_r-k_s+1) & \text{if } m \text{ is even,} \\ (k_\ell+k_t)^2+(m-k_\ell-k_t)(m-k_\ell-k_t+1) & \text{if } m \text{ is odd.} \end{cases}$$

- *if for all pairs (i,j) where $i,j \in \{1,2,3\}$ such that $\left|(k_i+k_j) - \lceil \frac{m}{2} \rceil\right| = M$ either $k_i + k_j > \lceil \frac{m}{2} \rceil$ for all such pairs (i,j) or $k_i + k_j < \lceil \frac{m}{2} \rceil$ for all such pairs (i,j) and (r,s) is one of such pairs, then*

$$(9) \quad EI(G) = (k_r+k_s)^2 + (m-k_r-k_s)(m-k_r-k_s+1).$$

Proof. If $M = 0$, then there is an even subgraph $C_{k_i+k_j}$ of order $k_i + k_j$ of G, where $i,j \in \{1,2,3\}$. Since the size of $C_{k_i+k_j}$ is $k_i + k_j = \lceil \frac{m}{2} \rceil$, it follows that $EI(G) = \binom{m+1}{2}$.

Now let $M \geq 1$. First assume that there exist at least two distinct pairs (r,s) and (ℓ,t) such that $(k_r+k_s) - \lceil \frac{m}{2} \rceil = M$ and $\lceil \frac{m}{2} \rceil - (k_\ell + k_t) = M$. Thus $k_r + k_s = \lceil \frac{m}{2} \rceil + M$ and $k_\ell + k_t = \lceil \frac{m}{2} \rceil - M$. Let $H_1 = C_{k_r+k_s}$ and $H_2 = C_{k_\ell+k_t}$. Then H_1 and H_2 are even subgraphs of size $k_r + k_s$ and $k_\ell + k_t$, respectively. By Observation 2.6, it follows that (8) holds.

Next, suppose that for all pairs (i,j) such that $\left|(k_i+k_j) - \lceil \frac{m}{2} \rceil\right| = M$ either $k_i + k_j > \lceil \frac{m}{2} \rceil$ for all such pairs (i,j) or $k_i + k_j < \lceil \frac{m}{2} \rceil$ for all such pairs (i,j). Let (r,s) be one of pairs. Then $H = C_{k_r+k_s}$ is an even subgraph of size $k_r + k_s$ in G. By Observation 2.6, it follows that (9) holds. ■

4. EULERIAN IRREGULARITIES OF COMPLETE BIPARTITE GRAPHS

An irregular Eulerian walk in a graph G of size m is said to be *optimal* if its length is $\binom{m+1}{2}$. A graph G is *optimal* if it contains an optimal irregular Eulerian walk. All optimal complete graphs and complete bipartite graphs have been determined in [1], which we state next.

Theorem 4.1 [1]**.** *For each integer $n \geq 2$, the complete graph K_n is optimal if and only if $n \geq 4$.*

Theorem 4.2 [1]**.** *For integers r and s with $2 \leq r \leq s$, the complete bipartite graph $K_{r,s}$ is optimal if and only if*

(i) *r and s are both even and $(r,s) \neq (2, 4k+2)$ for any nonnegative integer k or*

(ii) *at least one of r and s is odd and $rs \not\equiv 1, 2 \pmod 4$.*

Since K_2 is a tree and K_3 is a cycle, it follows by Theorem 1.1, Proposition 2.3

and Theorem 4.1 that

$$EI(K_n) = \begin{cases} 2 & \text{if } n = 2, \\ 9 & \text{if } n = 3, \\ \binom{\binom{n}{2}+1}{2} & \text{if } n \geq 4. \end{cases}$$

We now determine the Eulerian irregularity of a complete bipartite graph.

Theorem 4.3. *If the complete bipartite graph $K_{r,s}$ is not optimal where $2 \leq r \leq s$, then*

$$EI(K_{r,s}) = \begin{cases} \binom{rs+1}{2} + 6 & \textit{if } r \textit{ and } s \textit{ are both even,} \\ \binom{rs+1}{2} + 1 & \textit{if at least one of } r \textit{ and } s \textit{ is odd.} \end{cases}$$

Proof. Suppose that $G = K_{r,s}$ is not optimal where $2 \leq r \leq s$. By Theorem 4.2, either

- r and s are both even and $(r, s) = (2, 4k + 2)$ for some $k \geq 0$ or
- at least one of r and s is odd and $rs = 1, 2 \pmod 4$.

Let $m = rs$ be the size of G.

First, suppose that r and s are both even and $(r, s) = (2, 4k + 2)$ for some $k \geq 0$. Then $m = rs = 8k + 4$ and so $\frac{m}{2} = 4k + 2$. Since G contains the even subgraph $H = K_{2,2k+2}$ of size $4k + 4 = \frac{m}{2} + 2$, it follows by Lemma 2.2 that

$$EI(G) \leq (4k + 4)^2 + (4k)(4k + 1) = \binom{rs+1}{2} + 6.$$

Since G contains neither even subgraph of odd size $4k + 3$ nor even subgraph of odd size $4k+1$, it follows by Theorem 2.5, Observation 2.6 and Theorem 2.7 that $EI(G) \geq \binom{rs+1}{2} + 6$ and so $EI(G) = \binom{rs+1}{2} + 6$.

Next, suppose that at least one of r and s is odd and $rs = 1, 2 \pmod 4$. Denote the partite sets of G by

$$U = \{u_1, u_2, \ldots, u_r\} \text{ and } W = \{w_1, w_2, \ldots, w_s\}.$$

We consider three cases, according to the parity of r and s.

Case 1. r is odd and s is even. Since $rs \equiv 2 \pmod 4$ and $r \leq s$, it follows that $r = 2a + 1$ and $s = 4b + 2$, where $a, b \geq 1$ and $a \leq 2b$. Since G is not optimal by Theorem 4.2 and $m = rs$ is even, it follows by Observation 2.6 that $EI(G) \geq L(\frac{m}{2} + 1) = \binom{m+1}{2} + 1$ where $L(x)$ is defined in (2) for an integer x. That is,

$$EI(G) \geq \left(\frac{m}{2} + 1\right)^2 + \left(\frac{m}{2} - 1\right)\frac{m}{2}.$$

By Lemma 2.2, it remains to show that G contains an even subgraph of size $\frac{m}{2}+1$. Observe that

$$\frac{m}{2}+1=(2a+1)(2b+1)+1=4ab+2a+2b+2.$$

We consider two subcases, according to whether $a+b$ is odd or $a+b$ is even.

Subcase 1.1. $a+b$ is odd. First, suppose that $a \leq b$. Then $3b+a+1 \leq 4b+2$ and $\frac{m}{2}+1=4ab+2(a+b+1)$. Let $F_1=K_{2a,2b}$ be the subgraph of G induced by $\{u_1,u_2,\ldots,u_{2a}\} \cup \{w_1,w_2,\ldots,w_{2b}\}$ and let $F_2=K_{2,a+b+1}$ the subgraph of G induced by $\{u_1,u_2\} \cup \{w_{2b+1},\ w_{2b+2},\ldots,w_{3b+a+1}\}$. Then let H be the even subgraph consisting of F_1 and F_2 whose vertex set is $V(F_1)\cup V(F_2)$ and whose edge set $E(F_1)\cup E(F_2)$. Then the size of H is $\frac{m}{2}+1$.

Next, suppose that $b<a\leq 2b$. If a is even and b is odd, then $\frac{m}{2}+1=a(4b+2)+2(b+1)$. Let $F_1=K_{a,4b+2}$ with partite sets $\{u_1,u_2,\ldots,u_a\}$ and W and let $F_2=K_{2,b+1}$ with partite sets $\{u_{a+1},\ u_{a+2}\}$ and $\{w_1,w_2,\ldots,w_{b+1}\}$. Then let H be the even subgraph consisting of F_1 and F_2. If a is odd and b is even, then $\frac{m}{2}+1=(a-1)(4b)+6b+2(a+1)$. Let $F_1'=K_{a-1,4b}$ with partite sets $\{u_1,u_2,\ldots,u_{a-1}\}$ and $\{w_1,w_2,\ldots,w_{4b}\}$, let $F_2'=K_{2,3b}$ with partite sets $\{u_a,u_{a+1}\}$ and $\{w_1,\ w_2,\ldots,w_{3b}\}$ and let $F_3'=K_{a+1,2}$ with partite sets $\{u_1,u_2,\ldots,u_{a+1}\}$ and $\{w_{4b+1},w_{4b+2}\}$. Then let H be the even subgraph consisting of F_1', F_2' and F_3' and the size of H is $\frac{m}{2}+1$.

Subcase 1.2. $a+b$ is even. Then a and b are of the same parity. First, suppose that a and b are both odd, say $a=2p+1$ and $b=2q+1$ for some integers $p,q\geq 0$. Then $\frac{m}{2}+1=(2a)(2b)+2(2q)+2(2p)+6$. If $b=1$, then $a=1$ (since $r\leq s$) and so $G=K_{3,6}$. The even subgraph of G consisting of $C_4=(u_1,w_1,u_2,w_2,u_1)$ and $C_6=(u_1,w_3,u_2,w_4,u_3,w_5,u_1)$ has size 10. Thus, we may assume that $b\geq 2$ and so $3b+4\leq 4b+2$. Let $F_1=K_{2a,2b}$ with partite sets $\{u_1,u_2,\ldots,u_{2a}\}$ and $\{w_1,w_2,\ldots,w_{2b}\}$, let $F_2=K_{2,2q}$ with partite sets $\{u_1,u_2\}$ and $\{w_{2b+1},\ w_{2b+2},\ldots,w_{3b-1}\}$, let $F_3=K_{2p,2}$ with partite sets $\{u_1,u_2,\ldots,u_{2p}\}$ and $\{w_{3b},w_{3b+1}\}$ and let $F_4=C_6=(u_1,w_{3b+2},u_2,w_{3b+3},u_3,w_{3b+4},u_1)$. Then let H be the even subgraph consisting of F_1, F_2, F_3 and F_4 and the size of H is $\frac{m}{2}+1$.

Next, suppose that a and b are both even, say $a=2p$ and $b=2q$ for some integers $p,q\geq 1$. Then $\frac{m}{2}+1=(2a)(2b)+2[2(p-1)+2q]+6$. If $b=2$, then $a=2$ or $a=4$. Thus $G=K_{5,10}$ or $G=K_{9,10}$. For $K_{5,10}$, it follows that $\frac{m}{2}+1=26$ and let the even subgraph of G be consisted of $K_{2,10}$ with partite sets $\{u_1,u_2\}$ and $\{w_1,w_2,\ldots,w_{10}\}$ and $C_6=(u_3,w_1,u_4,w_2,u_5,w_3,u_3)$. For $K_{9,10}$, it follows that $\frac{m}{2}+1=46$ and let the even subgraph of G be consist of $K_{4,10}$ with partite sets $\{u_1,u_2,u_3,u_4\}$ and $\{w_1,w_2,\ldots,w_{10}\}$ and $C_6=(u_5,w_1,u_6,w_2,u_7,w_3,u_5)$. In each case, G has an even subgraph of size $\frac{m}{2}+1$.

We now assume that $b\geq 4$ and so $3b+5\leq 4b+2$. Let $F_1=K_{2a,2b}$ with partite sets $\{u_1,u_2,\ldots,u_{2a}\}$ and $\{w_1,w_2,\ldots,w_{2b}\}$, let $F_2=K_{2(p-1),2}$ with partite sets

$\{u_1, u_2, \ldots, u_{2(p-1)}\}$ and $\{w_{2b+1}, w_{2b+2}\}$, let $F_3 = K_{2,2q}$ with partite sets $\{u_1, u_2\}$ and $\{w_{2b+3}, w_{2b+4}, \ldots, w_{3b+2}\}$ and let $F_4 = C_6 = (u_1, w_{3b+3}, u_2, w_{3b+4}, u_3, w_{3b+5}, u_1)$. Then let H be the even subgraph consisting of F_1, F_2, F_3 and F_4 and the size of H is $\frac{m}{2} + 1$.

Case 2. r is even and s is odd. In this case, the size $m = rs$ is even. Furthermore, $r = 4a + 2$ and $s = 2b + 1$, where $a \geq 0$ and $b \geq 1$. Then

$$\frac{m}{2} + 1 = (2a)(2b) + 2(a + b + 1).$$

Since $r \leq s$, it follows that $a < b$ and so $a + b + 1 \leq 2b$. First, suppose that $a + b$ is odd. Let $F_1 = K_{2a,2b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{2b}\}$ and let $F_2 = K_{2,a+b+1}$ with partite sets $\{u_{2a+1}, u_{2a+2}\}$ and $\{w_1, w_2, \ldots, w_{a+b+1}\}$. Then let H be the even subgraph consisting of F_1 and F_2 and the size of H is $\frac{m}{2} + 1$.

Next, suppose that $a + b$ is even. First, assume that a and b are both odd, say $a = 2p + 1$ and $b = 2q + 1$ for some integers $p, q \geq 0$. If $a = 1$, then $G = K_{6,2b+1}$, where $b \geq 3$, and $\frac{m}{2} + 1 = 4b + 2(b + 2) = 4b + 2(b - 1) + 6$. Let $F_1 = K_{4,b}$ with partite sets $\{u_1, u_2, u_3, u_4\}$ and $\{w_1, w_2, \ldots, w_b\}$, let $F_2 = K_{2,b-1}$ with partite sets $\{u_5, u_6\}$ and $\{w_1, w_2, \ldots, w_{b-1}\}$ and let $F_3 = C_6 = (u_1, w_{b+1}, u_2, w_{b+2}, u_3, w_{b+3}, u_1)$. Then let H be the even subgraph consisting of F_1, F_2 and F_3 and the size of H is $\frac{m}{2} + 1$. Thus, we may assume that $a \geq 3$ and so $3a + 4 \leq 4a + 2$. Then $\frac{m}{2} + 1 = (2a)(2b) + 2(2p) + 2(2q) + 6$. Let $F_1' = K_{2a,2b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{2b}\}$, let $F_2' = K_{2p,2}$ with partite sets $\{u_{2a+1}, u_{2a+2}, u_{3a-1}\}$ and $\{w_1,\ w_2\}$, let $F_3' = K_{2,2q}$ whose partite sets $\{u_{3a}, u_{3a+1}\}$ and $\{w_1, w_2, \ldots, w_{2q}\}$ and let $F_4' = C_6 = (u_{3a+2}, w_1, u_{3a+3}, w_2, u_{3a+4}, w_3, u_{3a+2})$. Then let H be the even subgraph consisting of F_1', F_2', F_3' and F_4' and the size of H is $\frac{m}{2} + 1$.

Next, suppose that a and b are both even, say $a = 2p$ and $b = 2q$ for some integers $p, q \geq 1$. Then $\frac{m}{2} + 1 = (2a)(2b) + 2[2(p - 1) + 2q] + 6$. Let $F_1 = K_{2a,2b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{2b}\}$, let $F_2 = K_{2(p-1),2}$ whose partite sets $\{u_{2a+1}, u_{2a+2}, \ldots, u_{3a-2}\}$ and $\{w_1, w_2\}$, let $F_3 = K_{2,2q}$ with partite sets $\{u_{3a-1}, u_{3a}\}$ and $\{w_1, w_2, \ldots, w_{2q}\}$ and let $F_4 = C_6 = (u_{3a+1}, w_1, u_{3a+2}, w_2, u_{3a+3}, w_3, u_{3a+1})$. Then let H be the even subgraph consisting of F_1, F_2, F_3 and F_4 and the size of H is $\frac{m}{2} + 1$.

Case 3. r and s are both odd. Since $m = rs$ is odd, it follows by Observation 2.6 that $EI(G) \geq L(\lceil \frac{m}{2} \rceil - 1) = \binom{m+1}{2} + 1$. By Lemma 2.2, it remains to show that G contains an even subgraph of size $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2}$. Since $rs \equiv 1 \pmod 4$, either $r, s \equiv 3 \pmod 4$ or $r, s \equiv 1 \pmod 4$. We consider these two subcases.

Subcase 3.1. $r, s \equiv 3 \pmod 4$. Then $r = 4a + 3$ and $s = 4b + 3$, where

$0 \le a \le b$. Thus

$$\left\lceil \frac{m}{2} \right\rceil - 1 = \frac{m-1}{2} = 8ab + 6a + 6b + 4 = (2a)(4b) + 2(3a + 3b) + 4.$$

First, suppose that a and b are both even. If $a = b = 0$, then $G = K_{3,3}$ and $\frac{m-1}{2} = 4$. Let $H = C_4 = (u_1, w_1, u_2, w_2, u_1)$. Thus, we now assume that $b \ge a \ge 2$. Let $F_1 = K_{2a,4b}$ with partite sets $\{u_1, u_2, \dots, u_{2a}\}$ and $\{w_1, w_2, \dots, w_{4b}\}$, let $F_2 = K_{3a,2}$ with partite sets $\{u_1, u_2, \dots, u_{3a}\}$ and $\{w_{4b+1}, w_{4b+2}\}$, let $F_3 = K_{2,3b}$ with partite sets $\{u_{3a+1}, u_{3a+2}\}$ and $\{w_1, w_2, \dots, w_{3b}\}$ and let $F_4 = C_4 = (u_{3a+3}, w_1, u_{3a+4}, w_2, u_{3a+3})$. Then let H be the even subgraph consisting of F_1, F_2, F_3 and F_4 and the size of H is $\frac{m-1}{2}$.

Next, suppose that exactly one of a and b is odd. First, assume that a is even and b is odd, where then $a \ge 0$ and $b \ge 1$. If $a = 0$, then $G = K_{3,4b+3}$ and $\frac{m-1}{2} = 6b + 4 = 6(b-1) + 10$. Let $F_1 = K_{2,3(b-1)}$ with partite sets $\{u_1, u_2\}$ and $\{w_1, w_2, \dots, w_{3b-3}\}$, let $F_2 = C_4 = (u_1, w_{3b-2}, u_2, w_{3b-1}, u_1)$ and $F_3 = C_6 = (u_1, w_{3b}, u_2, w_{3b+1}, u_3, w_{3b+2}, u_1)$. Then let H be the even subgraph consisting of F_1, F_2 and F_3 and the size of H is $\frac{m-1}{2}$. If $a = 2$, then $G = K_{11,4b+3}$ where $b \ge 3$ and $\frac{m-1}{2} = 4(4b) + 6(b+1) + 10$. Let $F_1' = K_{4,4b}$ with partite sets $\{u_1, u_2, u_3, u_4\}$ and $\{w_1, w_2, \dots, w_{4b}\}$, let $F_2' = K_{2,3(b+1)}$ with partite sets $\{u_5, u_6\}$ and $\{w_1, w_2, \dots, w_{3(b+1)}\}$, $F_3' = C_{10} = (u_7, w_1, u_8, w_2, u_9, w_3, u_{10}, w_4, u_{11}, w_5, u_7)$. Then let H be the even subgraph consisting of F_1', F_2' and F_3' and the size of H is $\frac{m-1}{2}$. We now assume $a \ge 4$ and $3a + 7 \le 4a + 3$. Then $\frac{m-1}{2} = (2a)(4b) + 2(3a) + 2[3(b-1)] + 10$. Let $F_1'' = K_{2a,4b}$ with partite sets $\{u_1, u_2, \dots, u_{2a}\}$ and $\{w_1, w_2, \dots, w_{4b}\}$, let $F_2'' = K_{3a,2}$ with partite sets $\{u_1, u_2, \dots, u_{3a}\}$ and $\{w_{4b+1}, w_{4b+2}\}$, let $F_3'' = K_{2,3(b-1)}$ with partite sets $\{u_{3a+1}, u_{3a+2}\}$ and $\{w_1, w_2, \dots, w_{3b-3}\}$ and let $F_4'' = C_{10}$ be a cycle of order 10 in the subgraph $K_{5,5}$ of G with partite sets $\{u_{3a+3}, u_{3a+4}, \dots, u_{3a+7}\}$ and $\{w_1, w_2, \dots, w_5\}$. Then let H be the even subgraph consisting of F_1'', F_2'', F_3'' and F_4'' and the size of H is $\frac{m-1}{2}$.

Next, assume that a is odd and b is even, where then $1 \le a < b$. If $a = 1$, then $G = K_{7,4b+3}$ and $\frac{m-1}{2} = 2(4b) + 2(3b+2) + 6$. Let $F_1 = K_{2,4b}$ with partite sets $\{u_1, u_2\}$ and $\{w_1, w_2, \dots, w_{4b}\}$, let $F_2 = K_{2,3b+2}$ with partite sets $\{u_3, u_4\}$ and $\{w_1, w_2, \dots, w_{3b+2}\}$, let $F_3 = C_6 = (u_5, w_1, u_6, w_2, u_7, w_3, u_5)$. Then let H be the even subgraph consisting of F_1, F_2 and F_3 and the size of H is $\frac{m-1}{2}$. Thus, we now assume that $a \ge 3$. Then $\frac{m-1}{2} = (2a)(4b) + 6(a-1) + 6b + 10$. Let $F_1' = K_{2a,4b}$ with partite sets $\{u_1, u_2, \dots, u_{2a}\}$ and $\{w_1, w_2, \dots, w_{4b}\}$, let $F_2' = K_{3(a-1),2}$ with partite sets $\{u_1, u_2, \dots, u_{3a-3}\}$ and $\{w_{4b+1}, w_{4b+2}\}$, let $F_3' = K_{2,3b}$ with partite sets $\{u_{3a-2}, u_{3a-1}\}$ and $\{w_1, w_2, \dots, w_{3b}\}$, let $F_4' = C_4 = (u_{3a}, w_1, u_{3a+1}, w_2, u_{3a})$ and let $F_5' = C_6 = (u_{4a+1}, w_1, u_{4a+2}, w_2, u_{4a+3}, w_3, u_{4a+1})$. Then let H be the even subgraph consisting of F_1', F_2', F_3', F_4' and F_5' and the size of H is $\frac{m-1}{2}$.

Final, suppose that a and b are both odd. Let $a = 2p + 1$ and $b = 2q + 1$ for some integers $p, q \ge 0$. Then $\frac{m-1}{2} = (2a)(4b) + 6(a-1) + 6(b-1) + 16$. Let $F_1 = K_{2a,4b}$ with partite sets $\{u_1, u_2, \dots, u_{2a}\}$ and $\{w_1, w_2, \dots, w_{4b}\}$, let

$F_2 = K_{3(a-1),2}$ with partite sets $\{u_1, u_2, \ldots, u_{3a-3}\}$ and $\{w_{4b+1}, w_{4b+2}\}$, let $F_3 = K_{2,3(b-1)}$ with partite sets $\{u_{3a-2}, u_{3a-1}\}$ and $\{w_1, w_2, \ldots, w_{3b-3}\}$ and let $F_4 = K_{4,4}$ with $\{u_{3a}, u_{3a+1}, u_{3a+2}, u_{3a+3}\}$ and $\{w_1, w_2, w_3, w_4\}$. Then let H be the even subgraph consisting of F_1, F_2, F_3 and F_4 and the size of H is $\frac{m-1}{2}$.

Subcase 3.2. $r, s \equiv 1 \pmod 4$. Then $r = 4a+1$ and $b = 4b+1$ where $1 \le a \le b$. Thus

$$\left\lceil \frac{m}{2} \right\rceil - 1 = \frac{m-1}{2} = 8ab + 2a + 2b.$$

First, suppose that $a+b$ is even. Then $\frac{m-1}{2} = (2a)(4b) + 2(a+b)$. Let $F_1 = K_{2a,4b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{4b}\}$ and let $F_2 = K_{2,a+b}$ with partite sets $\{u_{2a+1}, u_{2a+2}\}$ and $\{w_1, w_2, \ldots, w_{a+b}\}$. Then let H be the even subgraph consisting of F_1 and F_2 and the size of H is $\frac{m-1}{2}$.

Next, suppose that $a+b$ is odd and so $a+b \ge 3$. If $a+b = 3$, then $G = K_{5,9}$ and $\frac{m-1}{2} = 22$. Let $F_1 = K_{2,8}$ with partite sets $\{u_1, u_2\}$ and $\{w_1, w_2, \ldots, w_8\}$ and $F_2 = C_6 = (u_3, w_1, u_4, w_2, u_5, w_3, u_3)$. Then let H be the even subgraph of size 22 consisting of F_1 and F_2 and the size of H is $\frac{m-1}{2}$. Thus we may assume that $a+b \ge 5$. If $a = 1$, then $b \ge 4$ and $G = K_{5,4b+1}$. Now $\frac{m-1}{2} = 2(4b) + 2(b-2) + 6$. Let $F_1' = K_{2,4b}$ with partite sets $\{u_1, u_2\}$ and $\{w_1, w_2, \ldots, w_{4b}\}$, $F_2' = K_{2,b-2}$ with partite sets $\{u_3, u_4\}$ and $\{w_1, w_2, \ldots, w_{b-2}\}$ and $F_3' = C_6 = (u_3, w_{b-1}, u_4, w_b, u_5, w_{b+1}, u_3)$. Then let H be the even subgraph consisting of F_1', F_2' and F_3' and the size of H is $\frac{m-1}{2}$.

Now assume that $a \ge 2$. Then $\frac{m-1}{2} = (2a)(4b) + 2(a+b-3) + 6$. Let $F_1'' = K_{2a,4b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{4b}\}$, let $F_2'' = K_{2,a+b-3}$ with partite sets $\{u_{2a+1}, u_{2a+2}\}$ and $\{w_1, w_2, \ldots, w_{a+b-3}\}$ and let $F_3'' = C_6 = (u_{2a+3}, w_1, u_{2a+4}, w_2, u_{2a+5}, w_3, u_{2a+3})$. Then let H be the even subgraph consisting of F_1'', F_2'' and F_3'' and the size of H is $\frac{m-1}{2}$. ■

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