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FAMILIES OF TRIPLES WITH HIGH MINIMUM DEGREE ARE HAMILTONIAN

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Abstract

In this paper we show that every family of triples, that is, a 3-uniform hypergraph, with minimum degree at least $(\frac{5-\sqrt{5}}{3}+\gamma)\binom{n-1}{2}$ contains a tight Hamiltonian cycle.

Keywords: 3-uniform hypergraph, Hamilton cycle, minimum vertex degree.

2010 Mathematics Subject Classification: 05D05, 05C65.

1. Introduction

Recently there has been a lot of interest in Dirac-type properties of uniform hypergraphs. With this name we describe a general class of problems and results linking minimum degrees of k-uniform hypergraphs to the existence of a Hamilton cycle or a (near) perfect matching, see, e.g., [7, 16, 12, 18, 8, 10, 5, 2, 3], and [11, 17, 14, 19, 1, 21, 4], resp. See [15] for a survey on this subject.

In this paper we restrict ourselves to families of triples, that is, to 3-*uniform hypergraphs* $H = (V, E)$, where $V := V(H)$ is a finite set of vertices (usually,

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$|V| = n$) and $E := E(H) \subseteq \binom{V}{3}$ is a family of 3-element subsets of V, or *triples*. We will call such an H a *3-graph* for short. Whenever convenient we will identify H with $E(H)$. We write $\overline{H}$ for the complement of H, that is, $\overline{H} = \binom{V}{3} \setminus H$.

There are several notions of a hypercycle. Here we consider only hypercycles whose triples form consecutive segments of a cyclic ordering of the vertices. For $l = 1, 2$, an *l-overlapping cycle* is a 3-graph whose vertices can be ordered cyclically in such a way that every triple forms a segment of this ordering and every two consecutive triples share l vertices. For $l = 1$ we call these cycles *loose*, while for $l = 2$ we call them *tight*. Note that in the former case the number of vertices must be even, and that in the latter case the number of triples equals the number of vertices. *Loose* and *tight paths* are defined in the same way, but with respect to a linear ordering of the vertices. For $l = 1, 2$, an l-overlapping cycle in H is *Hamiltonian* if it passes through all the vertices of H. For the sake of unification, a perfect matching (that is, a set of disjoint triples in H containing all the vertices) can be viewed as a Hamiltonian 0-overlapping cycle.

For a vertex $v \in V(H)$, let $H(v)$ denote *the link graph of v in H*, that is,

$$H(v) = \left\{ e \in \binom{V \setminus \{v\}}{2} : e \cup \{v\} \in H \right\}.$$

In particular, $|H(v)| = \deg_H(v)$, where $\deg_H(v)$ is the degree of vertex v in H. We set $\delta_1(H) = \min_v \deg_H(v)$ and observe that, trivially, $\delta_1(H) \le \binom{n-1}{2}$.

Besides the notion of vertex degree, in triple systems one can also define a pair degree. Given a 3-graph H and two vertices $u, v \in V(H)$, we denote by $N_H(u, v)$ the set of all triples of H which contain $\{u, v\}$. We call $\deg_H(u, v) = |N_H(u, v)|$ the degree of the pair $\{u, v\}$ in H. Let $\delta_2(H)$ denote the minimum pair degree in H and observe that $\delta_2(H) \le n - 2$.

The relation between the minimum degree $\delta_d(H)$, $d = 1, 2$, and the presence of a Hamiltonian l-overlapping cycle in a 3-graph, $l = 0, 1, 2$, is best depicted in terms of the extremal parameter $h_d^l(n)$.

Definition 1. Let d, l, and n satisfy $1 \le d \le 2$, $0 \le l \le 2$, and $3 - l$ divide n. We define $h_d^l(n)$ to be the smallest integer h such that every n-vertex 3-graph H satisfying $\delta_d(H) \ge h$ contains a Hamiltonian l-overlapping cycle.

The first Dirac-type result for 3-graphs was obtained by Katona and Kierstead who proved in [7] that

$$\left\lfloor \frac{n}{2} \right\rfloor \le h_2^2(n) \le \frac{5}{6}n + \frac{13}{6}.$$

Katona and Kierstead (implicitly) conjectured that their lower bound is the correct value of $h_2^2(n)$. Recently, this has been confirmed in [20]. Earlier, Kühn and Osthus proved in [12] that $h_2^1(n) \sim \frac{1}{4}n$. Very recently, Buss, Hàn, and Schacht proved in [2] that $h_1^1(n) \sim \frac{7}{16}\binom{n-1}{2}$. As far as perfect matchings are concerned,

l / d	0	1	2
1	$\binom{n-1}{2} - \binom{2n/3}{2} + 1 \sim \frac{5}{9}\binom{n-1}{2}$ [4, 13, 9]	$\sim \frac{7}{16}\binom{n-1}{2}$ [2]	$\sim (?)\binom{n-1}{2}$
2	$\frac{n}{2} + O(1)$ [17]	$\sim \frac{1}{4}n$ [12]	$\lfloor \frac{n}{2} \rfloor$ [20]

Table 1. Known values of $h_d^l(n)$.

$h_2^0(n) = \frac{n}{2} + O(1)$ ([17]) and $h_1^0(n) \sim \frac{5}{9}\binom{n-1}{2}$ ([4], see also Construction 1 and Theorem 12 below). This leaves only one case open: $d = 1$ and $l = 2$. (See Table 1 for a concise summary of those results.)

While proving a more general result, Glebov, Person, and Weps [3] showed that $h_1^2(n) \leq (1-\epsilon)\binom{n-1}{2}$, where the numerical value of ϵ is of the order of magnitute of 5×10^{-7}. In this paper we improve that bound.

Theorem 1. *For every $\gamma > 0$ there exists n_0 such that if $n \geq n_0$, then*

$$h_1^2(n) \leq \left(\frac{5-\sqrt{5}}{3} + \gamma\right)\binom{n-1}{2}.$$

Still, the upper bound on $h_1^2(n)$ provided by Theorem 1 seems to be far from the truth. Indeed, two different critical constructions yield the same lower bound of asymptotically only $\frac{5}{9}\binom{n-1}{2}$ (see Figure 1).

Construction 1. Let $H_1 = (V, E_1)$, where $V = X \cup Y$, $x := |X| = \lceil \frac{n}{3} \rceil - 1$, $y := |Y| = n - x$, and $E_1 = \left\{e \in \binom{V}{3} : e \cap X \neq \emptyset\right\}$. Then $\delta_1(H_1) = \binom{n-1}{2} - \binom{y-1}{2} \sim \frac{5}{9}\binom{n-1}{2}$. Suppose that H_1 has a tight Hamiltonian cycle C. Then X is a vertex cover of C. (Indeed, as no edge of H_1 is contained entirely in Y, the set X is a vertex cover of H_1.) But C is 3-regular, so $n = |C| \leq \sum_{x \in X} \deg_C(x) = 3x < n$, a contradiction.

Construction 2. Let $H_2 = (V, E_2)$, where $V = X \cup Y$, $x := |X| = \lceil \frac{n+1}{3} \rceil$, $y := |Y| = n - x$, and $E_2 = \left\{e \in \binom{V}{3} : |e \cap Y| \neq 1\right\}$.

Then $\delta_1(H_2) = \min\left\{\binom{x-1}{2} + \binom{y}{2}, \binom{y-1}{2} + x(y-1)\right\} = \binom{x-1}{2} + \binom{y}{2} \sim \frac{5}{9}\binom{n-1}{2}$. Suppose that H_2 has a tight Hamiltonian cycle C. Then X is an independent set in C. (Indeed, there is no tight path in H_2 connecting a triple in X with a vertex in Y.) Since no triple of H_2, and consequently of C, has exactly two vertices in X, and C is 3-regular, we have $n = |C| \geq \sum_{x \in X} \deg_C(x) = 3x > n$, a contradiction.

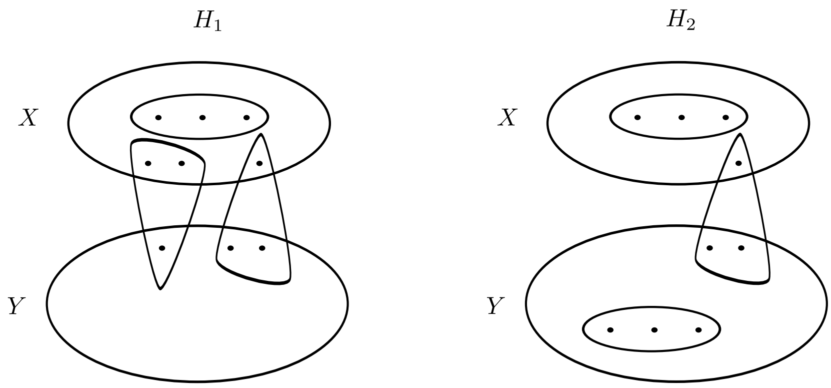

Figure 1. Two critical constructions.

Note that H_1 does not even have a perfect matching and so, for n divisible by 3, H_1 yields the lower bound on $h_1^0(n)$. Judging by the similarity between the existing results on Dirac thresholds for perfect matchings and for Hamiltonian cycles, it is tempting to conjecture that, indeed, $h_1^2(n) \sim h_1^0(n) \sim \frac{5}{9}\binom{n-1}{2}$.

On the other hand, H_2 does have a perfect matching (if $3|n$). More importantly, the pairs of vertices in H_2 form a disconnected structure. This suggests that any straightforward application of the absorbing method, as in [18, 20, 5] (see also [15]), which relies, among other tools, on a (pair) connecting lemma, may not work here.

2. Two Lemmas

Our proof of Theorem 1 deviates from the standard approach described in Section 3 (c.f. [15]), in that it relies on two new lemmas. In both we implicitly assume that $\gamma > 0$ is arbitrarily small and $n > n_0$, where $n_0 = n_0(\gamma)$ is sufficiently large. Pairs of vertices u, v with degree $\deg_H(u, v) \geq (\frac{1}{2}+\gamma)n$ will be called *large* (in H). From now on we will refer to tight paths and cycles as paths and cycles, resp. If P is a path with $t \geq 3$ vertices $v_1, \dots, v_t$ and $t-2$ edges $\{v_1, v_2, v_3\}, \dots, \{v_{t-2}, v_{t-1}, v_t\}$, then we call the ordered pairs (v_1, v_2) and (v_t, v_{t-1}) the *endpairs* of P, and we say that P *connects* its endpairs, or that P goes *between* them. The *length* of a path is defined as the number of its vertices.

Our first lemma replaces the Connecting Lemma (Lemma 2.13 in [15]). It turns out that when $\delta_1(H) \geq \frac{7}{8}\binom{n-1}{2}$, there are short paths between all large pairs of vertices. This lemma will be sufficient for our proof of Theorem 1, since we will make sure that the paths to be connected will indeed have large endpairs only.

Lemma 2. *For every $\gamma > 0$, if $\delta_1(H) \geq \frac{7}{8}\binom{n-1}{2}$ then for every pair of large pairs u, v and x, y of vertices there is a path on six vertices between (u, v) and (x, y).*

In the proof of Lemma 2 we will need the following fact whose simple proof is omitted.

Fact 1. *Let B and R be arbitrary, not necessarily disjoint sets, where $|B| \leq |R|$. Then*

$$|\{\{b, r\} : b \in B, r \in R\}| \geq \binom{|B|}{2}.$$

Proof of Lemma 2. By the assumption on $\delta_1(H)$,

$$|H(v) \cap H(y)| \geq \frac{3}{4}\binom{n-1}{2} - O(n).$$

Let us set $G = H(v) \cap H(y) - \{u, x\}$ and observe that $|V(G)| = n - 4$ and

$$|E(G)| \geq \frac{3}{4}\binom{n-1}{2} - O(n). \tag{1}$$

If there is an edge $\{w, z\} \in G$ such that $\{x, y, z\} \in H$ and $\{u, v, w\} \in H$ then the vertices x, y, z, w, v, u span the desired path (see Figure 2).

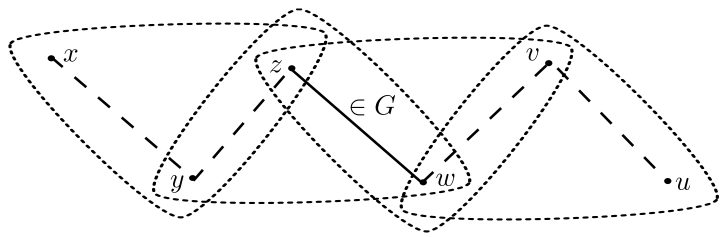

Figure 2. Illustration to the proof of Lemma 2.

Let us apply Fact 1 with $B = N_H(u, v) \cap V(G)$ and $R = N_H(x, y) \cap V(G)$ (we assume w.l.o.g. that $|B| \leq |R|$). As

$$|B| \geq (\tfrac{1}{2} + \gamma)n - 2,$$

by Fact 1 and (1) we conclude that, for sufficiently large n, the number of pairs $\{w, z\} \in \binom{V(G)}{2}$ such that $\{x, y, z\} \in H$ and $\{u, v, w\} \in H$ is at least

$$\binom{|B|}{2} \geq \frac{1}{2}\left(\frac{1}{2}+\gamma\right)^2 n^2 + O(n) > \left(\frac{1}{8}+\frac{\gamma}{2}\right)n^2 > \binom{n-4}{2} - |E(G)|,$$

which implies that at least one of these pairs is the desired edge $\{w, z\}$ of G. ■

The second lemma allows us to get down to a desirable position where all pairs of vertices with positive degrees are large without too much sacrifice of the vertex degrees.

Lemma 3. *If $\delta_1(H) \geq (\frac{5-\sqrt{5}}{3}+\gamma)\binom{n-1}{2}$ then there exists a spanning subhypergraph H' of H such that*
(i) $\delta_1(H') \geq (\frac{5}{9}+\gamma/2)\binom{n-1}{2}$ *and*
(ii) *for every pair $u, v \in V$ either $\deg_{H'}(u,v) = 0$ or $\deg_{H'}(u,v) \geq (\frac{1}{2}+\gamma)n$.*

Proof. Set $H_0 = H$ and consider the following "shaving" procedure which is iterated for as long as it is possible. The $(i+1)$st step, $i \geq 0$, can be described as follows. Suppose that we have already constructed a subhypergraph H_i, and that there exists a pair $e_{i+1} \in \binom{V}{2}$ with $0 < \deg_{H_i}(e_{i+1}) < (\frac{1}{2}+\gamma)n$. Then delete from H_i all edges which contain e_{i+1} and call the resulting subhypergraph H_{i+1}. Let t be the smallest integer such that

$$\left\{e \in \binom{V}{2} : 0 < \deg_{H_t}(e) < \left(\frac{1}{2}+\gamma\right)n\right\} = \emptyset.$$

We claim that $H' = H_t$ is the required subhypergraph of H. By the definition of t, it remains to show that $\delta_1(H') \geq (\frac{5}{9}+\gamma/2)\binom{n-1}{2}$.

For every vertex $v \in V$, let us estimate the difference $\deg_H(v) - \deg_{H'}(v)$, which counts the edges of H containing v and *lost* in the process of "shaving". For $j = 1, \ldots, t$, let S_j be the graph with vertex set $V = V(H)$ and edge set $\{e_1, \ldots, e_j\}$, and let $\Delta_j := \Delta(S_j)$ be the maximum degree in the graph S_j. Every lost edge of H must contain a pair e_i from S_t (see Figure 3). Let us first count those of them which contain a pair e_i such that $v \in e_i$. Let $d = \deg_{S_t}(v)$. Then the number of such edges of H is at most

$$\binom{d}{2} + d(n-1-d) < dn - d^2/2 \leq \Delta_t n - \Delta_t^2/2.$$

The number of remaining edges of H lost at v cannot exceed t, the number of pairs in S_t. In turn, trivially, $t \leq \Delta_t n/2$. Altogether,

$$\deg_H(v) - \deg_{H'}(v) \leq \Delta_t n - \Delta_t^2/2 + t \leq \frac{3}{2}\Delta_t n - \Delta_t^2/2. \tag{2}$$

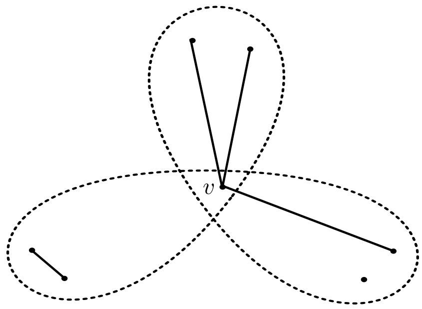

Figure 3. Types of edges "shaved" at vertex v
(pairs in S_t are drawn with straight lines).

Below we are going to show that $\Delta_t \leq (\frac{1}{2} - \frac{\sqrt{5}}{6})n$ which, by (2) and by our assumption on $\delta_1(H)$, immediately implies that for every $v \in V$

$$(3)\ \deg_{H'}(v) \geq \left(\frac{5-\sqrt{5}}{3} + \gamma\right)\binom{n-1}{2} - \frac{3}{2}\Delta_t n + \Delta_t^2/2 \geq \left(\frac{5}{9} + \gamma/2\right)\binom{n-1}{2}.$$

Thus, in order to complete the proof of Lemma 3, it remains to show the following claim.

Claim 4. $\Delta_t \leq (\frac{1}{2} - \frac{\sqrt{5}}{6})n$.

Proof. Suppose to the contrary that $\Delta_t > (\frac{1}{2} - \frac{\sqrt{5}}{6})n$ and let s be the smallest index such that

$$(4) \qquad \Delta_s = \left\lceil \left(\frac{1}{2} - \frac{\sqrt{5}}{6}\right) n \right\rceil.$$

Consider a vertex $v \in V$ with $\deg_{S_s}(v) = \Delta_s$. By the assumption on $\delta_1(H)$ we have

$$\deg_H(v) = |H(v)| \geq \left(\frac{5-\sqrt{5}}{3} + \gamma\right)\binom{n-1}{2}.$$

We will next show that the complement of the link, $\overline{H(v)} = \binom{V\setminus\{v\}}{2} \setminus H(v)$, satisfies

$$|\overline{H(v)}| > \left(\frac{\sqrt{5}-2}{3} - \gamma\right)\binom{n-1}{2},$$

a contradiction, since then $|H(v)| + |\overline{H(v)}| > \binom{n-1}{2}$.

Let $u_1, \dots, u_{\Delta_s}$ be the neighbors of v in S_s, connected to v at steps $t_1 \le \dots \le t_{\Delta_s}$ of the "shaving" procedure, that is, $e_{t_j+1} = \{v, u_j\}$, $j = 1, \dots, \Delta_s$. Because

$$0 < \deg_{H_{t_j}}(v, u_j) < (\tfrac{1}{2} + \gamma)n,$$

we have

$$\deg_{\overline{H_{t_j}}}(v, u_j) \ge n - 2 - \deg_{H_{t_j}}(v, u_j) > (\tfrac{1}{2} - \gamma)n - 2. \tag{5}$$

Let $\deg^-_{\overline{H_{t_j}}}(v, u_j)$ and $\deg^-_{\overline{H}}(v, u_j)$ stand for the number of neighbors of $\{v, u_j\}$ in, respectively, $\overline{H_{t_j}}$ and $\overline{H}$ within the set $V \setminus \{u_1, \dots, u_{j-1}\}$. By this definition and (5), we have

$$\deg^-_{\overline{H_{t_j}}}(v, u_j) \ge \deg_{\overline{H_{t_j}}}(v, u_j) - (j-1) \ge (\tfrac{1}{2} - \gamma)n - (j+1). \tag{6}$$

We also have

$$|\overline{H(v)}| \ge \sum_{j=1}^{\Delta_0} \deg^-_{\overline{H}}(v, u_j) \tag{7}$$

which is the starting point of our estimates.

In view of (6) and (7), we would like to compare the degrees $\deg^-_{\overline{H_{t_j}}}(v, u_j)$ and $\deg^-_{\overline{H}}(v, u_j)$. Clearly, $\overline{H_{t_j}} \supseteq \overline{H}$. But which triplets are counted by the difference

$$\deg^-_{\overline{H_{t_j}}}(v, u_j) - \deg^-_{\overline{H}}(v, u_j)\ ?$$

The answer comes from looking at the neighbors of u_j in S_{t_j} (see Figure 4).

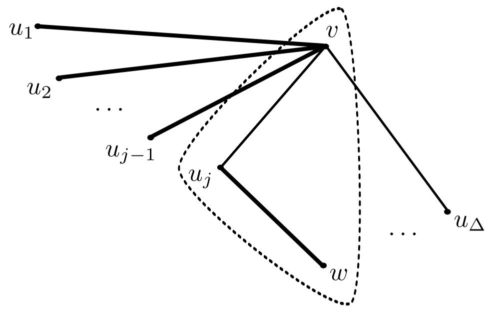

Figure 4. Illustration to the proof of Claim 4
(pairs in S_{t_j} are drawn with bold lines).

Suppose that we have $\{v, u_j, w\} \in H$, $w \notin \{u_1, \dots, u_{j-1}\}$. Then the only reason for $\{v, u_j, w\} \notin H_{t_j}$ is that the pair $\{u_j, w\}$ has been added to $S_{t'}$ at some earlier step $t' < t_j$. However, there are at most

$$\deg_{S_{t_j}}(u_j) \le \Delta(S_{t_j}) \le \Delta_s$$

such vertices w, and we conclude that

$$\deg^{-}_{\overline{H_{t_j}}}(v, u_j) - \deg^{-}_{\overline{H}}(v, u_j) \leq \Delta_s. \tag{8}$$

Using (4) and (6)–(8), we can finally bound $|\overline{H(v)}|$ as follows:

$$\begin{aligned} |\overline{H(v)}| &\overset{(7)}{\geq} \sum_{j=1}^{\Delta_s} \deg^{-}_{\overline{H}}(v, u_j) \overset{(8)}{\geq} \sum_{j=1}^{\Delta_s} \deg^{-}_{\overline{H_{t_j}}}(v, u_j) - \Delta_s^2 \\ &\overset{(6)}{\geq} \left(\frac{1}{2} - \gamma\right) n\Delta_s - \sum_{j=1}^{\Delta_s} (j+1) - \Delta_s^2 \\ &= \left(\frac{1}{2} - \gamma\right) n\Delta_s - \binom{\Delta_s + 2}{2} + 1 - \Delta_s^2 \overset{(8)}{>} \left(\frac{\sqrt{5}-2}{3} - \gamma\right) \binom{n-1}{2} \end{aligned} \tag{9}$$

for all $\gamma > 0$ and sufficiently large n. □

Remark 1. The constant in Claim 4 and, consequently, in Theorem 1 cannot be improved by our method. To see this, let us set $\Delta_t = \Delta_s = xn$ and assume that $\delta_1(H) \geq (c+\gamma)\binom{n-1}{2}$. We want to minimize c so that the proof of Lemma 3 still goes through, that is, both inequalities (3) and (9) must be satisfied. Omitting the lower order terms and ignoring γ, we can rewrite (3) and (9) as, resp., $c - 3x + x^2 \geq \frac{5}{9}$ and $x - 3x^2 \geq 1 - c$. To minimize c means to determine the $\min_{0<x<1} \max\{f(x), g(x)\}$, where $f(x) = -x^2 + 3x + \frac{5}{9}$ and $g(x) = 3x^2 - x + 1$. It turns out (see Figure 5) that this minimum is achieved at the smaller root of the equation $f(x) = g(x)$, that is, of the quadratic equation $x^2 - x + \frac{1}{9} = 0$, which, indeed, occurs at $x_0 = \frac{1}{2} - \frac{\sqrt{5}}{6}$ (note that $f(x_0) = g(x_0) = \frac{5-\sqrt{5}}{3}$).

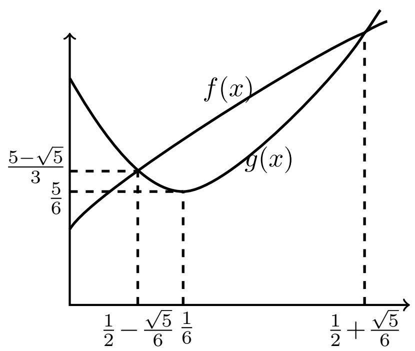

Figure 5. Graphical explanation of the constant $\frac{5-\sqrt{5}}{3}$.

3. Proof of Theorem 1

The proofs of several results mentioned in the Introduction use the absorbing path technique developed in [16, 18, 20]. An entire section in [15] is devoted to a detailed outline of the proof from [18], and our proof of Theorem 1 below follows that outline. In short, one begins with building an absorbing path A and putting aside a small reservoir set R. (These preliminary steps will be presented in Section 3.1.) Then a long cycle C containing A is created in the remaining hypergraph (using the reservoir R to connect several paths together) and, finally, utilizing the absorbing property of A, the cycle C is extended to a Hamiltonian cycle. (These major steps are shown in Section 3.2, except for a proof of the crucial Path Cover Lemma which is deferred to Section 4.)

3.1. Preliminary steps

Throughout, we implicitly assume that $\gamma > 0$ is arbitrarily small and $n > n_0$, where $n_0 = n_0(\gamma)$ is sufficiently large. Let $\delta_1(H) \geq \left(\frac{5-\sqrt{5}}{3} + \gamma\right)\binom{n-1}{2}$ and recall that a pair of vertices u, v of H is *large* if $\deg_H(u,v) \geq (\frac{1}{2} + \gamma)n$. We begin our proof of Theorem 1 by applying Lemma 3 to H, obtaining a spanning subhypergraph H' with the property that each pair u, v of vertices is large in H' if and only if $\deg_{H'}(u,v) > 0$. From now on, by a large pair we will always mean a pair large in H'. Since $\delta(H') > 0$, for each vertex u there are at least

$$(\tfrac{1}{2} + \gamma)n + 1 > \tfrac{1}{2}n \tag{10}$$

large pairs $\{u, v\}$. In the next step we find an absorbing path A in H. (We do not attempt to optimize the constants.) Comparing with the proof of Lemma 2.10 in [15], the only change appears in Claim 6 below, an analog of Claim 2.12 [15]—our lower bound on the number of absorbing 4-tuples is twice smaller than there.

Lemma 5. *There exists a path A in H (called* absorbing*) with $|V(A)| \leq \gamma^3 n$ such that for every subset $U \subset V \setminus V(A)$ of size $|U| \leq \gamma^7 n$ there is a path A_U in H with $V(A_U) = V(A) \cup U$ and such that A_U has the same endpairs as A.*

Proof. The path A will consist of disjoint absorbing 4-tuples taken from H' and "glued together", via Lemma 2, by disjoint paths of length six in H. We say that a 4-tuple of distinct vertices (x_1, x_2, x_3, x_4) *absorbs* v in H' if $\{x_1, x_2, x_3\} \in H'$, $\{x_2, x_3, x_4\} \in H'$, $\{x_1, x_2, v\} \in H'$, $\{x_2, v, x_3\} \in H'$, and $\{v, x_3, x_4\} \in H'$ (see Figure 6).

Claim 6. *For every $v \in V(H)$, there are at least $\gamma^2 n^4$ 4-tuples absorbing v in H'.*

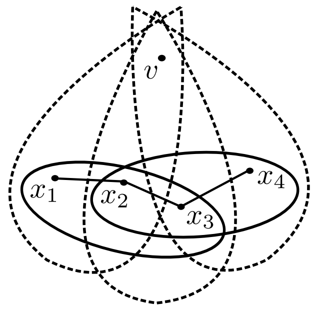

Figure 6. An absorbing 4-tuple.

Proof. Given a vertex v, we choose x_1 so that the pair $\{v, x_1\}$ is large, which can be done, by (10), in more than $n/2$ ways. Then, for any neighbor x_2 of $\{v, x_1\}$ in H' (and there are more than $n/2$ of them), we know that the pairs $\{x_1, x_2\}$ and $\{v, x_2\}$ are also large. Consequently, there are at least $2\gamma n$ vertices x_3 such that $\{x_1, x_2, x_3\} \in H'$ and $\{v, x_2, x_3\} \in H'$. Similarly, we argue that there are at least $2\gamma n$ choices of x_4. Altogether, we have at least $(n/2)^2(2\gamma n)^2 = \gamma^2 n^4$ 4-tuples absorbing v. □

To finish the proof of Lemma 5, we select randomly a family $\mathcal{F}'$ of ordered 4-tuples of distinct vertices of V, independently and with probability $p = \frac{\gamma^4}{n^3}$. With positive probability, $\mathcal{F}'$ satisfies:

- $|\mathcal{F}'| \leq 2n^4 p = 2\gamma^4 n$,
- there are at most $17n^7p^2$ pairs of intersecting 4-tuples in $\mathcal{F}'$ (because there are at most $16n^7$ such pairs in V).
- for every vertex v, by Claim 6, there are at least $\frac{1}{2}\gamma^2 n^4 p$ 4-tuples in $\mathcal{F}'$ which absorb v in H'.

By removing from $\mathcal{F}'$ all 4-tuples which do not absorb any vertex v as well as one 4-tuple of each intersecting pair, we obtain a subfamily $\mathcal{F}$ such that

- $|\mathcal{F}| \leq |\mathcal{F}'| = 2\gamma^4 n$,
- the 4-tuples in $\mathcal{F}$ form a disjoint family of paths in H',
- for every vertex v, there are at least $\frac{1}{2}\gamma^2 n^4 p - 17n^7p^2 \geq \gamma^7 n$ 4-tuples in $\mathcal{F}$ which absorb v in H'.

Using Lemma 2, we connect all paths in $\mathcal{F}$ into one path A of length at most $6 \times 2\gamma^4 n \leq \gamma^3 n$. Let $F_1, \ldots, F_t$, $t \leq 2\gamma^4 n$, be the paths (of length 4) in $\mathcal{F}$. Assume that, for some $i = 1, \ldots, t-1$, we have already connected $F_1, \ldots, F_i$ into a path

A_i of length $6i-2$. Let H_i be the subhypergraph of H obtained by removing from H all vertices of A_i along with all vertices of $F_{i+1}\cup\cdots\cup F_t$, except for one endpair e_{i+1} of F_{i+1} and the endpair e_i of F_i which is also an endpair of P_i. Since

$$\delta_1(H_i) \ge \delta_1(H) - 6tn \ge \delta_1(H) - 12\gamma^4 n^2 \ge \frac{7}{8}\binom{n-1}{2}$$

(the last inequality holding with a big margin) and

$$\min\{\deg_{H_i}(e_i), \deg_{H_i}(e_{i+1})\} \ge \left(\frac{1}{2}+\gamma\right)n - 2\gamma^4 n \ge \left(\frac{1}{2}+\frac{\gamma}{2}\right)n,$$

the assumptions of Lemma 2 are satisfied, and thus there is a path P_i of length 6 between e_i and e_{i+1} in H_i. The concatenation of the paths A_i, P_i, and F_{i+1} constitutes the path A_{i+1}. Finally, set $A = A_t$.

To see that A is indeed an absorbing path in H, consider an arbitrary subset $U \subseteq V \setminus V(A)$ of size $|U| \le \gamma^7 n$. Since for every $v \in U$ there are at least $\gamma^7 n$ 4-tuples F_i in A which absorb v, there is a one-to-one mapping $f : U \to \{1,\dots,t\}$ such that $F_{f(v)}$ absorbs v. Let $(x_1^v,\dots,x_4^v)$ be the vertices of the path $F_{f(v)}$. Then the path obtained from A by replacing, for each $v \in U$, the edges $\{x_1^v, x_2^v, x_3^v\}$ and $\{x_2^v, x_3^v, x_4^v\}$ with $\{x_1^v, x_2^v, v\}$, $\{x_2^v, v, x_3^v\}$, and $\{v, x_3^v, x_4^v\}$, is the desired path A_U. □

The next step in the proof of Theorem 1 is to put aside a reservoir set R which should be small, disjoint from the absorbing path A, quickly reachable from any large pair, as well as the induced subhypergraph $H[R]$ should satisfy the assumptions of Lemma 2.

Lemma 7. *There exists a set $R \subset V \setminus V(A)$ such that*
(a) $\frac{1}{4}\gamma^7 n \le |R| \le \frac{1}{2}\gamma^7 n$,
(b) *for every large pair e in H', we have $|N_{H'}(e) \cap R| \ge (\frac{1}{2}+\gamma/2)|R|$, and*
(c) $\delta_1(H[R]) \ge \frac{8}{9}\binom{|R|-1}{2}$.

Proof. Set $p = \frac{1}{3}\gamma^7$ and select a binomial random subset R of $V \setminus V(A)$ by including to R every element of $V \setminus V(A)$ independently, with probability p. The random variable $|R|$ has a binomial distribution with expectation $n'p$, where

$$(1-\gamma^3)n \le n' = |V \setminus V(A)| \le n. \tag{11}$$

By Chebyshev's inequality, with probability tending to 1 as $n \to \infty$,

$$|R| \sim n'p, \tag{12}$$

and thus (a) holds.

For every large pair e in H', the random variable $|N_{H'}(e) \cap R|$ is also binomially distributed, with expectation at least

$$\left(\left(\tfrac{1}{2}+\gamma\right)n - |V(A)|\right)p \geq \left(\tfrac{1}{2}+\gamma-\gamma^3\right)np.$$

Hence, by a standard application of Chernoff's bound (simultaneously for all large e), the random set R satisfies the condition of part (b) with probability tending to 1 as $n \to \infty$.

For part (c), fix a vertex v and consider a random variable X_v equal to the number of pairs of vertices $\{u, w\}$ such that $\{u, w\}$ is an edge of the link graph $H(v)$ and $\{u, w\} \subseteq R$. We apply to X_v Janson's inequality (see, e.g., [6], Theorem 2.14, page 31). We have

$$\mathbb{E}(X_v) = |H(v)|p^2 \geq \delta_1(H)p^2 \geq \frac{10}{11}\binom{np-1}{2},$$

and, using notation of [6], $\overline{\Delta} = \Theta(n^3)$, because $\overline{\Delta}$ is dominated by pairs of intersecting edges of $H(v)$. Thus,

$$\mathbb{P}\left(X_v \leq \frac{9}{10}\binom{np-1}{2}\right) \leq \exp\{-\Theta(n)\},$$

and, recalling (11) and (12), with probability tending to 1 as $n \to \infty$, for every v

$$X_v \geq \frac{9}{10}\binom{np-1}{2} \geq \frac{8}{9}\binom{|R|-1}{2}.$$

Consequently, the random set R also satisfies the condition of part (c). In summary, there exists a set $R \subset V \setminus V(A)$ which satisfies all three properties (a), (b), and (c). □

3.2. Major steps—the outline

To finish the proof of Theorem 1 we need to do three more things:

I. Build a collection of vertex-disjoint paths in $H'' = H'[V \setminus (V(A) \cup R)]$ which cover all vertices of $V(H'')$ except for a set T of at most $\frac{1}{2}\gamma^7 n$ vertices.

II. Using the reservoir R, connect the paths obtained in Step I as well as the absorbing path A to form a long cycle C in H, covering all vertices of V except for a set U of at most $\gamma^7 n$ vertices.

III. Absorb the set U of the leftover vertices into the cycle C to form a Hamiltonian cycle in H.

See Figures 7–9 for an illustration of this outline.

Step I is the most challenging one and we devote to it the entire next section, containing a proof of the following lemma.

Lemma 8 (Path Cover Lemma)**.** *For all $\beta > 0$ and $\rho > 0$ there exist integers n_0 and L such that every 3-graph H with $n > n_0$ vertices and $\delta(H) \geq \left(\frac{5}{9} + \beta\right)\binom{n-1}{2}$, contains a family of at most L vertex-disjoint paths, covering at least $(1-\rho)n$ vertices of H.*

Note that

$$\delta(H'') \geq \delta(H') - |V(A)|n - |R|n \geq \left(\tfrac{5}{9} + \gamma/3\right)\binom{n-1}{2}.$$

Therefore, applying Lemma 8 to H'' with $\beta = \gamma/3$ and $\rho = \frac{1}{2}\gamma^7$ yields the conclusion of Step I.

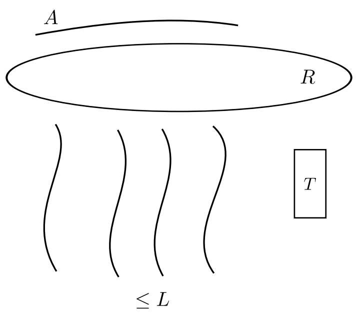

Figure 7. Outline of the proof of Theorem 1: Step I.

Step II is quite straightforward and based on Lemma 2 and the properties of R established in Lemma 7. Indeed, let $Q_1, \ldots, Q_\ell$, $\ell \leq L$, be the paths obtained in Lemma 8, and set $Q_0 = Q_{\ell+1} = A$, for convenience. Suppose that for some $i \in \{0, \ldots, \ell\}$ we have already connected $Q_0, \ldots, Q_i$ into one path Π_i using $6i$ vertices of R. Let the endpairs of Π_i be one endpair (a_1, a_2) of A and one endpair (x_1, x_2) of Q_i. Let (y_1, y_2) be an endpair of Q_{i+1} (if $i = \ell$, we take $y_j = a_j$, $j = 1, 2$). Let $R_i = R \setminus V(\Pi_i)$ be the set of $|R| - 6i$ remaining vertices of R. Since $|R| - |R_i| = 6i \leq 6L = O(1)$, for every large pair e in H', we have

$$|N_{H'}(e) \cap R_i| \geq (\tfrac{1}{2} + \gamma/2)|R| - 6L \geq (\tfrac{1}{2} + \gamma/3)|R|.$$

Hence, there exist $x_3, x_4, y_3, y_4 \in R_i$ such that all four triples $\{x_1, x_2, x_3\}$, $\{x_2, x_3, x_4\}$, $\{y_1, y_2, y_3\}$, $\{y_2, y_3, y_4\}$ belong to H'. Finally, note that

$$\delta_1(R_i) \geq \delta_1(R) - 6Ln \geq \frac{8}{9}\binom{|R|-1}{2} - 6Ln \geq \frac{7}{8}\binom{|R_i|-1}{2}.$$

Hence, we are in position to apply Lemma 2 (with $\gamma/3$) to $H'[R_i]$ and find a path S_i of length 6 which connects in R_i the large pairs (x_3, x_4) and (y_3, y_4). If $i < \ell$, the paths Π_i, S_i, and Q_{i+1}, together with the edges $\{x_1, x_2, x_3\}$, $\{x_2, x_3, x_4\}$, $\{y_1, y_2, y_3\}$, $\{y_2, y_3, y_4\}$ form the path Π_{i+1}. If $i = \ell$, they form the desired cycle C.

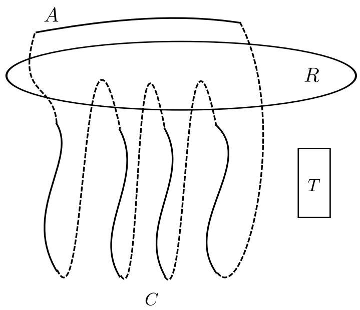

Figure 8. Outline of the proof of Theorem 1: Step II.

Step III follows immediately from the definition of the absorbing path given in the statement of Lemma 5. Indeed, let $U = V \setminus V(C)$, where C is the long cycle obtained in Step II. Then $|U| \le \gamma^7 n$ and there exists a path A_U such that $V(A_U) = V(A) \cup U$ and A_U has the same endpairs as A. Hence, one can replace A with A_U in C obtaining a Hamiltonian cycle in H.

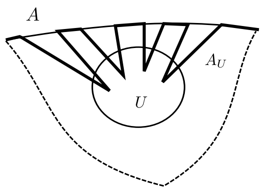

Figure 9. Outline of the proof of Theorem 1: Step III.

4. Path Cover Lemma

In this section we complete the proof of Theorem 1 by proving Lemma 8 stated at the end of the previous section. In doing so we will follow closely the proof of Lemma 2.2 from [18].

The proof of Lemma 8 relies on modifications of a couple of claims proved or stated in [18] for general k (Claims 4.2–4.4 there). Claim 4.5 in [18] is replaced by the result from [4] giving a Dirac threshold for perfect matchings in 3-graphs (Theorem 12 below).

A 3-graph H is 3-*partite* if there is a partition $V(H) = V_1 \cup V_2 \cup V_3$ such that every edge of H intersects each set V_i in precisely one vertex. A 3-partite 3-graph will be called here a $(3,3)$-*graph.* If, in addition, the vertex partition satisfies

$$|V_1| \le |V_2| \le |V_3| \le |V_1| + 1,$$

then the $(3,3)$-graph will be called *equitable.* Given a 3-graph H and three non-empty, disjoint subsets $A_i \subset V(H)$, $i = 1, \dots, 3$, we define $e_H(A_1, A_2, A_3)$ to be the number of edges in H with one vertex in each A_i, and the *density* of H with respect to (A_1, A_2, A_3) as

$$d_H(A_1, A_2, A_3) = \frac{e_H(A_1, A_2, A_3)}{|A_1||A_2||A_3|}.$$

For a $(3,3)$-graph H, we will write d_H for $d_H(V_1, V_2, V_3)$ and call it the *density of* H.

We say that a $(3,3)$-graph H is ε-*regular* if for all $A_i \subseteq V_i$ with $|A_i| \ge \varepsilon |V_i|$, $i = 1, 2, 3$, we have

$$|d_H(A_1, A_2, A_3) - d_H| \le \varepsilon.$$

Claim 9 [18]**.** *For all $0 < \varepsilon < \alpha < 1$, every ε-regular, equitable $(3,3)$-graph H on n vertices, n sufficiently large, and with density $d_H \ge \alpha$, contains a family $\mathcal{Q}$ of vertex-disjoint paths such that for each $P \in \mathcal{Q}$ we have $|V(P)| \ge \varepsilon(\alpha - \varepsilon)n/3$ and $\sum_{P \in \mathcal{Q}} |V(P)| \ge (1 - 2\varepsilon)n$.*

Claim 10 (Weak regularity lemma for 3-graphs)**.** *For all $\varepsilon > 0$ and every integer t_0 there exist T_0 and n_0 such that the following holds. For every 3-graph H on $n > n_0$ vertices there is, for some $t_0 \le t \le T_0$, a partition $V(H) = V_1 \cup \dots \cup V_t$ such that $|V_1| \le |V_2| \le \dots \le |V_t| \le |V_1| + 1$ and for all but less than $\varepsilon\binom{t}{3}$ triplets of partition classes $\{V_{i_1}, V_{i_2}, V_{i_3}\}$, the induced $(3,3)$-graph $H[V_{i_1}, V_{i_2}, V_{i_3}]$ of H is ε-regular.*

Given a partition like in Claim 10, we refer to the sets V_i as *clusters*, and define the *cluster* 3-*graph* K on the vertex set $[t] = \{1, \dots, t\}$ whose edges are all 3-element sets of indices $\{i_1, i_2, i_3\}$ such that $d_H(V_{i_1}, V_{i_2}, V_{i_3}) \ge \frac{\beta}{4}$ and $H[V_{i_1}, V_{i_2}, V_{i_3}]$ is ε-regular. Thus, K is the intersection

$$K = D \cap R(\varepsilon) \tag{13}$$

of two 3-graphs:

- D—consisting of all sets $\{i_1, i_2, i_3\} \subset [t]$ such that $d_H(V_{i_1}, V_{i_2}, V_{i_3}) \ge \frac{\beta}{4}$, and
- $R(\varepsilon)$—consisting of all sets $\{i_1, i_2, i_3\} \subset [t]$ such that $H[V_{i_1}, V_{i_2}, V_{i_3}]$ is ε-regular.

Claim 11. *If H is a 3-graph with $n > n_0$ vertices and $\delta(H) \geq \left(\frac{5}{9}+\beta\right)\binom{n-1}{2}$, $\varepsilon > 0$, $t \geq 12/\beta$, $V(H) = V_1 \cup \cdots \cup V_t$ is a partition as in Claim 10, and K is the cluster 3-graph defined above, then the number of vertices $w \in [t]$ with $\deg_K(w) \geq (\frac{5}{9}+\beta/2-\sqrt{\varepsilon})\binom{t-1}{2}$ is at least $(1-\sqrt{\varepsilon})t$.*

Proof. For clarity, we assume that t divides n and thus, for each $w = 1,\dots,t$, we have $|V_w| = n/t$. For every vertex $w \in K$, let X_w be the number of edges of H with one vertex in V_w and the other two vertices in two different clusters, V_x and V_y, such that $\{x,y,w\} \in D$. That is,

$$X_w = \sum\nolimits_{\{x,y\}\in N_D(w)} |H[V_w,V_x,V_y]|.$$

Then, on the one hand,

$$X_w \leq \deg_D(w)(n/t)^3,$$

while, on the other hand,

$$X_w \geq \frac{n}{t}\left(\frac{5}{9}+\beta\right)\binom{n-1}{2} - 3\binom{n/t}{3} - 3(t-1)\binom{n/t}{2}\frac{n}{t} - \frac{\beta}{4}\binom{t-1}{2}(n/t)^3.$$

Above, the first term is a lower bound on the number of edges of H with at least one vertex in V_w, except that the edges with two (three) vertices in V_w are counted twice (three times). We then subtract upper bounds on these exceptional edges, $2(t-1)\binom{n/t}{2}\frac{n}{t}$ and $3\binom{n/t}{3}$, respectively. In addition, we subtract $(t-1)\binom{n/t}{2}\frac{n}{t}$, an upper bound on the number of edges with one vertex in V_w and two other vertices in one other cluster. Finally, we subtract the edges in all subgraphs $H[V_w,V_x,V_y]$ with $\{w,x,y\} \notin D$. Observe that

$$3\binom{n/t}{3} \leq \frac{1}{t^2}\frac{n}{t}\binom{n-1}{2}$$

and

$$3(t-1)\binom{n/t}{2}\frac{n}{t} \leq 3\frac{t-1}{t^2}\frac{n}{t}\binom{n-1}{2}.$$

Also,

$$\frac{\beta}{4}\binom{t-1}{2}(n/t)^3 \leq \frac{\beta}{4}\frac{n}{t}\binom{n-1}{2}.$$

Hence,

$$X_w \geq \frac{n}{t}\left(\frac{5}{9}+\beta-\frac{3}{t}-\frac{\beta}{4}\right)\binom{n-1}{2} \geq \frac{n}{t}\left(\frac{5}{9}+\frac{\beta}{2}\right)\binom{n-1}{2}$$

and, consequently,

$$\deg_D(w) \geq \left(\frac{5}{9}+\frac{\beta}{2}\right)\frac{\binom{n-1}{2}}{(n/t)^2} \geq \left(\frac{5}{9}+\frac{\beta}{2}\right)\binom{t-1}{2}.$$

Since the number of irregular triples in K satisfies $|\overline{R(\varepsilon)}| < \varepsilon\binom{t}{3}$, at most $\sqrt{\varepsilon}t$ vertices $w \in K$ have $\deg_{\overline{R(\varepsilon)}}(w) \geq \sqrt{\varepsilon}\binom{t-1}{2}$. The claim follows. □

The last, but not least, ingredient of our proof of Lemma 8 is the following result about the existence of perfect matchings in 3-graphs with high minimum degree. As we are going to apply it to the cluster 3-graph, we change notation for the number of vertices from n to t.

Theorem 12 [4]. *For every $\beta > 0$ there exists t_1 such that every 3-graph H with $t \geq t_1$ vertices, $3|t$, and with $\delta(H) \geq (\frac{5}{9} + \beta)\binom{t-1}{2}$ contains a perfect matching.*

Now we can prove Lemma 8. Given β and ρ, choose $\varepsilon \leq \frac{1}{4}$ so that,

$$\frac{\beta}{6} - 3\sqrt{\varepsilon} \geq 0 \tag{14}$$

and

$$2\varepsilon + \sqrt{\varepsilon} \leq \rho. \tag{15}$$

Set also

$$t_0 = \max\{2t_1(\beta/3), 12/\beta\},$$

where $t_1 = t_1(\beta/3)$ is determined via Theorem 12. Let n_0 and T_0 be the constants determined by ε and t_0 via the weak regularity lemma for hypergraphs (Claim 10). We will prove Lemma 8 with this n_0 and

$$L = \frac{T_0}{\varepsilon\left(\frac{\beta}{4} - \varepsilon\right)}.$$

Apply Claim 10 to H with the above ε and t_0, obtaining a partition as described in that claim. (Assume again that $t|n$.) Let K be the cluster 3-graph defined prior to Claim 11 and let $W \subseteq [t]$ be the set of those clusters w of K for which

$$\deg_K(w) \geq \left(\frac{5}{9} + \frac{\beta}{2} - \sqrt{\varepsilon}\right)\binom{t-1}{2}.$$

Since $t \geq t_0 \geq 12/\beta$, by Claim 11 we have that $t' := |W| \geq (1 - \sqrt{\varepsilon})t$. Define $K' = K[W]$ and observe that, by (14),

$$\delta(K') \geq \left(\frac{5}{9} + \frac{\beta}{2} - \sqrt{\varepsilon}\right)\binom{t-1}{2} - \sqrt{\varepsilon}t(t-1) \geq \left(\frac{5}{9} + \frac{\beta}{3}\right)\binom{t'-1}{2}.$$

Note also that

$$t' \geq (1 - \sqrt{\varepsilon})t \geq (1 - \sqrt{\varepsilon})t_0 \geq 2(1 - \sqrt{\varepsilon})t_1 \geq t_1$$

(here we use the assumption $\varepsilon \le \frac{1}{4}$).
Thus, we are in position to apply Theorem 12 to K' and conclude that there is a perfect matching M in K' (we assume that $3|t'$). For every edge $e = \{x, y, z\} \in M$, let

$$H_e = H[V_x, V_y, V_z].$$

Recalling that $|V(H_e)| = 3(n/t)$, apply Claim 9 to each H_e, with the above ε and $\alpha = \frac{\beta}{4}$. As an outcome, we obtain a family $\mathcal{P}_e$ of paths such that for each $P \in \mathcal{P}_e$

$$|V(P)| \ge \frac{\varepsilon}{3}\left(\frac{\beta}{4} - \varepsilon\right)|V(H_e)| = \varepsilon\left(\frac{\beta}{4} - \varepsilon\right)\frac{n}{t} := l, \tag{16}$$

and

$$\sum\nolimits_{P \in \mathcal{P}_e} |V(P)| \ge (1 - 2\varepsilon)|V(H_e)|$$

(see Figure 10). Consider the union of all these families, $\mathcal{P} = \bigcup_{e \in M} \mathcal{P}_e$. Since, clearly, $|M| \le t/3$, and at most $\sqrt{\varepsilon} n$ vertices of H are not covered by the clusters of M, using (15), we conclude that $\mathcal{P}$ covers all but at most

$$|M| \times 2\varepsilon|V(H_e)| + \sqrt{\varepsilon} n = |M| \times 6\varepsilon(n/t) + \sqrt{\varepsilon} n \le \left(2\varepsilon + \sqrt{\varepsilon}\right) n \le \rho n$$

vertices of H. Moreover, since by (16) each path in $\mathcal{P}$ has length at least l and $t \le T_0$, we have

$$|\mathcal{P}| \le n/l = \frac{t}{\varepsilon\left(\frac{\beta}{4} - \varepsilon\right)} \le L.$$

This completes the proof of Lemma 8.

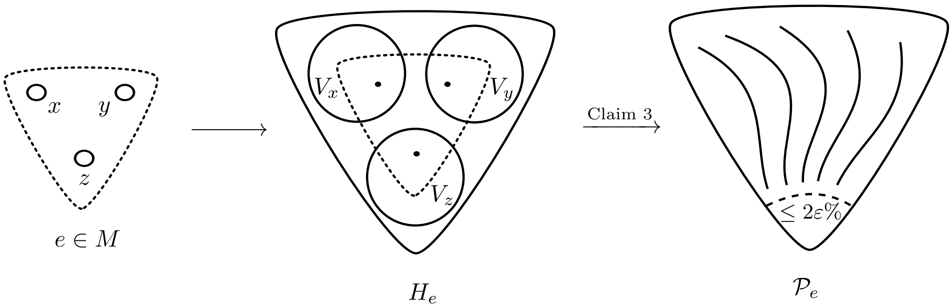

Figure 10. Illustration to the proof of Lemma 8.

5. FINAL COMMENTS

As we have tried to explain in Remark 1 at the end of Section 2, our method cannot be stretched out to yield a bound on $h_1^2(n)$ better than $\frac{5-\sqrt{5}}{3}\binom{n-1}{2} \sim$

$0.92\binom{n-1}{2}$. However, recently, together with Schacht and Szemerédi, we have been trying to improve this bound by applying other variants of Lemmas 1 and 2. More specifically, we redefine a large pair to have degree just at least $(1/3+\gamma)n$, and apply "the shaving procedure" only to the edges with all three pairs small. This approach shows the potential to reduce the bound to approximately $0.8\binom{n-1}{2}$, still not quite satisfactory in view of the lower bound of $\frac{5}{9}\binom{n-1}{2}$.

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