

ON DEGREE SETS AND THE MINIMUM ORDERS IN BIPARTITE GRAPHS

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Abstract

For any simple graph G , let $D(G)$ denote the degree set $\{deg_G(v) : v \in V(G)\}$. Let S be a finite, nonempty set of positive integers. In this paper, we first determine the families of graphs G which are unicyclic, bipartite satisfying $D(G) = S$, and further obtain the graphs of minimum orders in such families. More general, for a given pair (S, T) of finite, nonempty sets of positive integers of the same cardinality, it is shown that there exists a bipartite graph $B(X, Y)$ such that $D(X) = S$, $D(Y) = T$ and the minimum orders of different types are obtained for such graphs.

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1. INTRODUCTION

All graphs considered here are finite, undirected, without loops and without multiple edges. We denote the vertex-set, and the edge-set of a graph G by

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$V(G)$ and $E(G)$, respectively. We let $\overline{G}$ denote the complement of G . The girth of G is the length of a shortest cycle in G . For any two disjoint graphs G and H , $G \cup H$, $G + H$, and $G \circ H$ denote the *union*, the *sum*, and the *corona* of G and H , respectively, as defined in [1]. A *bipartite graph* is that graph G whose vertex set can be partitioned into two subsets X, Y such that every edge of G has one end in X and other end in Y , and we denote this graph by $B(X, Y)$, and call (X, Y) as *bipartition* of $V(G)$. Moreover, if any vertex in X is joined to every vertex in Y , then G is called the *complete bipartite graph* and is denoted by $K(p_1, p_2)$, where $|X| = p_1$, and $|Y| = p_2$. In other words, $K(p_1, p_2) = p_1 K_1 + p_2 K_1$. For any connected graph G , we write kG for the graph G with k components, each component being isomorphic to G . The *degree set* $D(G)$ of a graph G is the set of degrees of the vertices of G . For a finite, nonempty set S of positive integers, we shall write $\mu(S)$ to represent the minimum order of a graph G such that $D(G) = S$. If $S = \{a_1, a_2, \dots, a_n\}$, where $n \geq 1$ and $1 \leq a_1 < a_2 < \dots < a_n$, then it will be convenient to write $\mu(S)$ as simply $\mu(a_1, a_2, \dots, a_n)$. A unicyclic graph is a connected graph that contains exactly one cycle. A *caterpillar* is a tree of order ≥ 3 such that the removal of all its pendant vertices produces a path.

In [2, 3, 4], degree sets are investigated mainly for trees, planar graphs, k -(edge) connected graphs, unicyclic graphs and k -degenerate graphs. Pirzada *et al.* [5] studied degree sets in bi- and tri-partite graphs, and Tripathi *et al.* [6] determined the least order of a graph with a given degree set. Given a finite, nonempty set S of positive integers, we determine now the families of graphs G which are unicyclic and bipartite satisfy the condition $D(G) = S$. Further, we obtain the graphs of minimum orders in such families. More general, for given a pair (S, T) of finite, nonempty sets of positive integers of the same cardinality, it is shown that there exists a bipartite graph $B(X, Y)$ such that $D(X) = S$ and $D(Y) = T$ and also the minimum order of different types are obtained for such graphs.

2. RESULTS

First, we restate a result of [2] for bipartite graphs.

Theorem 1. *Let $S = \{a_1, a_2, \dots, a_n\}$, $n \geq 1$, be a set of integers with $1 \leq a_1 < a_2 < \dots < a_n$. Then there exists a tree (i.e., connected, bipartite, acyclic graph) having degree set S if and only if $a_1 = 1$. Moreover, if $a_1 = 1$, then $\mu(a_1, a_2, \dots, a_n) = \sum_{i=1}^n (a_i - 1) + 2$.*

Notice that every even cycle C_{2m} , $m \geq 2$, is bipartite with $D(C_{2m}) = \{2\}$ and moreover, $\mu(2) = 4$. In the following result, we obtain the minimum order of a connected unicyclic bipartite graph.

Theorem 2. Let $S = \{a_1, a_2, \dots, a_n\}$, $n \geq 2$, be a set of integers with $1 \leq a_1 < a_2 < \dots < a_n$. Then there exists a connected unicyclic bipartite graph B with $D(B) = S$ if and only if either (a) or (b) below holds:

- (a) For $n = 2$, $a_1 = 1$ and $a_2 \geq 3$. Moreover, $\mu(S) = 4(a_2 - 1)$ if $a_1 = 1$ and $a_2 \geq 3$.
- (b) For $n \geq 3$, $a_1 = 1$. Furthermore, if $a_1 = 1$, then

$$\mu(S) = \begin{cases} 3a_2 + a_3 - 4 & \text{if } n = 3, \\ 2a_2 + a_3 + a_4 - 4 & \text{if } n = 4, \\ \sum_{i=2}^n (a_i - 1) & \text{if } n \geq 5. \end{cases}$$

Proof. Suppose B is a connected unicyclic bipartite graph with $D(B) = S$. Then B is not an even cycle. Notice that every connected unicyclic bipartite graph other than an even cycle has at least one vertex of degree 1, and in addition, there exists no connected, unicyclic bipartite graph whose degree set is $\{1, 2\}$. This implies that $a_1 = 1$ and $a_2 \geq 3$.

We prove the converse for $n \geq 2$. For $n = 2$, assume $S = \{a_1, a_2\}$ is a set of integers with $1 = a_1 < a_2$, and $a_2 \geq 3$. We observe that a graph $F_m = C_{2m} \circ \overline{K}_{a_2-2}$ for some $m \geq 2$, is connected unicyclic bipartite and has the degree set $\{1, a_2\}$. In fact, since F_2 is the smallest such graph satisfying the required property, $\mu(S) = 4(a_2 - 1)$. However, (b) can be proved by observing that any unicyclic connected bipartite graph B is an even cycle with trees attached to its vertices. Further, such a B can be replaced by B^* which is an even cycle with stars attached at its vertices except at one vertex where a caterpillar is attached. Then for a B^* to have the minimum order, it can be shown as follows.

Suppose $S = \{a_1, a_2, \dots, a_n\}$, $n \geq 3$, is the set of integers with $1 = a_1 < a_2 < \dots < a_n$. If $3 \leq n \leq 5$, then we shall construct a connected unicyclic bipartite graph B with the smallest possible girth 4 and $D(B) = S$ as follows: Consider the cycle C_4 whose vertex set is $\{v_1, v_2, v_3, v_4\}$ and distinguish three cases depending upon n .

1. $n = 3$. Then adjoin $a_2 - 2$, $a_3 - 2$, $a_2 - 2$ and $a_2 - 2$ end-edges, respectively, at v_1, v_2, v_3 and v_4 . Thus, $\mu(S) = 3a_2 + a_3 - 4$.
2. $n = 4$. Then as above, adjoin $a_2 - 2$, $a_3 - 2$, $a_4 - 2$, and $a_2 - 2$ end-edges, respectively, at v_1, v_2, v_3 and v_4 . In this case, $\mu(S) = 2a_2 + a_3 + a_4 - 4$.
3. $n = 5$. Then adjoin $a_2 - 2$, $a_3 - 2$, $a_4 - 2$, and $a_5 - 2$ end-edges, respectively, at v_1, v_2, v_3 and v_4 . So, $\mu(S) = \sum_{i=2}^5 (a_i - 1)$, where $a_1 = 1$.

Now, assume by induction on n that the result is true for all $n \leq m$. Let $n = m + 1$. Then $S = S' \cup \{a_{m+1}\}$, where $S' = \{a_1, a_2, \dots, a_m\}$ with $1 = a_1 < a_2 < \dots < a_m$. By the induction hypothesis, there exists the smallest connected unicyclic,

bipartite graph $B(a_m)$ whose degree set is S' , and $\mu(S') = \sum_{i=1}^m (a_i - 1)$. Finally, a connected unicyclic bipartite graph $B(a_n)$ of the smallest order is obtained from $B(a_m)$ by adjoining just $a_{m+1} - 1$ new end-edges at any vertex of degree 1 in $B(a_m)$. The order of this graph $B(a_n)$ is certainly $\sum_{i=1}^{m+1} (a_i - 1)$, and its degree set is S .

Now, suppose that B is any connected unicyclic bipartite graph with p vertices and q edges such that $D(B) = S = \{a_1, a_2, \dots, a_n\}$ with $a_1 = 1 < a_2 < \dots < a_n$ and $n \geq 2$. Necessarily, B contains at least the following facts depending on n :

- i. When $n = 2$. 4 vertices of degree a_2 , and $(p - 4)$ vertices of degree $a_1 = 1$.
- ii. When $n = 3$. 3 vertices of degree a_2 , one vertex of degree a_3 and $p - 4$ vertices of degree $a_1 = 1$.
- iii. When $n = 4$. 2 vertices of degree a_2 , one vertex of degree a_i for $3 \leq i \leq 4$, and $p - 4$ vertices of degree $a_1 = 1$.
- iv. When $n \geq 5$. One vertex of degree a_i , for $2 \leq i \leq n$, and $p - n + 1$ vertices of degree $a_1 = 1$.

Since the sum of the degrees of the vertices of B is $2q$, and $p = q$ for a connected unicyclic bipartite graph, we have

$$2p = 2q \geq \begin{cases} 4a_2 + (p - 4) & \text{if } n = 2, \\ 3a_2 + a_3 + (p - 4) & \text{if } n = 3, \\ 2a_2 + a_3 + a_4 + (p - 4) & \text{if } n = 4, \\ \sum_{i=2}^n a_i + (p - n + 1) & \text{if } n \geq 5 \end{cases}$$

or

$$p \geq \begin{cases} 4(a_2 - 1) & \text{if } n = 2, \\ 3a_2 + a_3 - 4 & \text{if } n = 3, \\ 2a_2 + a_3 + a_4 - 4 & \text{if } n = 4, \\ \sum_{i=2}^n (a_i - 1) & \text{if } n \geq 5. \end{cases}$$

Therefore, the minimum order of a connected unicyclic bipartite graph B having degree set B is

$$\mu(S) = \begin{cases} 4(a_2 - 1) & \text{if } n = 2, \\ 3a_2 + a_3 - 4 & \text{if } n = 3, \\ 2a_2 + a_3 + a_4 - 4 & \text{if } n = 4, \\ \sum_{i=2}^n (a_i - 1) & \text{if } n \geq 5. \end{cases}$$

■

For any nonempty subset U of $V(G)$, $D(U)$ denotes the set of degrees of vertices in U . Thus, the degree set of a bipartite graph B with a bipartition (X, Y) is the set $D(B)$ which is the union of the sets of degrees in X and Y , i.e., $D(B) = D(X) \cup D(Y)$.

Lemma 3. *For any given positive integer n , there exists a complete bipartite graph B with bipartition (X, Y) such that $D(B) = \{n\}$ if and only if $n = |X| = |Y|$.*

Proof. The proof is obvious and hence it is omitted. ■

Given a pair (S, T) of finite, nonempty sets having positive integers, $\mu(S \cup T) = \min\{|G| : G \in \mathcal{G}\}$, where $\mathcal{G}$ is the family of all bipartite graphs G with $D(G) = (S \cup T)$. The following theorem shows the existence of a bipartite graph $B(X, Y)$ such that $D(X) = S$, $D(Y) = T$, and also provides the minimum of such graph when it is connected.

Theorem 4. *Let $S = \{a_1, a_2, \dots, a_n\}$ and $T = \{b_1, b_2, \dots, b_n\}$, $n \geq 1$ be sets of integers with $1 \leq a_1 < a_2 < \dots < a_n$, and $1 \leq b_1 < b_2 < \dots < b_n$. Then there exists a bipartite graph $B(X, Y)$ such that $D(X) = S$, and $D(Y) = T$. Furthermore, the bipartite graph $B(X, Y)$ is connected if and only if the minimum order $\mu(S \cup T) = a_n + b_n$, where $|X| = a_n$ and $|Y| = b_n$.*

Proof. We begin with the trivial fact that $\cup_{i=1}^n (\overline{K}_{a_i} + \overline{K}_{b_i})$ is a bipartite graph which satisfies the required property. In the connected case, we just proceed by induction on n . For $n = 1$, we observe that a graph $G(a_1; b_1) = \overline{K}_{a_1} + \overline{K}_{b_1}$ is the appropriate bipartite graph with bipartition (Y, X) such that $D(X) = \{a_1\} = S$, and $D(Y) = \{b_1\} = T$. Moreover, it has the required property. For $n = 2$, construct a bipartite graph H with bipartition (Y, X) as under: Let $Y = U_1 \cup U_2$, $X = V_1 \cup V_2$ with $U_1 \cap U_2 = \emptyset$, $V_1 \cap V_2 = \emptyset$, $|U_1| = a_1$, $|U_2| = a_2 - a_1$, $|V_1| = b_1$, $|V_2| = b_2 - b_1$, and let each vertex of U_1 is joined to each vertex of V_1 and V_2 , and further each vertex of U_2 is joined to each vertex of V_1 . Then H is certainly a connected bipartite graph with the required property: $D(X) = S = \{a_1, a_2\}$, $D(Y) = T = \{b_1, b_2\}$ and $|X| = |V_1| + |V_2| = b_2$, $|Y| = |U_1| + |U_2| = a_2$, so that $|H| = |X| + |Y| = a_2 + b_2$.

Let $n \geq 3$. Assume for any two sets S and T , each containing m positive integers, where $3 \leq m < n$, the result is true; in other words, by induction hypothesis, there exists a connected bipartite graph $B_1(Y_1, X_1)$ such that $|X_1| = b_{n-1} - b_1$, and $|Y_2| = a_{n-1} - a_1$, where

$$\begin{aligned} D(X_1) &= \{a_i - a_1 : 2 \leq i \leq n-1\}, \\ D(Y_1) &= \{b_j - b_1 : 2 \leq j \leq n-1\}. \end{aligned}$$

To complete the inductive step, we construct the desired connected bipartite graph K as follows: Firstly, we consider the graph $B_1(Y_1, X_1)$, and add a new complete bipartite graph $B_2(Y_2, X_2)$ (where $|X_2| = b_1$ and $|Y_2| = a_1u$) by joining every vertex of X_1 to each vertex of Y_2 , and every vertex of X_2 to every vertex of Y_1 . To this resulting graph, further we add another new graph $\overline{K}_{a_n-a_{n-1}} \cup \overline{K}_{b_n-b_{n-1}}$, by joining every vertex of X_2 to every vertex of $\overline{K}_{a_n-a_{n-1}}$, and every vertex of Y_2 to every vertex of $\overline{K}_{b_n-b_{n-1}}$. This results a bipartite graph K of order $a_n + b_n$ which satisfies the required property. The result now follows by the principle of mathematical induction.

Next, let G be any bipartite graph with bipartition (M, N) such that $D(M) = S$ and $D(N) = T$. Suppose G is connected. Since a_n and b_n are the largest elements of S and T , respectively, there exists at least one pair $(x, y) \in M \times N$ such that $\deg(x) = a_n$ and $\deg(y) = b_n$. Hence $|M| + |N| \geq a_n + b_n = |X| + |Y|$. Thus, $B(X, Y)$ has the minimum order in the class of all bipartite graphs $G(M, N)$ such that $D(M) = S$ and $D(N) = T$. On the other hand, we have already constructed above an appropriate bipartite graph of order $a_n + b_n$ which holds the desired property. Thus, the minimum order of the connected bipartite graph $B(X, Y)$ is precisely $a_n + b_n$.

Conversely, suppose the minimum order of $B(X, Y)$ is $a_n + b_n$, where $|X| = b_n$, and $|Y| = a_n$. Since for some $x \in X$, $\deg_B(x) = a_n = |Y|$, and for some $y \in Y$, $\deg_B(y) = b_n = |X|$, $B(X, Y)$ is obviously connected. ■

The proof of the proceeding theorem also provides the following result.

Corollary 5. Let $S = \{a_1, a_2, \dots, a_n\}$ and $T = \{b_1, b_2, \dots, b_n\}$, $n \geq 1$ be sets of different positive integers such that $a_1 < a_2 < \dots < a_n$, and $b_1 < b_2 < \dots < b_n$. Then there exists a connected, bipartite graph $B(X, Y)$ of order $a_n + b_n$ such that $D(X) = S$ and $D(Y) = T$, where $|X| = b_n$ and $|Y| = a_n$.

Corollary 6. Let $S = \{a_1, a_2, \dots, a_n\}$, $n \geq 1$, be a set of integers with $1 \leq a_1 < a_2 < \dots < a_n$. Then there exists a connected, bipartite graph B with bipartition (X, Y) such that $D(X) \neq D(Y)$, and $D(B) = S$. Moreover, the minimum order of B with $D(B) = S$ is

$$\mu(S) = \begin{cases} a_{\frac{n}{2}} + a_n & \text{if } n \text{ is even,} \\ a_{\lceil \frac{n}{2} \rceil} + a_n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If $n = 2m$ for some $m \geq 1$, then we have two sets R and T of different positive integers of the same cardinalities, where $R = \{a_1, a_2, \dots, a_m\}$, and $T = \{a_{m+1}, a_{m+2}, \dots, a_n\}$.

Otherwise, we consider two sets of positive integers of the same cardinalities as follows: $R = \{a_1, a_2, \dots, a_{m+1}\}$, and $T = \{a_1, a_{m+2}, \dots, a_n\}$.

In either case, the result follows by the direct application of Theorem 4 on R and T . ■

Corollary 7. *Let $S = \{a_1, a_2, \dots, a_n\}$, $n \geq 1$, be a set of integers with $1 \leq a_1 < a_2 < \dots < a_n$. Then there exists a bipartite graph B such that $D(B) = S$. Furthermore, $B(X, Y)$ is a connected, bipartite graph such $D(X) = D(Y) = S$ if and only if its minimum order is $2a_n$ so that $|X| = |Y| = a_n$.*

Proof. Let $S = T = \{a_1, a_2, \dots, a_n\}$, $n \geq 1$, be a set of integers with $1 \leq a_1 < a_2 < \dots < a_n$. The result immediately follows by the application of Theorem 4. ■

The immediate consequence of the above corollary is the following result:

Corollary 8. *Let $S = \{a_1, a_2, \dots, a_n\}$, $n \geq 1$, be a set of integers with $a_1 < a_2 < \dots < a_n$. Then there exists a connected, bipartite graph $B(X, Y)$ of order $2a_n$ such that $D(X) = D(Y) = S$, where $|X| = |Y| = a_n$.*

3. CONCLUSION

- 1 For arbitrary set S of positive integers, the problem of determining a bicyclic, tricyclic and in general multi-cyclic bipartite graph that satisfies the property stated in Theorem 2 is open.
- 2 For arbitrary sets S and T of positive integers with different cardinalities, the problem of determining a bipartite graph that holds the property stated in Theorem 4 is open.

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