Discussiones Mathematicae
Graph Theory 34 (2014) 331–352
doi:10.7151/dmgt.1740

A DECOMPOSITION OF GALLAI MULTIGRAPHS

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Abstract

An edge-colored cycle is *rainbow* if its edges are colored with distinct colors. A *Gallai* (multi)graph is a simple, complete, edge-colored (multi)graph lacking rainbow triangles. As has been previously shown for Gallai graphs, we show that Gallai multigraphs admit a simple iterative construction. We then use this structure to prove Ramsey-type results within Gallai colorings. Moreover, we show that Gallai multigraphs give rise to a surprising and highly structured decomposition into directed trees.

Keywords: edge coloring, Gallai multigraph.

2010 Mathematics Subject Classification: 05C15.

1. Introduction

We assume throughout that all multigraphs are simple (no loops), complete (each pair of vertices is connected by at least one edge), finite, and edge-colored. We treat graphs as a special type of multigraph in which no pair of vertices is connected by more than one edge. Define $\overline{uv}$ to be the set of colors present on edges between vertices u and v. Call an edge-colored cycle *rainbow* if its edges are colored distinctly.

Tibor Gallai [10] showed that colored complete graphs lacking rainbow triangles can be disconnected by the removal of two colors and as a corollary, gave an elegant iterative construction of all such graphs. For this reason, we call a (multi)graph lacking rainbow triangles *Gallai.* Gyárfás and Simonyi [12] considered the problem of finding monochromatic stars and spanning trees in Gallai graphs while Ball, Pultr, and Vojtěchovský [2] characterized Gallai graphs in which each triangle contains precisely two colors. Mubayi and Diwan [3] also studied Gallai graphs but from the perspective of edge density in the different colors.

It follows from a simple inductive argument that Gallai multigraphs lack rainbow n-cycles for all n. Precious little progress has been made toward understanding edge-colored graphs lacking rainbow n-cycles for a fixed n. Ball *et al.* [2] gave algebraic results about the sequence (n : G lacks rainbow n-cycles) as a monoid, and Vojtěchovský [13] extended the work of Alexeev [1] to find the densest arithmetic progression contained in this sequence. In the other direction, Frieze and Krivelevich [5] showed that there is a constant c such that any edge-coloring of K_n in which no color appears more than cn times contains rainbow cycles of length k for all $3 \leq k \leq n$. We hope that a deeper study of Gallai multigraphs might shed some light on rainbow C_n-free colorings.

We show in Section 2 that Gallai multigraphs have practically the same iterative construction as Gallai graphs, with the only difference being an additional criterion for multiedge creation. The key observation powering this construction is that, like Gallai graphs, any Gallai multigraph can be disconnected by the removal of at most two colors and thus can be decomposed into components connected by at most two colors. We also provide a characterization of Gallai multigraphs that are maximal in the sense of edge addition.

After extending the previously known decomposition of Gallai graphs to the multigraph case, we use this structure in Section 3 to aid in considering the Ramsey problem within Gallai multigraphs. In particular, we prove Ramsey-type results for finding small monochromatic paths and cycles in Gallai multigraphs. A survey of this type of problem can be found in [8] with an updated version maintained at [9].

In Section 4, we explore an alternative and surprising decomposition of Gallai multigraphs into highly structured directed trees. To do this, we first describe the structure of a Gallai multigraph G by defining a sequence $M_n(G) = (\mathcal{V}_n, \mathcal{E}_n, \mathcal{A}_n)$ of edge-colored mixed graphs. We then show that the final iteration of this sequence applied to a subgraph H of G breaks into weak components that are rooted trees.

We close with Section 5 by suggesting possible extensions of the ideas contained in this work.

1.1. Basic notation

We denote vertices using lowercase letters such as u, v, and w, sets of vertices using uppercase letters from the end of the alphabet such as U, V, and W, and colors using uppercase letters from the beginning of the alphabet such as A, B, and C. Given two sets of vertices, U and V, we write UV for the set of edges connecting vertices of U to vertices of V. This notation will also be used with singletons, u and v, to refer to the edges connecting u and v. We extend our notation $\overline{uv}$ and denote the set of colors present in a set of edges, say UV, by writing $\overline{UV}$ when there is no risk of ambiguity. Otherwise we refer explicitly to the coloring at hand, i.e. $\rho[UV]$. If $\overline{UV} = \{A\}$, we will often shorten notation by writing $\overline{UV} = A$.

Many of our results will be stated in terms of mixed graphs. A *mixed graph* is a triple $M = (V, E, A)$ with vertices V, undirected edges E, and directed edges A. We say M is *complete* if every pair of distinct vertices is connected by exactly one directed or undirected edge. The *weak components* of a directed graph are the components of the graph that results from replacing each directed edge with an undirected edge. For our purposes, the *weak components* of a mixed graph $M = (V, E, A)$ will be the weak components of the directed graph (V, A). Note that this notion of component disregards undirected edges.

We use the term *rooted tree* to refer to a directed graph that is transitive and whose transitive reduction forms a tree in the usual sense. If (V, A) is a rooted tree, then its *root*, written 1_V, is the unique vertex having the property that there is a directed edge from 1_V to every other vertex in V.

2. Constructions of Gallai Graphs and Multigraphs

Implicit in his seminal work on transitively orientable graphs, Gallai [10] proved that every Gallai graph contains a set of at most two colors that, when removed, disconnects the graph.

Lemma 1 [10]**.** *If G is a Gallai graph having more than one vertex, then G can be disconnected by the removal of two colors.*

It follows easily from Lemma 1 that the following construction yields all Gallai graphs. Let $\mathcal{G}$ be the family of graphs defined inductively by:

1. The single vertex graph is in $\mathcal{G}$.
2. Fix colors A and B and graphs $\{G_i : 1 \leq i \leq t\} \subseteq \mathcal{G}$. For each $1 \leq i \neq j \leq t$, connect G_i and G_j by either A or B. The resulting graph is in $\mathcal{G}$.

Theorem 2 [10]**.** *$\mathcal{G}$ is the family of all Gallai graphs.*

Lemma 1 is the key to Gallai's construction. Notice that the task of disconnecting a Gallai multigraph by the removal of colors becomes inherently more difficult with the addition of multiedges. Nonetheless, as we show in Lemma 3, the removal of two colors still suffices to disconnect any Gallai multigraph.

Lemma 3. *If M is a Gallai multigraph having more than two vertices, then M can be disconnected by the removal of two colors.*

Once we have established Lemma 3, the following construction is easily seen to yield all Gallai multigraphs.

Let $\mathscr{M}$ be the family of multigraphs defined inductively by:

1. Any multigraph with fewer than three vertices is in $\mathscr{M}$.
2. Fix colors A and B and graphs $\{M_i : 1 \leq i \leq t\} \subseteq \mathscr{M}$. For each $1 \leq i \neq j \leq t$, connect M_i and M_j by either A or B. If $|M_i| = |M_j| = 1$, we may also connect M_i and M_j by both A and B. The resulting multigraph is in $\mathscr{M}$.

Theorem 4. *$\mathscr{M}$ is the family of all Gallai multigraphs.*

Proof. It is clear that the construction does not introduce rainbow triangles.

Suppose then that M is a Gallai multigraph having three or more vertices and let A and B be colors whose removal disconnects M into $\{M_i : 1 \leq i \leq t\}$ for $t \geq 2$.

For $1 \leq i \neq j \leq t$, we argue that M_i and M_j must be connected by A or B and not both. Let u_1u_2 be an edge in M_i colored neither A nor B. For each v in M_j, to avoid a rainbow triangle, the colors of u_1v and u_2v must agree. Since these edges span two distinct components, their colors must be A or B. By definition each pair of vertices in M_i is connected by a path whose edge-colors fall outside A and B. It thus follows that v is connected to all of M_i by either A or B. Repeating this argument for every v in M_j and then reversing the roles of M_i and M_j, we have that M_i and M_j must be connected by all A or all B. Hence, for $1 \leq i \neq j \leq t$, M_i and M_j must be connected by a single color A or B except in the case when $|M_i| = |M_j| = 1$. Since we now allow multiedges, it is also possible that the single vertices of M_i and M_j are connected by both A and B.

Thus M can be constructed from smaller Gallai multigraphs in line (2) of the construction. ■

***Proof of Lemma* 3.** Let G be a Gallai multigraph with at least three vertices and fix any vertex $v \in V(G)$. We argue that either

1. v and G are connected by at most two colors (and thus their removal disconnects G with v as one component) or

2. any colors whose removal disconnects $G - v$ also suffice to disconnect G.

If G has only three vertices, the claim is easily checked. Suppose then that $|V(G)| \geq 4$ and that v is connected to G by at least three colors.

Case 1. $G - v$ can be disconnected by the removal of a single color A. Let $G_1, \ldots, G_k$ be the remaining components of $G - v$ upon removal of the color A. By assumption, there are vertices u and w in $V(G - v)$ and colors B and C, distinct from A, such that $B \in \overline{uv}$ and $C \in \overline{wv}$ (it could happen that $u = w$). To avoid a rainbow triangle, it must be the case that u and w fall in the same component of $G - v$, say G_1, and that v is connected to each of the remaining components by only the color A. Thus the removal of edges of color A disconnects G into components $G_1 + v, G_2, \ldots, G_k$.

Case 2. $G - v$ requires the removal of two colors, say A and B, to be disconnected. Let $G_1, \ldots, G_k$ be the components of $G - v$ upon removal of colors A and B. Note that each pair of distinct components is connected by a single color A or B. If v is connected to one of these components, say G_1, by no color other than A or B, then the removal of colors A and B disconnects G with G_1 as one of the components. Otherwise, v must be connected to some vertex in G_i by a third color C_i distinct from A and B for each $1 \leq i \leq k$. Moreover, to avoid a rainbow triangle, it must be the case that $C_i = C_j$ for each $1 \leq i, j \leq k$. Let C be this common color.

To avoid rainbow triangles, every edge incident with v must be colored A, B, or C. Thus we may select $u \in V(G - v)$ such that $A \in \overline{vu}$. Without loss of generality, we may assume $u \in G_1$. Since $G - v$ cannot be disconnected by the removal of just the color A, there is another component, say G_2, such that G_1 and G_2 are connected by B. Fix $w \in G_2$, such that $\overline{vw} = C$. The vertices u, v, w now form a rainbow triangle.

It must then be the case that v and G_i are connected by just the colors A and B for some i and thus G can be disconnected by the removal of A and B. ■

We call uv *isolated* if for every $w \notin \{u, v\}$, $\overline{uw} = \overline{vw}$ and $|\overline{uw}| = 1$. Notice that if uv is isolated we can reduce the multigraph by collapsing the edge(s) uv. Likewise, given any multigraph, we can arbitrarily introduce new isolated edges without introducing rainbow triangles. We therefore call a multigraph *reduced* if it contains no isolated edges.

We call the (multi)edge uv *maximal* if no new color can be added to $\overline{uv}$ without introducing a rainbow triangle. Here we allow the possibility that uv has "all possible colors" and thus is maximal. Likewise, a multigraph with coloring ρ is *maximal* if uv is maximal for all vertices u and v. Note that, in order for a multigraph to be maximal, it must either be a singleton or contain at least three vertices. With these definitions, a very similar result can be obtained for maximal (reduced) Gallai multigraphs. Also, if a multigraph is maximal, it is also necessarily reduced. Let $\mathscr{M}$ be the set of Gallai multigraphs obtained from the following process:

1. A single vertex is a (trivially) maximal Gallai multigraph.

2. Let $M_1, M_2, \ldots, M_t$ be a set of maximal Gallai multigraphs such that $\sum |M_i| \geq 3$ and let A and B be two colors. Let M be the graph obtained from $\bigcup M_i$ by adding all edges from M_i to M_j in either color A or color B such that if $|M_i| = |M_j| = 1$, then there exists a set M_k with $E(M_i, M_k) \neq E(M_j, M_k)$. Between every two sets M_i and M_j satisfying $|M_i| = |M_j| = 1$ so $\{u\} = M_i$ and $\{v\} = M_j$, we insert the multiedge uv in both A and B.

Theorem 5. *The set $\mathscr{M}$ defined above is the set of all maximal* (*reduced*) *Gallai multigraphs.*

Proof. It is clear that this construction must produce a maximal Gallai multigraph.

Suppose then that M is a maximal Gallai multigraph with at least 3 vertices. By Theorem 4 there exists a partition of M into $M_1, M_2, \ldots, M_t$ such that there exist two colors A and B with the following properties:

- If $|M_i| = |M_j| = 1$, then the multiedge between M_i and M_j has one or both of the colors A and B.

- If $|M_i| \geq 2$ or $|M_j| \geq 2$, then either all edges between M_i and M_j are A, or they are all B.

Choose such a partition with the most pieces and call such a partition a *supreme* Gallai partition. Such a partition has no set M_i with $|M_i| = 2$ since otherwise the multiedge within M_i can have any number of colors, making M not maximal.

Claim 6. *If M is maximal, then every piece of a supreme Gallai partition is also maximal.*

Proof. Consider a maximal Gallai partition M, and suppose some piece M_i of a supreme Gallai partition is not maximal. Then there exist vertices u and v in M_i such that a color A can be added to uv without producing a rainbow triangle within M_i while simultaneously forming a rainbow triangle uvw for some w in $M - M_i$. Thus uw and vw have distinct colors B and C. However, this contradicts Theorem 4 since edges between pieces must be the same color, provided one of the pieces has more than one vertex. □

By Claim 6, all pieces $M_1, \ldots, M_t$ are maximal. By the maximality of these pieces, we cannot add any additional edges within any piece. Furthermore, by Theorem 4, we cannot add any more colors between any two pieces so long as one of them has size larger than 1. Thus the only thing we must ensure is that both colors A and B are present between every pair of pieces consisting of one vertex each. By the properties of a supreme Gallai partition, the presence of such edges in M would never create a rainbow triangle. Furthermore, by the maximality of M, both colors A and B must appear between any two singleton pieces of M. ■

In looking at maximal Gallai multigraphs, one may wonder if a colored (non multi)graph could possibly be a maximal Gallai multigraph. It turns out this is not the case, as seen in the following proposition.

Proposition 7. *Every maximal Gallai multigraph on at least three vertices has at least three multiedges which form two monochromatic triangles on the same three vertices.*

Proof. Suppose there exists a maximal Gallai multigraph with no multiedge present. Let G be the smallest such example and consider a supreme Gallai partition of G. Since G is maximal and $|G| > 1$, we actually get $|G| \geq 3$. Our goal is to show that there is a Gallai partition of either G or a subgraph of G which contains at least three parts of order 1. Certainly if this supreme Gallai partition of G contains at least three parts of order 1, we're done so suppose not. Then there must exist a part H of order at least 3 and we may then consider a supreme Gallai partition of H. Since $|G|$ is finite, such a process must terminate and thus, must produce three parts of order 1.

Suppose A and B are the colors used in this Gallai partition. Then, between every pair of vertices in our parts of order 1, there must be edges of both A and B since G is maximal. Thus, there are at least three multiedges in G forming two monochromatic triangles. ■

3. Multigraph Gallai-Ramsey Numbers

Recently, there has been increasing interest in the area of Gallai-Ramsey theory. In its general form, let $gr_k(G : H)$ be defined as the smallest integer n such that every k-coloring of a complete graph on n vertices contains either a rainbow G or a monochromatic H, where there exists an example on $n-1$ vertices with no such subgraph. Since these numbers are bounded from above by the classical Ramsey number for finding a monochromatic H, they must exist.

Let the *multigraph Gallai-Ramsey number* $mgr_k(G:H)$ be the smallest integer n such that every (reduced) maximal rainbow G-free multigraph on n vertices contains a monochromatic H where there exists an example on $n-1$ vertices with no such subgraph. Since every such multigraph contains a complete graph, these numbers are bounded from above by the aforementioned Gallai-Ramsey numbers and thus exist. By Proposition 7, it follows immediately that $mgr_k(K_3:K_3) = 3$ whereas $gr_k(K_3:K_3) = 5^{k/2}$ when k is even (and a similar number when k is odd, see [11]) so we already see that $mgr \neq gr$ in this case. In general, by the above observation, we easily get the following relationship between mgr and gr.

Fact 8. For any graphs G and H and for any integer k,

$$mgr_k(G:H) \leq gr_k(G:H).$$

The general behavior of the function $gr_k(G:H)$ when $G = K_3$ was established in [11] by Gyárfás, Sárközy, Sebő and Selkow.

Theorem 9 [11]**.** *Let H be a fixed graph with no isolated vertices. If H is not bipartite, then $gr_k(K_3:H)$ is exponential in k. If H is bipartite but not a star, then $gr_k(K_3:H)$ is linear in k.*

Note that if H is a star, the Gallai-Ramsey number is simply a constant depending only on the size of the star and not the number of colors. A similar result holds for maximal multigraphs.

Corollary 10. *Let H be a fixed graph with no isolated vertices. If H is not bipartite and not K_3, then $mgr_k(K_3:H)$ is exponential in k. If H is bipartite but not a star, then $mgr_k(K_3:H)$ is linear in k.*

Proof. The upper bounds follow from Theorem 9 and Fact 8. The case when $H = K_3$ is established by the observation above so suppose H is not bipartite and not K_3 and consider the following inductive construction. Let G_2 be the graph on three vertices consisting of an edge in each of colors 1 and 2 between every pair of vertices. To construct G_{i+1}, make two copies of G_i and insert all edges in color $i+1$ between these copies. It can be easily shown that this colored multigraph is maximal, reduced and contains no monochromatic H.

Next suppose H is bipartite and not a star. Letting $\alpha(H)$ denote the size of the largest independent set of vertices in H, let $a = |H| - \alpha(H)$. Since H is not a star, we see that $a \geq 2$. Then consider the following inductive construction.

Again let G_2 be the graph on three vertices consisting of an edge in each of colors 1 and 2 between every pair of vertices. To construct G_{i+1}, add $a-1$ vertices to G_i with all new edges having color $i+1$. It is easy to see that this construction is maximal, reduced and contains no monochromatic H. ■

For the next two results, we will use the following construction as a sharpness example. Let G_2 contain three vertices which induce a monochromatic triangle in each of colors 1 and 2. We then build G_{i+1} from G_i by adding a new vertex with all new edges from this vertex to G_i having color $i+1$. This multigraph certainly contains no rainbow triangle and no monochromatic C_4 or monochromatic path of order at least 4. It is also easy to see that this multigraph is maximal as no edge can be added within this structure in any color without creating a rainbow triangle.

Next we consider mgr for a monochromatic C_4. A study of this problem for general cycles in the graph context can be found in [4, 7]. In particular, in [4], it was shown that $gr_k(K_3 : C_4) = k + 4$. By Fact 8, we immediately see that $mgr_k(K_3 : C_4) \leq k + 4$ but it turns out that this bound is not sharp as seen in our next result.

Theorem 11. *For $k \geq 3$, we have $mgr_k(K_3 : C_4) = k + 2$.*

Proof. For the lower bound, consider the construction of G_k above. Since $|G_k| = k + 1$, this shows that $mgr_k(K_3 : C_4) > k + 1$.

Let M be a maximal Gallai multigraph on $k + 2$ vertices and suppose M contains no monochromatic C_4. By Proposition 7, there exists a set of three vertices T which induce a monochromatic triangle in two different colors, say red and blue. Let v be a vertex in $G - T$. In order to avoid a monochromatic C_4, v must have at least one edge of a color i other than red or blue to T. Then, to avoid a rainbow triangle, v must have all edges in color i to T. Next suppose $w \neq v$ is another vertex in $G - T$. Similarly, all edges between w and T must have color j for some j other than red and blue but also, if $i = j$, then there is a monochromatic C_4 using v, w and two vertices of T. Thus, each vertex of $G-T$ must have a unique color, distinct from red and blue, to T. Since there are only k colors available, there must be at most $k-2$ vertices outside T for a total of at most $k+1$ vertices in G, a contradiction. Thus, $mgr_k(K_3 : C_4) \leq k + 2$, completing the proof. ■

Next we consider some small cases of Gallai multigraph Ramsey numbers for paths. A study of this problem for general paths in the graph context can be found in [4]. In particular, in [4], it was proven that $gr(K_3 : P_4) = k + 3$. By Fact 8, this means that $mgr_k(K_3 : P_4) \leq k+3$ but it turns out this bound is also not the best possible as seen in the following result.

Theorem 12. *For $k \geq 3$, the following hold:*
(1) $mgr_k(K_3 : P_4) = k + 2$,
(2) $mgr_k(K_3 : P_5) = k + 3$,
(3) $mgr_k(K_3 : P_6) = 2k + 2$,
(4) $mgr_k(K_3 : P_7) = 2k + 3$.

Proof. First suppose we are finding a monochromatic P_4. For the lower bound, consider the construction of G_k above. Since $|G_k| = k + 1$, this shows that $mgr_k(K_3 : P_4) > k + 1$. The upper bound follows from Theorem 11 since P_4 is a subgraph of C_4.

For P_5, we consider the same construction as above except redefining G_2 to have 4 vertices which induce a K_4 in each of two colors. G_{i+1} is then constructed from G_i as before. This graph certainly contains no monochromatic P_5 and $|G_k| = k + 2$ so this shows $mgr_k(K_3 : P_5) \geq k + 3$. For the upper bound, let G be a maximal Gallai multigraph on at least $k + 3$ vertices and let T be a double-triangle (say in red and blue) as implied by Proposition 7. For every $v \in G \setminus T$, if v has an edge to T that is not red or blue, say green, then all edges from v to T must be green to avoid a rainbow triangle. Furthermore, there must not exist another vertex in $G \setminus T$ with any green edges to T to avoid creating a green P_5. Finally, there is at most one vertex in $G \setminus T$ with a red or blue edge to T since this vertex would also have all other red and blue edges to T and more than one such vertex would again create a monochromatic P_5. Thus, $|G| \leq 3+(k-2)+1 = k+2$, a contradiction.

For P_6 and P_7, the lower bound is given by a similar construction. Start with G_2 defined to be a set of 5 or 6 respectively vertices inducing a complete graph in both colors 1 and 2. G_{i+1} is then constructed from G_i by adding two vertices with all incident edges using color $i + 1$. This multigraph has $2k + 1$ (respectively $2k + 2$) vertices and certainly contains no rainbow triangle but also contains no monochromatic P_6 or P_7 respectively. It remains to observe that this coloring is also maximal.

For the upper bound for P_6, let G be a maximal Gallai multigraph on at least $2k$ vertices and let T be a double-triangle (say in red and blue) as implied by Proposition 7. For each vertex $v \in G \setminus T$, if v has an edge of a new color (say green) to T, then v must have only green edges to T to avoid a rainbow triangle. In order to avoid producing a monochromatic P_6, there can be at most two vertices in $G \setminus T$ with the same colors on edges to T. There can also be at most 2 vertices with red and blue edges to T. This means that $|G| \leq 3 + 2(k - 2) + 2 = 2k + 1$, a contradiction.

For the upper bound for P_7, let G be a maximal Gallai multigraph on at least $2k + 1$ vertices and let T be a double-triangle (say in red and blue) as implied by Proposition 7. First suppose there is at least one vertex in $G \setminus T$ with at least one red or blue edge to T. Then, to avoid a rainbow triangle, v must have all other red and blue edges to T. To avoid creating a monochromatic P_7, there can be at most 3 such vertices with red or blue edges to T. As in the previous cases, there can be at most 2 vertices in $G \setminus (T \cup \{v\})$ with edges of a single color (other than red or blue) to $T \cup \{v\}$. Thus, $|G| \leq 3 + 3 + 2(k-2) = 2k+2$, a contradiction.

This means we may assume there is no vertex $v \in G \setminus T$ with red or blue edges to T. Then there can be at most 3 vertices in $G \setminus T$ with edges in a single color (other than red or blue) to T to avoid creating a monochromatic P_7. If there are two colors, each with 3 vertices having all edges to T, the edges between these sets of vertices must be in these same two colors to avoid a rainbow triangle but this easily creates a monochromatic P_7. Thus, there can be at most one color i with 3 vertices having all edges in color i to T, all other colors having at most 2 such vertices. This implies that $|G| \leq 3+3+2(k-3) = 2k$, again a contradiction. ■

Note that this proof does not extend immediately to an upper bound for $mgr_k(K_3 : P_8)$ since additional argument would be needed to bound the number of vertices all having green edges to T.

4. Decomposition of Gallai Multigraphs

We now develop an iterative decomposition of Gallai multigraphs into directed trees.

4.1. Basic techniques: maximality and dominance

Let $(G = (V, E), \rho)$ be a Gallai multigraph (recall that ρ is an edge coloring on G). We will in all cases assume that distinct edges connecting the same vertices are colored distinctly. We also think of $V \subseteq \mathbb{N}$ and thus having a natural ordering. We say that $(G = (V, E), \rho)$ is *uniformly colored* if for all $e_i, e_j \in E$, we have $\rho(e_i) = \rho(e_j)$.

Let $(G = (V, E), \rho)$ be a maximal Gallai multigraph. For $u, v \in V$, notice that $|\overline{uv}| \geq 3$ if and only if uv is isolated. Therefore, if G is reduced, $|\overline{uv}| = 1$ or 2 for all $u, v \in V$. Furthermore, if G is not reduced, we can reach a reduced Gallai multigraph by successively collapsing isolated edges of G.

Lemma 13. *Suppose $(G = (V, E), \rho)$ is a maximal Gallai multigraph. If $u, v \in V$ and $A \in \overline{uv}$, then for all $B \notin \overline{uv}$, there is $w \in V - \{u, v\}$ and $C \notin \{A, B\}$ such that either $A \in \overline{uw}$ and $C \in \overline{wv}$ or $C \in \overline{uw}$ and $A \in \overline{wv}$.*

Proof. Note that if $|V| = 2$, then $\overline{uv}$ consists of "all possible colors" and we thus vacuously satisfy any claim about $B \notin \overline{uv}$. Assume then that $|V| \geq 3$.

Since G is maximal and $B \notin \overline{uv}$, we can find $w \neq u, v$ such that u, v, w would form a rainbow triangle if B were to be added to $\overline{uv}$. Thus we may find $X \in \overline{uw}$ and $Y \in \overline{vw}$ such that X, Y, and B are distinct. However, since $A \in \overline{uv}$, $|\{X, Y, A\}| \leq 2$ and thus $A = X$ or $A = Y$. Let C be the other color. ∎

While Lemma 14 will follow from our general decomposition result, Theorem 15, we present it separately here because of its importance in understanding the most basic structure of a maximal reduced Gallai multigraph.

Lemma 14. *The vertices of a reduced maximal Gallai multigraph that are connected by two edges form uniformly colored cliques.*

Proof. Let $(G=(V, E), \rho)$ be a reduced maximal Gallai multigraph. Let $u, v, w \in V$. Suppose $\overline{uv} = \{A, B\}$ and $\overline{vw} = \{C, D\}$. If $\{A, B\} \neq \{C, D\}$, then we find a rainbow triangle no matter the colors of $\overline{uw}$. Suppose then that $\overline{uv} = \overline{vw} = \{A, B\}$. Certainly $\overline{uw} \subseteq \{A, B\}$. Suppose $\overline{uw} = A$. Since $B \notin \overline{uw}$, by Lemma 13, we may find $x \in V - \{u, w\}$ such that, without loss of generality, $A \in \overline{ux}$ and $C \in \overline{wx}$. To avoid a rainbow triangle in v, x, w we must have $\overline{vx} = C$. But this leads to a rainbow triangle in u, x, v. ∎

Our main result given in Theorem 15 is primarily an explanation of how each of these uniformly colored cliques are connected, and the following relation on sets of vertices plays a central role in this analysis. Let $(G = (\mathcal{V}, \mathcal{E}), \rho)$ be a Gallai multigraph. For $U, V \subseteq \mathcal{V}$ disjoint, we say that U *dominates* V and write $U \to V$ if and only if $|\overline{UV}| > 1$ and

1. $U = \{u\}, V = \{v\}$, and $u < v$ or
2. $|U| > 1$ or $|V| > 1$ and for every $u \in U$ and $v \in V$, $\overline{uv} = \overline{uV}$.

Given $U, V \subseteq \mathcal{V}$, we write $\Sigma(U, V)$ for the map from U to the powerset of $\overline{UV}$ defined by $u \mapsto \overline{uV}$. When $U \to V$, $\Sigma(U, V)$ completely describes the relationship between U and V and we call it the *signature of* $U \to V$.

Note that if $U \to V$ and $V \to U$, then every pair of vertices between U and V are connected by the same multiple colors. As we will see, our analysis will not encounter this situation because we will quickly deal only with cases when the vertices of U and V are connected by single edges. We also note the fact that if $U \to V$, then for every $v \in V$, we have $U \to v$.

Given a reduced maximal Gallai multigraph $(G = (\mathcal{V}, \mathcal{E}), \rho)$, we will describe its structure through a sequence of edge-colored mixed graphs $M_n(G) = (\mathcal{V}_n, \mathcal{E}_n, \mathcal{A}_n)$ defined as follows:

1. $\mathcal{V}_0 := \mathcal{V}$,
2. $\mathcal{A}_0 := \{(u, v) \in \mathcal{V}^2 : u \to v\}$, and

3. $\mathcal{E}_0 := \{\{u, v\} \in [\mathcal{V}]^2 : |\rho[uv]| = 1\}$,

and for $n \geq 1$

(1′) $\mathcal{V}_n$ is the partition of $\mathcal{V}$ induced by the weak components of $M_{n-1}(G)$,

(2′) $\mathcal{A}_n := \{(U, V) \in \mathcal{V}_n^2 : U \to V\}$, and

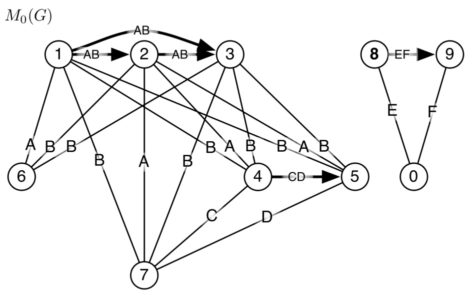
(3′) $\mathcal{E}_n := \{\{U, V\} \in [\mathcal{V}_n]^2 : |\rho[UV]| = 1\}$.

For each n, ρ induces a list edge-coloring, ρ', of $\mathcal{E}_n \cup \mathcal{A}_n$ by $\rho'(e) = \rho[UV]$ where $e = (U, V)$ or $e = \{U, V\}$. Likewise, Σ induces a partition of $\mathcal{A}_n$ by $(U_1, V_1) \sim_\Sigma (U_2, V_2)$ if and only if $\Sigma(U_1, V_1) = \Sigma(U_2, V_2)$. Note that $(U_1, V_1) \sim_\Sigma (U_2, V_2)$ if and only if $U_1 = U_2$ and $\rho[uV_1] = \rho[uV_2]$ for all $u \in U_1$.

Figure (1) shows an example of this sequence for a particular Gallai multigraph. For readability, we show only those edges in $M_n(G)$ that contribute to the formation of directed edges in $M_{n+1}(G)$. The hash marks on the directed edges in $M_1(G)$ indicate whether the signatures agree or disagree. Notice that vertex 8 is bold. In the notation to be introduced immediately proceeding Lemma 18, this particular vertex will be identified as $\mathbf{1}_{V(G)}$ and has the property of being connected to the rest of G by two colors, E and F.

G

	0	1	2	3	4	5	6	7	8	9
0		F	F	F	F	F	F	F	E	F
1			AB	AB	B	B	A	B	E	E
2				AB	A	A	B	A	E	E
3					B	B	B	B	E	E
4						CD	A	C	E	E
5							A	D	E	E
6								A	E	E
7									E	E
8										EF
9										



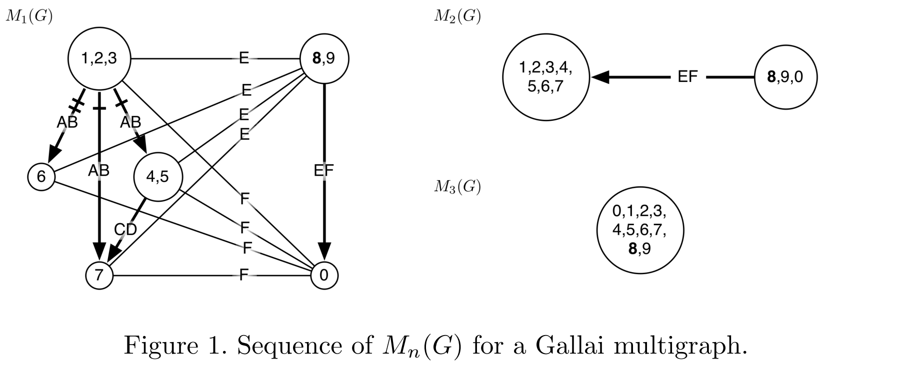


Figure 1. Sequence of $M_n(G)$ for a Gallai multigraph.

4.2. Decomposition of maximal Gallai multigraphs

We may now state our main result.

Theorem 15. *Let G be a reduced maximal Gallai multigraph, H an induced subgraph of G, and $M_n(H) = (\mathcal{V}_n, \mathcal{E}_n, \mathcal{A}_n)$ the sequence described above. Then*
(1) *$M_n(H)$ is complete,*
(2) $|\rho'(e)| = \begin{cases} 1 & \text{if} \quad e \in \mathcal{E}_n, \\ 2 & \text{if} \quad e \in \mathcal{A}_n, \end{cases}$
(3) *the weak components of $M_n(H)$ form rooted trees, and*
(4) *if $(U, V), (V, W) \in \mathcal{A}_n$, then $(U, V) \sim_\Sigma (U, W)$*
for all $n \geq 0$.

For convenience, if $M_k(H)$ has properties (1)–(4) for all $k \leq n$, we will say that H has the *tree property* for n.

4.3. Proof of Theorem 15

Throughout this section, we assume $(G = (\mathcal{V}, \mathcal{E}), \rho)$ is a reduced maximal Gallai multigraph and H an induced subgraph of G.

Lemma 16. *Suppose $U, V, W \subseteq \mathcal{V}$ disjoint, $U \to V$, and $\{A, B\} = \overline{UV}$.*
(1) *If $\overline{UW} = C \notin \{A, B\}$, then $\overline{VW} = C$.*
(2) *If $\overline{VW} = C \notin \{A, B\}$, then either $C \in \overline{UW}$ or $U \to W$ and $\Sigma(U, V) = \Sigma(U, W)$. If we also know that $U \to W, W \to U$, or $|\overline{UW}| = 1$ and we know that U always dominates with the same colors (i.e., whenever $U \to U'$, then $\overline{UU'} = \{A, B\}$), then either $\overline{UW} = C$ or $U \to W$ and $\Sigma(U, V) = \Sigma(U, W)$.*
(2') *If W is a single vertex, we need only require $C \in \overline{VW}$ in (2).*

Proof. (1) Fix $v \in V$ and $w \in W$. Since $U \to V$, we may select $u_A, u_B \in U$ such that $A \in \overline{u_A v}$ and $B \in \overline{u_B v}$. Observe that the triangle w, u_A, v forces $\overline{vw} \subseteq \{A, C\}$ while w, u_B, v forces $\overline{vw} \subseteq \{B, C\}$. Thus $\overline{vw} = C$. Since v and w were arbitrary, $\overline{VW} = C$.

(2) Fix $u \in U, w \in W, v \in V$. Since $\overline{UV} = \{A, B\}$, we are in one of the following cases: $\overline{uv} = A$, $\overline{uv} = B$, or $\overline{uv} = \{A, B\}$. If $\overline{uv} = \{A, B\}$, then the fact that $\overline{vw} = C$ forces $\overline{uw} = C$ and thus $C \in \overline{UW}$. If $\overline{uv} = A$, then $\overline{uw} \subseteq \{A, C\}$ and thus if $C \notin \overline{UW}$, then $\overline{uw} = A$. Likewise, if $\overline{uv} = B$, then either $C \in \overline{uw}$ or $\overline{uw} = B$. We thus have either $C \in \overline{UW}$ or $U \to W$ and $\Sigma(U, V) = \Sigma(U, W)$.

Suppose we also know that we are in one of the following cases:

(i) $U \to W$ and $\overline{UW} = \{A, B\}$,

(ii) $W \to U$, or

(iii) $|\overline{UW}| = 1$.

Again, if $C \notin \overline{UW}$, then we must be in case (i). We would like to conclude that if $C \in \overline{UW}$, then we are in case (iii) and thus $\overline{UW} = C$. Suppose $W \to U$. To avoid a rainbow triangle, $\overline{UW} \subseteq \{A, B, C\}$. In particular, since $|\overline{UW}| \geq 2$, $A \in \overline{UW}$ or $B \in \overline{UW}$. Assume $A \in \overline{UW}$ and fix $w \in W$ such that $A \in \overline{wU}$. We may then choose $u \in U$ and $v \in V$ such that $B \in \overline{uv}$ but now u, v, w is a rainbow triangle. Thus $W \not\to U$ and $\overline{UW} = C$.

($2'$) Let $W = \{w\}$ and $V = \{v\}$. Note that we are very nearly in the original setup of part (2). In particular, we still have $U \to V$, $\overline{UV} = \{A, B\}$, and each pair of vertices between V and W is connected by the color C. The only place where this relaxation might affect the proof in (2) is at the beginning where we consider $\overline{uv} = \{A, B\}$. We now know only that $C \in \overline{VW}$ but it follows immediately that $\overline{VW} = C$ and the proof follows as before. ■

Lemma 17. *H has the tree property for* 0.

Proof. It is clear that $M_0(H)$ is complete. The rest of the claim is essentially a restatement of lemma 14. By the definition of dominance between single vertices, each complete, uniformly colored clique from Lemma 14 becomes a linear ordered set of vertices and thus a rooted tree. In this context, property (4) of Theorem 15 is simply the observation that these cliques are uniformly colored. ■

Before proceeding, we introduce some convenient notation. Elements of $\mathcal{V}_n$ are by definition subsets of $V(H)$. We will however at times want to speak of their structure as rooted trees. For $U \in \mathcal{V}_n$, we write $\Upsilon(U)$ to refer to the set of elements of $\mathcal{V}_{n-1}$ contained in U and 1_U to refer to the root of $\Upsilon(U)$. Notice that $1_U \in \mathcal{V}_{n-1}$ has its own tree structure and thus we may refer to 1_{1_U}, $1_{1_{1_U}}$, etc. We may continue this recursion until we reach a single vertex. We write $\mathbf{1}_U$ to refer to this single vertex. Similarly, for $u \in V(H)$, we write $[u]_n$ to refer to the unique $U \in \mathcal{V}_n$ containing u. Lastly, we point out how this notation fits together. For $U \in \mathcal{V}_n$, $[\mathbf{1}_U]_n = U$, $[\mathbf{1}_U]_{n-1} = 1_U$, $[\mathbf{1}_U]_{n-2} = 1_{1_U}, \ldots$, and $[\mathbf{1}_U]_0 = \mathbf{1}_U$.

We also associate a set of colors with each member of $\mathcal{V}_n$ as follows. For $u \in \mathcal{V}_0$, $\widehat{u} := \cup_{u \to v} \overline{uv}$ and for $U \in \mathcal{V}_n$ with $n > 0$, $\widehat{U} := \widehat{1_U}$. Lemma 18 demonstrates the importance of this notation.

Lemma 18. *Suppose H has the tree property for n and $(U, V) \in \mathcal{A}_{n+1}$. Then U always dominates with the same two colors $\widehat{U}$, i.e.*

(i) $\overline{UV} = \widehat{U}$ *and*

(ii) $|\widehat{U}| = 2$.

Proof. It is clear that $|\widehat{U}| = 2$ since $\widehat{U}$ is defined inductively and dominance between two vertices must be with exactly two colors. Likewise, (i) certainly holds for $U, V \in \mathcal{V}_0$. Since H has the tree property for n and $U \to V$, $|\overline{U'V}| = 1$ for all $U' \in \Upsilon(U)$ and $|\overline{UV}| \geq 2$. Suppose we find $C \in \overline{UV} - \widehat{U}$. Then fix $U_C \in \Upsilon(U)$ such that $\overline{U_C V} = C$. If $U_C = 1_U$, then for each $U' \in \Upsilon(U) - \{1_U\}$, we may apply part (1) of Lemma 16 with $1_U \to U'$, $\overline{1_U U'} = \widehat{U}$, and $\overline{1_U V} = C \notin \widehat{U}$ to conclude that $\overline{U'V} = C$. We thus have $\overline{UV} = C$, a contradiction.

Suppose then that $1_U \neq U_C$ and thus $1_U \to U_C$. By induction, $\overline{1_U U_C} = \widehat{U}$. We may now apply part (2) of Lemma 16 with $1_U \to U_C$, $\overline{1_U U_C} = \widehat{U}$, and $\overline{U_C 1_V} = C \notin \widehat{U}$ to get that either $C \in \overline{1_U 1_V}$ or $1_U \to 1_V$. The former has already been ruled out while the latter contradicts the assumption that 1_U and 1_V were in different components of $\mathcal{V}_n$. ■

The following lemma is useful because it allows us to locate a vertex in U that is connected to the rest of U by only the colors contained in $\widehat{U}$.

Lemma 19. *If H has the tree property for n and $U \in \mathcal{V}_{n+1}$, then $\overline{\mathbf{1}_U U} = \widehat{U}$.*

Proof. By Lemma 18, $|\widehat{U}| = 0$ or 2. Note that $|\widehat{U}| = 0$ if and only if $U = \{\mathbf{1}_U\}$. In this case, $\overline{\mathbf{1}_U U} = \widehat{U} = \emptyset$. We now proceed with the assumption that $|\widehat{U}| = 2$ and thus $|U| \geq 2$. For $n = 0$, U is a nontrivial uniformly colored clique, in which case $\widehat{U}$ is by definition $\overline{\mathbf{1}_U U}$.

For $n \geq 1$, by induction $\overline{1_U \mathbf{1}_{1_U}} = \widehat{1_U} = \widehat{U}$. But since $\mathbf{1}_{1_U} = \mathbf{1}_U$, we have $\overline{1_U \mathbf{1}_U} = \widehat{U}$ and thus $\widehat{U} \subseteq \overline{\mathbf{1}_U U}$. By Lemma 18, $\overline{1_U U'} = \widehat{U}$ for every $U' \in \Upsilon(U) - \{1_U\}$. Finally, given that $\mathbf{1}_U \in 1_U$, we have $\overline{\mathbf{1}_U U'} \subseteq \overline{1_U U'} = \widehat{U}$. Thus $\overline{\mathbf{1}_U U} = \widehat{U}$. ■

Lemmas 20 and 21 will be used in situations where a tree is connected to another tree or vertex by a color not present in the dominating colors of the first tree.

Lemma 20. *Suppose H has the tree property for n, $U, V \in \mathcal{V}_{n+1}$ distinct, and $V' \in \Upsilon(V)$ such that $C \in \overline{UV'} - \widehat{U}$. Then $\overline{UV'} = C$.*

Proof. Suppose that $U' \in \Upsilon(U) - \{1_U\}$ such that $\overline{U'V'} = C$. Applying part (2) of Lemma 16 with $1_U \to U'$ and $\overline{U'V'} = C \notin \overline{1_U U'}$, we get that $\overline{1_U V'} = C$ or $1_U \to V'$. Since 1_U and V' are in different components of $\mathcal{V}_{n+1}$, we must be in the case $\overline{1_U V'} = C$.

For each $U' \in \Upsilon(U) - \{1_U\}$ we may apply part (1) of Lemma 16 with $1_U \to U'$ and $\overline{1_U V'} = C \notin \overline{1_U U'}$ to get that $\overline{U'V'} = C$ and thus $\overline{UV'} = C$. ■

Lemma 21. *Suppose H has the tree property for n, $U \in \mathcal{V}_{n+1}$, and $v \in \mathcal{V}$ such that $\overline{\mathbf{1}_U v} = C \notin \widehat{U}$. Then $\overline{Uv} = C$.*

Proof. First observe that $v \notin U$ since by Lemma 19, $\overline{\mathbf{1}_U U} = \widehat{U}$. Next let k be maximal such that $\overline{[\mathbf{1}_U]_k v} = C$. If $k = n+1$, we are done. Suppose $k < n+1$ and select $u \in [\mathbf{1}_U]_{k+1} - [\mathbf{1}_U]_k$. We may apply (1) of Lemma 16 with $[\mathbf{1}_U]_k \to u$ and

$\overline{[\mathbf{1}_U]_k v} = C \notin \overline{[\mathbf{1}_U]_k u}$ to get that $\overline{uv} = C$. Thus $\overline{[\mathbf{1}_U]_{k+1} v} = C$, which violates the maximality of k. ■

Note that in Lemma 21 we do not require that v be in $V(H)$ but rather in the larger set $\mathcal{V}$.

Lemma 22. *For $n \geq 0$, suppose H has the tree property for n and $U, V \in \mathcal{V}_{n+1}$ such that $\overline{UV} \subseteq \widehat{U} = \widehat{V} = \{A, B\}$ Then the following statements are equivalent.*

(1) *$U \to V$,*

(2) *there is $x \in \mathcal{V} - (U \cup V)$ such that $U \to x$ and $\overline{Vx} = C \notin \{A, B\}$, and*

(3) *$V \not\to U$ and $\overline{UV} = \{A, B\}$.*

Furthermore, when the statements are true, $\Sigma(U, V) = \Sigma(U, x)$.

Proof. ($1 \Rightarrow 2$) We may assume $\overline{1_U 1_V} = A$. Since G is maximal and $B \notin \overline{\mathbf{1}_U \mathbf{1}_V}$, there is $x \in \mathcal{V}$ and $C \notin \{A, B\}$ such that $A \in \overline{\mathbf{1}_U x}$ and $C \in \overline{x \mathbf{1}_V}$ or $C \in \overline{\mathbf{1}_U x}$ and $A \in \overline{x \mathbf{1}_V}$. Notice that if $C \in \overline{\mathbf{1}_U x}$, since $C \notin \widehat{U}$, $|\mathbf{1}_U x| = 1$ and thus $C = \overline{\mathbf{1}_U x}$. Likewise, if $C \in \overline{\mathbf{1}_V x}$, then $C = \overline{\mathbf{1}_V x}$. Furthermore, in either case, since $C \notin \overline{\mathbf{1}_U U} = \overline{\mathbf{1}_V V}$, we must conclude that $x \in \mathcal{V} - (U \cup V)$.

Suppose we are in the latter case, i.e. $C = \overline{\mathbf{1}_U x}$ and $A \in \overline{x \mathbf{1}_V}$. By Lemma 21, $\overline{Ux} = C$. We may then apply part (1) of Lemma 16 with $U \to V$ and $\overline{Ux} = C \notin \overline{UV}$ to get that $\overline{Vx} = C$, which contradicts our assumption that $A \in \overline{\mathbf{1}_V x}$.

We must then be in the former case, i.e. $A \in \overline{\mathbf{1}_U x}$ and $C = \overline{x \mathbf{1}_V}$ and, again by Lemma 21, $\overline{Vx} = C$. Note that if we can show that $C \notin \overline{Ux}$, we may then apply part (2) of Lemma 16 with $U \to V$ and $\overline{xV} = C \notin \overline{UV}$ to get that $U \to x$ and that $\Sigma(U, V) = \Sigma(U, x)$, which is exactly what we would like to prove.

To this end, suppose $C \in \overline{Ux}$ and let k be minimal such that $C \in \overline{[\mathbf{1}_U]_k x}$. If $k = 0$, we have that $C \in \overline{\mathbf{1}_U x}$ and it again follows that $\overline{Ux} = C$, which is a contradiction. Thus $k > 0$. Fix $u \in [\mathbf{1}_U]_k$ such that $C \in \overline{ux}$. Since k is minimal, $u \in [\mathbf{1}_U]_k - [\mathbf{1}_U]_{k-1}$ and thus $[\mathbf{1}_U]_{k-1} \to u$. Recall that $\widehat{[\mathbf{1}_U]_i} = \widehat{U}$ for all $i \leq n$ and thus $\overline{[\mathbf{1}_U]_{k-1} u} = \{A, B\}$.

We may now apply part ($2'$) of Lemma 16 with $[\mathbf{1}_U]_{k-1} \to u$ and $C \in \overline{ux}$ to get that either $C \in \overline{[\mathbf{1}_U]_{k-1} x}$ or $[\mathbf{1}_U]_{k-1} \to x$ and $\Sigma([\mathbf{1}_U]_{k-1}, x) = \Sigma([\mathbf{1}_U]_{k-1}, u)$. By the minimality of k, we must be in the latter case. Then we may apply part (2) of Lemma 16 with $[\mathbf{1}_U]_{k-1} \to x$ and $C = \overline{xV}$ to get that either $C \in \overline{[\mathbf{1}_U]_{k-1} V}$ or $[\mathbf{1}_U]_{k-1} \to V$. Both of theses cases contradict the assumption that $\overline{1_U V} = A$. Thus $C \notin \overline{Ux}$.

($2 \Rightarrow 3$) We may apply part (2) of Lemma 16 with $U \to x$ and $\overline{xV} = C \notin \overline{Ux}$ to get that either $C \in \overline{UV}$ or $U \to V$ and $\Sigma(U, x) = \Sigma(U, V)$. Since $C \notin \overline{UV} \subseteq \{A, B\}$, we are left in the case $U \to V$ and thus $\overline{UV} = \widehat{U} = \{A, B\}$ and $V \not\to U$. We note here that we are using our earlier observation about dominance that except in trivial cases $U \to V$ implies $V \not\to U$.

$(3 \Rightarrow 1)$. Again we may assume $\overline{1_U 1_V} = A$. As argued in $(1 \Rightarrow 2)$, we may find $x \in \mathcal{V} - (U \cup V)$ such that either $A \in \overline{\mathbf{1}_U x}$ and $\overline{Vx} = C$ or $\overline{Ux} = C$ and $A \in \overline{\mathbf{1}_V x}$ for some $C \notin \{A, B\}$.

Suppose we are in the latter case. Since $V \not\to U$ and $\overline{UV} = \{A, B\}$, there must be $U_A, U_B \in \Upsilon(U)$ and $V' \in \Upsilon(V)$ such that $\overline{U_A V'} = A$ and $\overline{U_B V'} = B$. Then x, U_A, V' forces $\overline{xV'} \subseteq \{A, C\}$ while $xU_B V'$ forces $\overline{xV'} \subseteq \{B, C\}$. Thus $\overline{xV'} = C$ and $C \in \overline{Vx}$.

We may then let k be minimal such that $C \in \overline{[\mathbf{1}_V]_k x}$. If $k = 0$, then $C \in \overline{\mathbf{1}_V x}$ and by Lemma 21 $\overline{Vx} = C$, which contradicts our assumption that $A \in \overline{\mathbf{1}_V x}$. Therefore $k > 0$. As before, we select $v \in [\mathbf{1}_V]_k - [\mathbf{1}_V]_{k-1}$ such that $C \in \overline{vx}$. We now apply part $(2')$ of Lemma 16 with $[\mathbf{1}_V]_{k-1} \to v$ and $C \in \overline{xv}$ to get that either $C \in \overline{[\mathbf{1}_V]_{k-1} x}$ or $[\mathbf{1}_V]_{k-1} \to x$. By the minimality of k, we must be in the latter case and we may apply part (2) of Lemma 16 with $[\mathbf{1}_V]_{k-1} \to x$ and $C = \overline{1_U x}$ (recall our assumption that $\overline{Ux} = C$) to get that either $C \in \overline{1_U [\mathbf{1}_V]_{k-1}}$ or $[\mathbf{1}_V]_{k-1} \to 1_U$. Both of these cases contradict the assumption that $\overline{1_U 1_V} = A$.

We therefore may assume that $A \in \overline{\mathbf{1}_U x}$ and $\overline{Vx} = C$. It is either the case that $U \to V$ or $U \not\to V$. If we suppose that $U \not\to V$, then we are in the case just handled with the roles of U and V reversed. Since that assumption leads to a contradiction, we have that $U \to V$. ∎

We have now developed sufficient technical tools to address the main points of Theorem 15.

Lemma 23. *If H has the tree property for n, then $M_{n+1}(H)$ is complete.*

Proof. Let $U, V \in \mathcal{V}_{n+1}$. If $|\overline{UV}| = 1$, then $\{U, V\} \in \mathcal{E}_{n+1}$. Suppose then that $|\overline{UV}| > 1$. Notice that if $\widehat{U} = \emptyset$, then U is a single vertex and thus $V \to U$ and $(V, U) \in \mathcal{A}_{n+1}$.

We may therefore assume $|\widehat{U}| = |\widehat{V}| = 2$ and consider the following cases.

Case 1. $\widehat{U} \neq \widehat{V}$ and $|\overline{UV}| > 2$. We may then select $C_U, C_V \in \overline{UV}$ distinct such that $C_U \notin \widehat{U}$ and $C_V \notin \widehat{V}$ and $U' \in \Upsilon(U), V' \in \Upsilon(V)$ such that $C_V \in \overline{U'V}$ and $C_U \in \overline{UV'}$. By Lemma 20, $\overline{U'V} = C_V$ and $\overline{UV'} = C_U$ and thus $C_V = \overline{U'V'} = C_U$, a contradiction.

Case 2. $\widehat{U} \neq \widehat{V}$ and $|\overline{UV}| = 2$. Without loss of generality, we may assume there is $C \in \overline{UV} - \widehat{U}$. Let $\overline{UV} = \{C, D\}$. Select $V_C \in \Upsilon(V)$ such that $C \in \overline{UV_C}$. By Lemma 20, $\overline{UV_C} = C$. Now select $V_D \in \Upsilon(V)$ such that $D \in \overline{UV_D}$. Observe that if $C \in \overline{UV_D}$, then $D \notin \overline{UV_D} = C$. Thus $\overline{UV_D} = D$. Since $\overline{UV} = \{C, D\}$ and every element of $\Upsilon(V)$ is of the type V_C or V_D, we have accounted for every element of $\Upsilon(V)$ and thus $V \to U$, i.e. $(V, U) \in \mathcal{A}_{n+1}$.

Case 3. $\widehat{U} = \widehat{V} = \{A, B\}$ and $C \in \overline{UV} - \{A, B\}$. Fix $U_C \in \Upsilon(U)$ such that $C \in \overline{U_C V}$. By Lemma 20, $\overline{U_C V} = C$ and thus for every $V' \in \Upsilon(V), C \in \overline{UV'}$.

Applying Lemma 20 again, gives us that $\overline{UV'} = C$ and thus $\overline{UV} = C$. Thus $\{U, V\} \in \mathcal{E}_{n+1}$.

Case 4. $\overline{UV} = \widehat{U} = \widehat{V}$. If $V \not\to U$, apply $(3 \Rightarrow 1)$ from Lemma 22 to get that $U \to V$. ■

Lemma 24. *Suppose H has the tree property for n. The weak components of $M_{n+1}(H)$ are transitive, and if $(U, V), (V, W) \in \mathcal{A}_{n+1}$, then $(U, V) \sim_\Sigma (U, W)$.*

Proof. Let $U, V, W \in \mathcal{V}_{n+1}$ such that $U \to V$ and $V \to W$. By Lemma 18, $|\widehat{U}| = |\widehat{V}| = 2$. We consider two cases: $\widehat{U} \neq \widehat{V}$ and $\widehat{U} = \widehat{V}$.

Suppose $\widehat{U} \neq \widehat{V}$ and let $A \in \widehat{U} - \widehat{V}$. Fix $U_A \in \Upsilon(U)$ such that $\overline{U_A V} = A$ and $V_1, V_2 \in \Upsilon(V)$ such that $\overline{V_1 W} \neq \overline{V_2 W}$. Fix $W' \in \Upsilon(W)$. We have that U_A, V_1, W' forces $\overline{U_A W'} \subseteq \{A, \overline{V_1 W'}\}$ while U_A, V_2, W' forces $\overline{U_A W'} \subseteq \{A, \overline{V_2 W'}\}$ and thus $\overline{U_A W'} = A$. Since W' was arbitrary, $\overline{U_A W} = A$. Note that since $|\overline{U_A W}| = 1$, we have ruled out the possibility that $W \to U$. By Lemma 23, we will be done if we can show that $|\overline{UW}| > 1$. Observe that we could also choose $U_B \in \Upsilon(U)$ such that $\overline{U_B V} = B \neq A$. If it happens that $B \notin \widehat{V}$, by the same reasoning as above $\overline{U_B W} = B$ so that $\{A, B\} \subseteq \overline{UW}$ and thus $U \to W$ and $\Sigma(U, V) = \Sigma(U, W)$.

Suppose then that $\widehat{U} = \{A, B\}$ and $\widehat{V} = \{B, C\}$. We can now find $U_A, U_B \in \Upsilon(U)$ such that $\overline{U_A W} = A$ and $\overline{U_B W} \subseteq \{B, C\}$. Therefore $|\overline{UW}| > 1$ and by Lemma 23 we have that $U \to W$. By Lemma 18, $\overline{UW} = \widehat{U} = \{A, B\}$ and thus $\overline{U_B W} = B$. Therefore, $\Sigma(U, V) = \Sigma(U, W)$.

We now consider the case $\widehat{U} = \widehat{V} = \{A, B\}$. Note that $\overline{UW} \subseteq \{A, B\}$ since we may otherwise easily form a rainbow triangle. We now have the setup for $(1 \Rightarrow 2)$ of Lemma 22 with $U \to V$ and have $x \in \mathcal{V} - (U \cup V)$ such that $U \to x$, $\Sigma(U, V) = \Sigma(U, x)$, and $\overline{xV} = C \notin \{A, B\}$. Applying part (1) of Lemma 16 to $V \to W$ and $\overline{Vx} = C \notin \overline{VW}$, we have that $\overline{xW} = C$. Now apply part (2) of Lemma 16 with $U \to x$ and $\overline{xW} = C \notin \overline{Ux}$ to get that either $C \in \overline{UW}$ or $U \to W$ and $\Sigma(U, W) = \Sigma(U, x) = \Sigma(U, V)$. We have already ruled out the former while the latter is what we sought to prove. ■

Lemma 25. *Suppose H has the tree property for n. The weak components of $M_{n+1}(H)$ form rooted trees.*

Proof. After Lemma 24, we need only show that for $U_1, U_2, V \in \mathcal{V}_{n+1}$ distinct, if $U_1 \to V$ and $U_2 \to V$, then either $U_1 \to U_2$ or $U_2 \to U_1$. By Lemma 23, it suffices to show $|\overline{U_1 U_2}| > 1$.

Suppose $|\overline{U_1 U_2}| = 1$. Observe that if $|\overline{U_1 V} \cup \overline{U_2 V} \cup \overline{U_1 U_2}| > 2$, then we must find a rainbow triangle in U_1, U_2, V. Thus we may assume $\overline{U_1 U_2} = A \in \widehat{U_1} = \widehat{U_2} = \{A, B\}$. As in the proof of Lemma 22, since $\widehat{U_1} = \widehat{U_2}$, we may select $x \in \mathcal{V} - (U_1 \cup U_2)$ such that $\overline{U_1 x} = C \notin \{A, B\}$ and $A \in \overline{\mathbf{1}_{U_2} x}$ (or with the roles of U_1 and U_2 reversed). We may then apply part (1) of Lemma 16 with $U_1 \to V$

and $\overline{U_1x} = C \notin \overline{U_1V}$ to get that $\overline{Vx} = C$ and apply part (2) of Lemma 16 with $U_2 \to V$ and $\overline{Vx} = C \notin \overline{U_2V}$ to get that either $C \in \overline{U_2x}$ or $U_2 \to x$.

First we consider the case $C \in \overline{U_2x}$. As before, let k be minimal such that $C \in \overline{[\mathbf{1}_{U_2}]_k x}$. If $k = 0$, by Lemma 21, $\overline{U_2x} = C$, which contradicts our assumption that $A \in \overline{\mathbf{1}_{U_2}x}$. Thus $k > 0$ and we may select $u \in [\mathbf{1}_{U_2}]_k - [\mathbf{1}_{U_2}]_{k-1}$ such that $C \in \overline{ux}$. We may apply part $(2')$ of Lemma 16 with $[\mathbf{1}_{U_2}]_{k-1} \to u$ and x to get that either $C \in \overline{[\mathbf{1}_{U_2}]_{k-1}x}$ or $[\mathbf{1}_{U_2}]_{k-1} \to x$. By the minimality of k, we must be in the latter case. Note that $B \in \overline{[\mathbf{1}_{U_2}]_{k-1}x} = \widehat{U_2} = \{A, B\}$. Thus by our assumptions that $\overline{U_1U_2} = A$ and $\overline{U_1x} = C$, we may locate a rainbow triangle in U_1, $[\mathbf{1}_{U_2}]_{k-1}$, x.

We turn now to the second case, $U_2 \to x$. We may apply part (2) of Lemma 16 with $U_2 \to x$ and $\overline{xU_1} = C \notin \overline{U_2x}$ to get that either $C \in \overline{U_1U_2}$ or $U_2 \to U_1$, which both contradict our assumption that $\overline{U_1U_2} = A$. Thus $U_1 \to U_2$ or $U_2 \to U_1$. ■

Taking Lemmas 18, 23, 24, and 25 together we have proved Theorem 15.

4.4. Alternative proof of Lemma 3

We now point out how it follows from Theorem 15 that every Gallai multigraph having more than two vertices can be disconnected by the removal of two colors.

Proof. We first observe that it suffices to prove the claim for reduced Gallai multigraphs. Let G be a Gallai multigraph with more than two vertices. If G is not reduced, then we may form a smaller Gallai multigraph G' by collapsing an isolated edge in G. If removing the colors A and B disconnects G', then it also disconnects G. We may continue collapsing isolated edges until we reach either a single vertex or a non-trivial reduced Gallai multigraph. In the former case, prior to collapsing the final edge, we had two vertices connected by at most two colors. Thus these two colors suffice to disconnect G.

We will thus be done if we can establish the claim for reduced Gallai multigraphs, and, in turn, it suffices to establish the claim for reduced maximal Gallai multigraphs. Suppose then that G is a reduced maximal Gallai multigraph. By Theorem 15, the sequence $M_n(G)$ terminates at either a single vertex or a non-trivial Gallai graph. In the latter case, we are done by Lemma 1.

Suppose then that $M_n(G)$ terminates at a single vertex. It then follows from Theorem 15 and Lemma 18 that $\mathbf{1}_{V(G)}$ is connected to G by two colors and thus the removal of these colors disconnects G. ■

5. Related Future Work

One may ask if the assumption that the multigraph is complete is necessary for these results. The definitions of *maximal* and *reduced* may have to be adjusted, but perhaps similar structure and results can be provided for non-complete Gallai multigraphs. Unfortunately, a complete bipartite graph with as many multiedges as desired will never contain a rainbow triangle (or any odd cycle) so one would have to add other restrictions.

Another avenue for possible extension is to consider forbidding rainbow graphs other than the triangle. It has been noted in [8] that consideration of a rainbow C_4 appears to be a very difficult problem but, in following the approach of [6], one might consider a triangle with pendant edges or other similar structures. Since we are considering multigraphs, we wonder if anything fundamentally different might happen if the forbidden rainbow structure is itself a multigraph.

Certainly a consideration of other graphs H in the search for $mgr_k(K_3 : H)$ would also be of interest.

Acknowledgment

The authors would like to thank the anonymous referees for helpful suggestions, including the present form of the proof of Lemma 3.

References

[1] B. Alexeev, *On lengths of rainbow cycles*, Electron. J. Combin. **13(1)** (2006) #105

[2] R.N. Ball, A. Pultr and P. Vojtěchovský, *Colored graphs without colorful cycles*, Combinatorica **27** (2007) 407–427. doi:10.1007/s00493-007-2224-6

[3] A. Diwan and D. Mubayi, *Turán's theorem with colors*, preprint, 2007.

[4] R.J. Faudree, R. Gould, M. Jacobson and C. Magnant, *Ramsey numbers in rainbow triangle free colorings*, Australas. J. Combin. **46** (2010) 269–284.

[5] A. Frieze and M. Krivelevich, *On rainbow trees and cycles*, Electron. J. Combin. **15** (2008) #59.

[6] S. Fujita and C. Magnant, *Extensions of Gallai-Ramsey results*, J. Graph Theory **70** (2012) 404–426. doi:10.1002/jgt.20622

[7] S. Fujita and C. Magnant, *Gallai-Ramsey numbers for cycles*, Discrete Math. **311** (2011) 1247–1254. doi:10.1016/j.disc.2009.11.004

[8] S. Fujita, C. Magnant and K. Ozeki, *Rainbow generalizations of Ramsey theory: a survey*, Graphs Combin. 26 (2010) 1–30. doi:10.1007/s00373-010-0891-3

[9] S. Fujita, C. Magnant and K. Ozeki. *Rainbow generalizations of Ramsey theory: a survey*, (2011) updated. http://math.georgiasouthern.edu/∼cmagnant

[10] T. Gallai, *Transitiv orientierbare Graphen*, Acta Math. Acad. Sci. Hungar. **18** (1967) 25–66.
doi:10.1007/BF02020961

[11] A. Gyárfás, G. Sárközy, A. Sebő and S. Selkow, *Ramsey-type results for Gallai colorings*, J. Graph Theory **64** (2010) 233–243.

[12] A. Gyárfás and G. Simonyi, *Edge colorings of complete graphs without tricolored triangles*, J. Graph Theory **46** (2004) 211–216.
doi:10.1002/jgt.20001

[13] P. Vojtěchovský, *Periods in missing lengths of rainbow cycles*, J. Graph Theory **61** (2009) 98–110.
doi:10.1002/jgt.20371

Received 28 November 2012
Revised 9 April 2013
Accepted 9 April 2013