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RANK NUMBERS FOR BENT LADDERS

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Abstract

A ranking on a graph is an assignment of positive integers to its vertices such that any path between two vertices with the same label contains a vertex with a larger label. The rank number of a graph is the fewest number of labels that can be used in a ranking. The rank number of a graph is known for many families, including the ladder graph $P_2 \times P_n$. We consider how "bending" a ladder affects the rank number. We prove that in certain cases the rank number does not change, and in others the rank number differs by only 1. We investigate the rank number of a ladder with an arbitrary number of bends.

Keywords: graph colorings, rankings of graphs, rank number, Cartesian product of graphs, ladder graph, bent ladder graph.

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1. Introduction

A coloring $f : V(G) \to \{1, 2, \ldots, k\}$ is a *k-ranking* of G if $f(u) = f(v)$ implies every $u - v$ path contains a vertex w such that $f(w) > f(u)$. The *rank number* of a graph, $\chi_r(G)$, is the minimum k such that G has a k-ranking. A k-ranking that uses $\chi_r(G)$ labels will be referred to as a *χ_r-ranking*. When the value of k is clear we will refer to a k-ranking simply as a ranking.

Research on rank numbers was sparked by its applications to the scheduling of manufacturing systems, Cholesky factorizations of matrices and VLSI layout [11, 14]. The optimal tree node ranking problem is identical to the problem of generating a minimum height node separator tree for a tree graph. Node separator trees are extensively used in VLSI layout [11]. These models are suitable for communication networks design where information flow between nodes needs to be monitored. Similar models are applicable in the design of management organizational structures. A matrix application was observed by Kloks, Müller, and Wong [10].

It was shown by Bodlaender *et al.* [2] that for a given bipartite graph G and a positive integer t, deciding if $\chi_r(G) \leq t$ is NP-Complete. However rank numbers have been determined for several families of graphs including: paths, cycles, split graphs, complete multipartite graphs, Möbius ladder graphs, caterpillars, powers of paths and cycles, and some grid graphs [1, 2, 3, 4, 6, 7, 12], and [13].

In 2009, Novotny, Ortiz, and Narayan [12] determined the rank number of the ladder graph $L_n = P_2 \times P_n$ and showed $\chi_r(P_2 \times P_n) = \lfloor \log_2 (n+1) \rfloor + \left\lfloor \log_2 \left(n + 1 - \left(2^{\lfloor \log_2 n \rfloor - 1}\right)\right) \right\rfloor + 1 = \lfloor \log_2 (n+1) \rfloor + \left\lfloor \log_2 \left(\frac{2(n+1)}{3}\right) \right\rfloor + 1$.

This result was also shown by Chang, Kuo, and Lin [4]. We consider how the rank number behaves if the ladder has one or more 'bends'. It turns out that in many cases the rank number does not change, and in others it differs by only 1. In this paper we determine rank numbers for the two extreme cases of bent ladders: the first where there is a single bend (L-shaped) and in the other the number of bends is maximized (similar to a staircase).

2. Preliminaries

We begin by recalling a definition of Ghoshal, Laskar, and Pillone [6].

Definition 1. A k-ranking is minimal if decreasing any label violates the ranking property.

The operation of a reduction was introduced by Ghoshal, Laskar, and Pillone [6].

Definition 2. Given a graph G and a set $S \subseteq V(G)$ the *reduction* of G is a graph G^*_S such that $V(G^*_S) = V(G) - S$ and for vertices u and v, $\{u, v\} \in E(G^*_S)$

if and only if there exists a $u - v$ path in G with all internal vertices belonging to S.

We present a generalization of Lemma 5 in [12] that will be used for bent ladders and staircase ladders. A 1-*bridge* is a set of two adjacent vertices x and y along with four edges that connect two graphs together as shown in Figure 1. Recall that a *vertex separating set* of a connected graph G is a set of vertices whose removal disconnects G. A graph is k-*connected* if any vertex separating set contains at least k vertices.

Lemma 3. *Let G be the union of two 2-connected graphs H_1 and H_2 that are connected by a 1-bridge, where $\chi_r(H_1) = \chi_r(H_2)$. Then $\chi_r(G) \geq \chi_r(H_1) + 2$.*

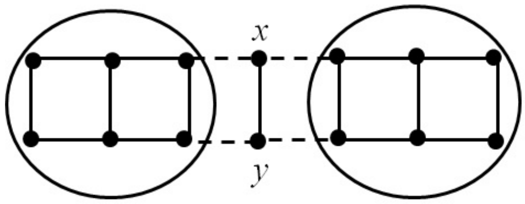

Figure 1. A 1-bridge.

Proof. Assume that $\chi_r(H_1) = \chi_r(H_2)$. Let the two added vertices be labeled x and y. We consider cases for different minimal rankings of G. We will show in each case there is a vertex with a label greater than or equal to $\chi_r(H_1) + 2$.

Case (i). There exists a vertex in each copy of L_s labeled $\chi_r(H_1)$. Since the highest two labels are unique in the ranking, we have $\chi_r(G) \geq \chi_r(H_1) + 2$.

Case (ii). There exists a vertex in each copy of L_s labeled $\chi_r(H_1) + 1$. Since the vertex with the highest label must be unique it follows that $\chi_r(G) \geq \chi_r(H_1) + 2$.

Case (iii). There exists a vertex u in one copy of H_1 labeled $\chi_r(H_1)$ and one vertex v in the other copy of H_1 labeled $\chi_r(H_1) + 1$. Without loss of generality assume v is in the copy of H_1 on the right side. Since the ranking of G is minimal the vertices in the copy of H_1 on the left side include labels $1, 2, \ldots, \chi_r(H_1)$ and vertices in the copy of H_1 on the right side include labels $1, 2, \ldots, \chi_r(H_1) - 1$, $\chi_r(H_1) + 1$. Let w and z be the two vertices in G labeled $\chi_r(H_1) - 1$. Note that there are two edge disjoint paths from w to x and two edge disjoint paths from z to x. Hence there must be a path from w to z that avoids both u and v. Hence either x or y must be labeled at least $\chi_r(H_1) + 2$. ■

We define L_n to be *critical* if $\chi_r(L_n) = \chi_r(L_{n-1}) + 1$ for $n \geq 2$. It was shown by Novotny, Ortiz, and Narayan [12] that a ladder L_n is critical if and only if $n = 2^k - 1$, or $2^k + 2^{k-1} - 1$ for any $k \geq 1$.

Lemma 4. *Let $k \geq 2$ and $n = 2^k - 2$ or $2^k + 2^{k-1} - 2$. Then $\chi_r(L_n) - \chi_r(L_{\frac{n}{2}}) = 1$.*

Proof. Recall that $\chi_r(P_2 \times P_n) = \lfloor \log_2(n+1) \rfloor + \left\lfloor \log_2\left(\frac{2(n+1)}{3}\right)\right\rfloor + 1$.

Case (i). $n = 2^k - 2$.

$$\begin{aligned}\chi_r(L_n) - \chi_r(L_{\frac{n}{2}}) &= \left(\lfloor \log_2(2^k - 2 + 1)\rfloor + \left\lfloor \log_2\left(\tfrac{2(2^k-2+1)}{3}\right)\right\rfloor\right)\\ &\quad - \left(\lfloor \log_2(2^{k-1} - 1 + 1)\rfloor + \left\lfloor \log_2\left(\tfrac{2(2^{k-1}-1+1)}{3}\right)\right\rfloor\right)\\ &= \lfloor \log_2(2^k - 1)\rfloor - \lfloor \log_2(2^{k-1})\rfloor\\ &\quad + \left\lfloor \log_2\left(\tfrac{2(2^k-2+1)}{3}\right)\right\rfloor - \left\lfloor \log_2\left(\tfrac{2(2^{k-1}-1+1)}{3}\right)\right\rfloor\\ &= \left\lfloor \log_2\left(\tfrac{2^{k+1}-2}{3}\right)\right\rfloor - \left\lfloor \log_2\left(\tfrac{2^k}{3}\right)\right\rfloor.\end{aligned}$$

Since there is only one power of 2 between $\frac{2^{k+1}-2}{3}$ and $\frac{2^k}{3}$, $\left\lfloor \log_2\left(\frac{2^{k+1}-2}{3}\right)\right\rfloor - \left\lfloor \log_2\left(\frac{2^k}{3}\right)\right\rfloor = 1$.

Case (ii). $n = 2^k + 2^{k-1} - 2$.

$$\begin{aligned}\chi_r(L_n) - \chi_r\left(L_{\frac{n}{2}}\right) &= \left(\lfloor \log_2(2^k + 2^{k-1} - 2 + 1)\rfloor + \left\lfloor \log_2\left(\tfrac{2(2^k+2^{k-1}-2+1)}{3}\right)\right\rfloor\right)\\ &\quad - \left(\lfloor \log_2(2^{k-1} + 2^{k-2} - 1 + 1)\rfloor + \left\lfloor \log_2\left(\tfrac{2(2^{k-1}+2^{k-2}-1+1)}{3}\right)\right\rfloor\right)\\ &= \lfloor \log_2(2^k + 2^{k-1} - 2 + 1)\rfloor - \lfloor \log_2(2^{k-1} + 2^{k-2} - 1 + 1)\rfloor\\ &\quad + \left\lfloor \log_2\left(\tfrac{2(2^k+2^{k-1}-2+1)}{3}\right)\right\rfloor - \left\lfloor \log_2\left(\tfrac{2(2^{k-1}+2^{k-2}-1+1)}{3}\right)\right\rfloor\\ &= \lfloor \log_2(2^k + 2^{k-1} - 2 + 1)\rfloor - \lfloor \log_2(2^{k-1} + 2^{k-2} - 1 + 1)\rfloor\\ &\quad + \left\lfloor \log_2\left(\tfrac{3\cdot 2^k-2)}{3}\right)\right\rfloor - \left\lfloor \log_2\left(\tfrac{3\cdot 2^{k-1})}{3}\right)\right\rfloor\\ &= 1 + (k-1) - (k-1) = 1.\end{aligned}$$

■

3. BENT LADDERS

We define a *bent ladder* $BL_n(a,b)$ to be the union of L_a and L_b that are joined at a right angle with a single L_2, so that $n = a + b + 2$. We note that b is implicitly determined by n. An example of a bent ladder is shown in Figure 2.

Theorem 5. *Let $BL_n(a,b)$ be the bent ladder composed of L_a, L_b, and L_2 where $n = a + b + 2$.*

$$\text{Then } \chi_r(BL_n(a,b)) = \begin{cases} \chi_r(L_n) - 1 & \text{if } n = 2^k - 1 \text{ for some } k \in \mathbb{Z}_+ \\ & \quad \text{and } a \equiv 2 \text{ or } 3 \pmod 4, \\ \chi_r(L_n) & \text{otherwise.} \end{cases}$$

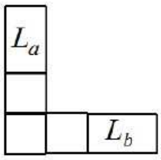

Figure 2. The bent ladder $BL_n(a)$.

We use a series of lemmas to prove the result.

Lemma 6. $\chi_r(BL_n) \leq \chi_r(L_n)$ *for all* n.

Proof. It was shown in [12] that there exists a χ_r-ranking of L_n where the label 1 is placed on alternating vertices. We consider such a labeling here. Label the vertices in L_a as they are in the first a rungs of L_n. Label the vertices of L_b as they are in the last b rungs of L_n. The remaining four vertices are labeled as shown in Figure 2. The remaining two rungs at the bend will have two vertices labeled 1. Let d, e be the other two labels. Without loss of generality assume $d < e$ and e is on the rung adjacent to a vertex in L_a. Let c be the label on the rung of L_b adjacent to the vertex on the joining L_2 that is not labeled 1.

Now we show that the labeling f of BL_n is a ranking. It may be helpful to refer to Figure 3. We consider two vertices x and y where $f(x) = f(y)$. If x and y are both in L_a or both in L_b then the ranking condition must be met. Finally consider the case where $x \in V(L_a)$ and $y \in V(L_b)$. There are two x, y paths in L_n one passing through d and another passing through e. Hence $e > d > f(x)$. Since d or e will be on the path from x to y in BL_n the ranking property is met. Hence the labeling f of BL_n is a ranking. ∎

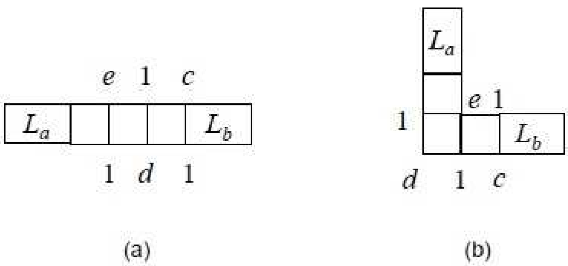

Figure 3. Transforming a ladder into a single bent ladder

We give the following definition that will be used in the next lemma. Given a vertex x we say that x has *path access* to i if there exists a path from x to a vertex labeled i that avoids any vertex with a label larger than i.

Lemma 7. *Let $k \geq 2$ and $n = 2^k - 2$ or $2^k + 2^{k-1} - 2$. Then in any χ_r-ranking of L_n, the highest two labels occur diagonally opposite in the central two rungs, and there is a vertex labeled 1 on each end of the ladder that has path access to each of the labels $i = 2, \ldots, \chi_r(L_n)$.*

Proof. The lemma is true for all χ_r-rankings of L_2 and L_4. Suppose the lemma holds for L_n. Consider L_{2n+2}. Not placing the labels $\chi_r(L_n)+1$ and $\chi_r(L_n)+2$ on the center two rungs will leave the ladder $L_{\frac{n}{2}}$ to be labeled with only $\chi_r(L_n) - 2$ labels which is impossible by Lemma 4.

We define a sequence $\{g_n\}$ that will be used in the following lemma. Let $h_i = \alpha + 1$ where 2^α is the highest power of 2 that divides i. Then replace each $t \geq 2$ in $\{h_n\}$ with the terms $2t - 2$ and $2t - 1$ in either order. Finally add 1 to each of the terms to get the sequence $\{g_n\}$. ■

Lemma 8. *Let f be minimal χ_r-ranking of $L_{2^k+2^{k-1}-2}$ where $v_{i,j}$ is the vertex in the i-th row and j-th column. Then $f(v_{i,j}) = 1$ if $i+j$ is even and $f(v_{i,j}) = g_j$ if $i + j$ is odd.*

Proof. This lemma is true for all χ_r-rankings of L_4. Suppose the lemma holds for L_n. Consider L_{2n+2}. Note that if $n = 2^k + 2^{k-1} - 2$ then $2n + 2 = 2^{k+1} + 2^k - 2$. By Lemma 8 the highest two labels must lie on opposite corners of the center two rungs. The remaining structure follows by induction. ■

We illustrate an example of a labeling in Figure 4.

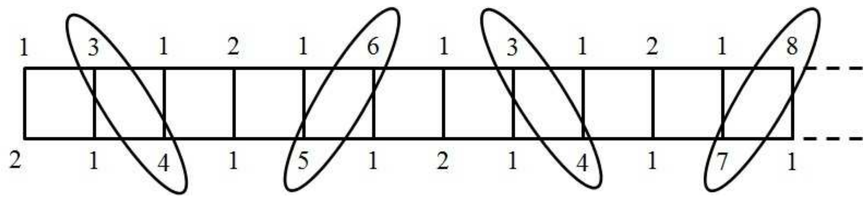

Figure 4. Note that labels within an oval may be interchanged.

We next define a sequence $\{w_n\}$ that will be used in the upcoming Lemma. Let $z_i = \alpha + 1$ where 2^α is the highest power of 2 that divides i. Then replace each $t \geq 1$ in $\{z_n\}$ with the terms $2t$ and $2t + 1$ in either order to obtain $\{w_n\}$.

Lemma 9. *Let h be a χ_r-ranking of L_{2^k-2}. Let $v_{i,j}$ be the vertex in the i-th row and j-th column. Then if $i + j$ is odd then $h(v_{i,j}) = w_j$. If $j \equiv 1 \pmod 4$ and $i = 1$ or $j \equiv 2 \pmod 4$ and $i = 2$ then $h(v_{i,j}) = 1$. If $j \equiv 3 \pmod 4$ and $i = 1$ or $j \equiv 0 \pmod 4$ and $i = 2$ then $h(v_{i,j}) = 1$ or 2.*

Proof. Observe that the lemma holds for L_2. Then note that if $n = 2^k - 2$ then $2n+2 = 2^{k+1} - 2$. Suppose the lemma holds for L_n. Consider L_{2n+2}. By Lemma 8 the highest two labels must lie on opposite corners of the center two rungs. The remaining structure follows by induction. ∎

We illustrate an example of a labeling in Figure 5.

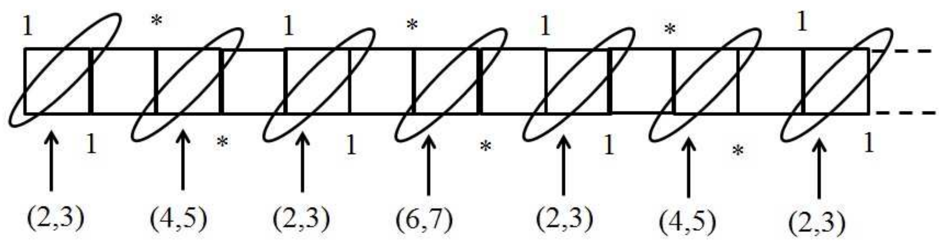

Figure 5. An example of a labeling.

Lemma 10. *$\chi_r(L_n) - 1 \le \chi_r(BL_n) \le \chi_r(L_n)$ for all n.*

Proof. Contract the three vertices x,y, and z at the bend in BL_n into a single vertex labeled with $m = \max\{x, y, z\}$, and note that this gives a valid ranking of L_{n-1}. See Figure 6.

Hence $\chi_r(BL_n) \ge \chi_r(L_{n-1})$. By Lemma 6, $\chi_r(L_{n-1}) \le \chi_r(BL_n) \le \chi_r(L_n)$. Noting that $\chi_r(L_n) = \chi_r(L_{n-1})$ or $\chi_r(L_{n-1}) + 1$ gives the desired result. ∎

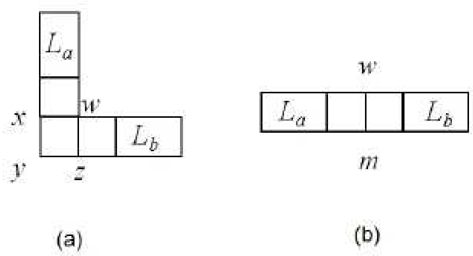

Figure 6. A valid ranking of L_{n-1}.

The combination of Lemmas 6 and 10 gives the rank numbers for all BL_n where L_n is not a critical ladder. We consider the following case involving the non-critical ladder L_{10}.

Example 11. Let $n = 10$. Recall that $\chi_r(L_9) = \chi_r(L_{10}) = 5$. By Lemma 6 we have $\chi_r(BL_{10}) \le \chi_r(L_{10}) = 5$. Lemma 10 implies that the labels in any k-ranking of BL_{10} can be used to form a k-ranking of L_9. Then we have $5 = \chi_r(L_9) \leq \chi_r(BL_{10}) \leq \chi_r(L_{10}) = 5$. Hence $\chi_r(BL_{10}) = 5$.

However we see in this next example that this approach cannot be extended to critical ladders.

Example 12. Let $n = 11$. Recall that $\chi_r(L_{10}) = 5$ and $\chi_r(L_{11}) = 6$. Lemmas 6 and 10 give that $5 = \chi_r(L_{10}) \leq \chi_r(BL_{10}) \leq \chi_r(L_{11}) = 6$. Hence $5 \leq \chi_r(BL_{10}) \leq 6$.

As a result we must consider cases of $\chi_r(BL_n)$ where L_n is a critical ladder separately. We address these cases in the next three lemmas.

Lemma 13. *For $k \geq 2$, $\chi_r\left(BL_{2^k+2^{k-1}-1}\right) = \chi_r\left(L_{2^k+2^{k-1}-1}\right)$.*

Proof. It was shown in [12] that $\chi_r(L_{2^j+2^{j-1}-1}) = 2j+1$. We proceed by induction on k. For the base case $k = 2$, it is easy to verify that $\chi_r(BL_5) = \chi_r(L_5) = 5$. Assume that $\chi_r(BL_{2^j+2^{j-1}-1}) = \chi_r(L_{2^j+2^{j-1}-1})$ for some j. Consider $BL_{2^{j+1}+2^j-1}$ as one copy of $BL_{2^j+2^{j-1}-1}$ and one copy of $L_{2^j+2^{j-1}-1}$ joined by a 1-bridge. By induction we have that $\chi_r(BL_{2^j+2^{j-1}-1}) = \chi_r(L_{2^j+2^{j-1}-1}) = 2j+1$. Application of the bridge lemma gives that $\chi_r(BL_{2^{j+1}+2^j-1}) = \chi_r(L_{2^{j+1}+2^j-1}) = 2(j+1)+1$. ■

In our next two lemmas we investigate $\chi_r\left(BL_{2^k-1}\right)$. Let BL_{2^k-1} be composed of ladders L_a and L_b joined by a L_2. We make the following observations which will be helpful in the proofs of Lemmas 14 and 15. We have that $a+b = 2^k - 3 \equiv 1 \pmod 4$. In Lemma 14 we consider the case where $a \equiv 0 \pmod 4$ (which implies that $b \equiv 1 \pmod 4$). In Lemma 15 we consider the case where $a \equiv 2 \pmod 4$ (which implies that $b \equiv 3 \pmod 4$).

Lemma 14. *Let $k \geq 3$. Consider BL_{2^k-1} as two ladders, L_a and L_b, joined by an L_2. If $a \equiv 0 \pmod 4$ or $a \equiv 1 \pmod 4$, then $\chi_r\left(BL_{2^k-1}\right) = \chi_r\left(L_{2^k-1}\right)$.*

Proof. Recall that $\chi_r(L_{2^j-1}) = 2j$ [12]. We proceed by induction on k. For the base case $k = 3$, there is only one bent ladder where $a \equiv 0 \pmod 4$ or $a \equiv 1 \pmod 4$. This is precisely the case where $a = 1$ and $b = 4$. Since this graph is composed of two copies of L_3 joined by a 1-bridge, we have that $\chi_r(BL_7) = \chi_r(L_7) = 6$. Assume that $\chi_r(BL_{2^j-1}) = \chi_r(L_{2^j-1})$ for some j. Consider $BL_{2^{j+1}-1}$ as one copy of BL_{2^j-1} and one copy of L_{2^j-1} joined by an L_1. By induction we have that $\chi_r(BL_{2^j-1}) = \chi_r(L_{2^j-1}) = 2j$. Application of the Lemma 4 gives that $\chi_r(BL_{2^{j+1}-1}) = \chi_r(L_{2^{j+1}-1}) = 2(j+1)$. ■

Lemma 15. *Consider BL_{2^k-1} as two ladders, L_a and L_b, joined by an L_2. If $a \equiv 2 \pmod 4$ or $a \equiv 3 \pmod 4$, then $\chi_r\left(BL_{2^k-1}\right) = \chi_r\left(L_{2^k-1}\right) - 1$.*

Proof. Without loss of generality suppose that $a \equiv 2 \pmod 4$. By Lemma 10, $\chi_r\left(BL_{2^k-1}\right) = \chi_r\left(L_{2^k-1}\right)$ or $\chi_r\left(L_{2^k-1}\right) - 1$. We exhibit an explicit ranking using $\chi_r\left(L_{2^k-1}\right) - 1 = \chi_r\left(L_{2^k-2}\right)$ labels. Rank L_{2^k-2} with χ_r labels choosing all vertices marked with a star to be 1. Then at the a-th rung of L_{2^k-2}, relabel the vertex labeled 1 with 2. If the vertices of the $(a-1)$ rung are 1 and 3 this gives a ranking. Otherwise if the vertices of the $(a-1)$ rung are 1 and 2 then the the vertices of the $(a-2)$ rung are 1 and 3; exchanging the labels 2 and 3 on these 2 rungs gives a ranking. Expand the vertex labeled 2 at the a-th rung into three vertices x,y, and z as follows, where $x = 1$, $y = 2$, $z = 1$. Note that this is a ranking of BL_{2^k-1} using $\chi_r\left(L_{2^k-1}\right) - 1$ labels. ■

***Proof of Theorem* 5.** Note that the theorem holds for $n = 2, 3$, and 4. We proceed by induction on j for all values of n in the interval $2^j + 2^{j-1} - 1 \le n \le 2^{j+1} + 2^j - 1$. Suppose $n = 2^j + 2^{j-1} - 1$. By Lemma 13, $\chi_r(BL_n) = \chi_r(L_n)$. Suppose $2^j + 2^{j-1} \le n \le 2^{j+1} - 2$. Since $\chi_r\left(L_{2^k+2^{k-1}-1}\right) = \chi_r\left(L_{2^{j+1}-2}\right)$ by Lemma 6 we have $\chi_r(BL_n) \le \chi_r(L_n) = \chi_r\left(L_{2^j+2^{j-1}-1}\right)$. Hence $\chi_r(BL_n) = \chi_r\left(L_{2^j+2^{j-1}-1}\right) = \chi_r(L_n)$. If $n = 2^{j+1} - 1$ then by Lemmas 14 and 15 the claim holds. If $2^{j+1} \le n \le 2^{j+1} + 2^j - 2$ then contracting the three vertices x,y, and z at the bend of BL_n gives a ranking of L_{n-1}. Note that in this case, $\chi_r(L_n) = \chi_r(L_{n-1})$. Hence $\chi_r(BL_n) \ge \chi_r(L_{n-1}) = \chi_r(L_n)$. Finally by Lemma 6, $\chi_r(BL_n) = \chi_r(L_n)$. This completes the inductive step. ■

Corollary 16. *For all $n \ne 2^k - 1$, $\chi_r(BL_n) = \chi_r(L_n)$ regardless of where the ladder is bent.*

4. STAIRCASE LADDERS

In this section we investigate ladders with a maximum number of bends. We call these graphs *staircase ladders.* We define a staircase ladder SL_n to be a graph with $n-1$ subgraphs $S_1, S_2, \ldots, S_{n-1}$ each of which are isomorphic to C_4. The staircase ladder is placed on a grid with the vertices of the subgraphs as follows: $v(S_1) = \{(0,0),(0,1),(1,1),(1,0)\}$, $v(S_2) = \{(1,0),(1,1),(2,1),(2,0)\}$, $v(S_3) = \{(1,1),(1,2),(2,2),(2,1)\}$, $v(S_4) = \{(2,2),(2,3),(3,3),(3,2)\}$. For $0 \le j \le \left\lceil \frac{n-1}{2} \right\rceil$, $v(s_{2j+1}) = \{(j,j),(j,j+1),(j+1,j+1),(j+1,j)\}$. For $0 \le j \le \left\lfloor \frac{n-1}{2} \right\rfloor$, $v(s_{2j}) = \{(j+1,j),(j+1,j+1),(j+2,j+1),(j+2,j)\}$.

The graph of SL_8 is shown in Figure 7. The staircase ladders SL_n has $n-1$ induced subgraphs isomorphic to C_4 (squares).

Theorem 17. *We have*
$$\chi_r(SL_n) = \begin{cases} \chi_r(L_{n+1}) & \text{if } n = 2^k + 2^{k-1} - 2 \text{ for some } k \ge 3, \\ \chi_r(L_n) & \text{otherwise.} \end{cases}$$

We use a series of lemmas to establish the result.

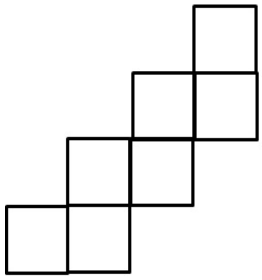

Figure 7. The graph SL_8.

Lemma 18. *For all* $n \geq 1$, $\chi_r(SL_{2n+2}) \geq \chi_r(SL_n) + 2$.

Proof. This follows from Lemma 4. ∎

Lemma 19. *For all* $j \geq 2$, $\chi_r\left(SL_{2^j-2}\right) \geq \chi_r(L_{2^j-2})$.

Proof. It is clear that the result holds for $j = 2$. Suppose the statement holds for $j - 1$. By Lemma 18, $\chi_r\left(SL_{2^j-2}\right) \geq \chi_r(SL_{2^{j-1}-2}) + 2 \geq \chi_r(L_{2^{j-1}-2}) + 2 = \chi_r\left(L_{2^j-2}\right)$. Hence the lemma holds for all j. ∎

Lemma 20. *For all* $n \geq 1$, $\chi_r(SL_n) \leq \chi_r\left(P^2_{n+1}\right) + 1 = \chi_r\left(L_{n+1}\right)$.

Proof. Consider the following labeling of SL_n. Label all vertices of degree 2 with 1, except for the bottom left and top right corners. The reduction of this graph is P^2_{n+1}. Labeling the remaining vertices using the labels $\{2, 3, \ldots, \chi_r\left(P^2_{n+1}\right) + 1 = \chi_r\left(L_{n+1}\right)\}$ gives the desired result. ∎

We recall the labeling h of L_n defined in Lemma 9. For a staircase graph SL_n let $v_{1,j}$ be the j-th vertex of the path along the top of the staircase and let $v_{2,j}$ be the j-th vertex along the bottom of the staircase graph. We then label the vertices of the staircase using a labeling σ where $\sigma(v_{1,j}) = h(v_{1,j})$ and $\sigma(v_{2,j}) = h(v_{2,j})$ and $\sigma(v_{2,2i+1}) = 1$ for all $1 < i < \left\lfloor \frac{n}{2} \right\rfloor$. An example of a staircase labeled with σ is given in Figure 8.

Lemma 21. *For all* $j \geq 2$, $\chi_r\left(SL_{2^j-2}\right) = \chi_r(L_{2^j-2})$, $\chi_r\left(SL_{2^j-1}\right) = \chi_r\left(SL_{2^j-2}\right)$ $+1 = \chi_r(L_{2^j-1})$. *Furthermore in a* χ_r*-ranking of* $\chi_r\left(SL_{2^j-2}\right)$ *the corner vertices with label* 1 *have path access to all labels in the set* $\{1, 2, \ldots, \chi_r\left(SL_{2^j-2}\right) + 1\}$, *and every* χ_r*-ranking of* $\chi_r\left(SL_{2^j-2}\right)$ *has the recursive structure defined by* σ.

Proof. Note that the lemma is true for all χ_r-rankings of $\chi_r\left(SL_2\right)$ and $\chi_r\left(SL_6\right)$. Suppose the lemma holds for SL_{2^j-2}, for all $k \geq 3$. Consider $SL_{2^{j+1}-2}$ as the union of two copies of SL_{2^j-2} connected by four central vertices, label the vertex of degree 4 as $\chi_r\left(SL_{2^j-2}\right) + 1$, and label the other two vertices 1 if they are adjacent to a vertex labeled 2, or 1 or 2 otherwise. Note that this is a ranking

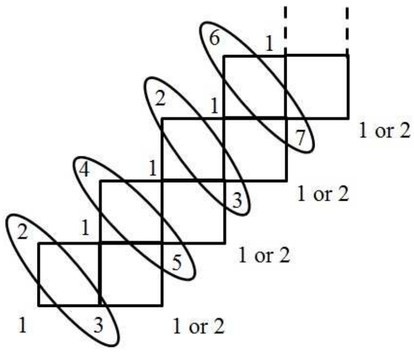

Figure 8. Note that labels within an oval may be interchanged.

of $SL_{2^{j+1}-2}$ using $\chi_r\left(SL_{2^j-2}\right)+2$ labels having the recursive structure described above. Note that the vertices labeled 1 on the ends have path access to all labels $\{1,2,\ldots,\chi_r\left(SL_{2^j-2}\right)+2\}$. Hence $\chi_r\left(SL_{2^{j+1}-2}\right) \le \chi_r\left(SL_{2^j-2}\right)+2 = \chi_r\left(L_{2^j-2}\right) = \chi_r\left(L_{2^{j+1}-2}\right)$. By Lemma 19, $\chi_r\left(SL_{2^{j+1}-2}\right) = \chi_r\left(L_{2^{j+1}-2}\right)$.

Let $r=\chi_r\left(SL_{2^j-2}\right)$. We next prove that there does not exist a χ_r-ranking of $SL_{2^{j+1}-2}$ with a different structure that the one given above. Consider $SL_{2^{j+1}-2}$ as the union of two copies of SL_{2^j-1} sharing a single vertex a, plus an extra vertex d. See Figure 9.

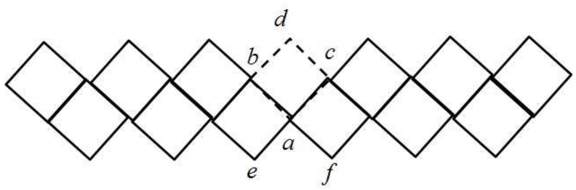

Figure 9. Joining of two staircase ladder graphs.

If either copy of SL_{2^k-2} uses r labels then a and d must have labels greater than r. Hence we will only consider the case where one of the top two labels is in each copy. If a is neither $r+1$ nor $r+2$, and these labels occur once in the left and right copies of SL_{2^j-1}. Note that the label r must be unique. Note that $a=r$, otherwise there will exist a ranking of SL_{2^j-1} that uses r labels.

First we consider the case where one of the vertices b or c is labeled using $r+1$ or $r+2$. Then SL_{2^j-2} can be ranked with r labels, which is a contradiction.

Next we consider the case where neither b or c is labeled $r+1$. Then since the removal of r, $r+1$, and $r+2$ do not disconnect the graph, there is a path between vertices labeled $r-1$ from each copy of SL_{2^j-1}. To show $\chi_r\left(SL_{2^{j+1}-1}\right)=\chi_r\left(SL_{2^{j+1}-2}\right)+1$, suppose $\chi_r\left(SL_{2^{j+1}-1}\right)=\chi_r\left(SL_{2^{j+1}-2}\right)$. Consider $SL_{2^{j+1}-1}$ as a copy of $SL_{2^{j+1}-2}$ connected to an extra rung, L_1. The extra rung is connected to a corner vertex of $SL_{2^{j+1}-2}$ labeled 1 with path access to $\{1, 2, \ldots, \chi_r(SL_{2^{j+1}-2})\}$ which is a contradiction. Hence $\chi_r(SL_{2^{j+1}-1}) \geq \chi_r(SL_{2^{j+1}-2}) + 1 = \chi_r(L_{2^{j+1}-2}) + 1 = \chi_r(L_{2^{j+1}-1}) = \chi_r(L_{2^{j+1}})$. By Lemma 20, $\chi_r(SL_{2^{j+1}-1}) = \chi_r(L_{2^{j+1}}) = \chi_r(SL_{2^{j+1}-1})$. This completes the inductive step. ■

Lemma 22. *For $j \geq 3$, $\chi_r(SL_{2^j+2^{j-1}-2}) = \chi_r(L_{2^j+2^{j-1}-1})$.*

Proof. We first show that $\chi_r(SL_{10}) \geq 7$. Let SL_4' be the graph consisting of SL_4 along with a pendant edge as shown in Figure 10.

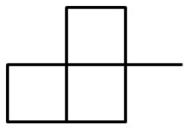

Figure 10. SL_4 with a pendant edge.

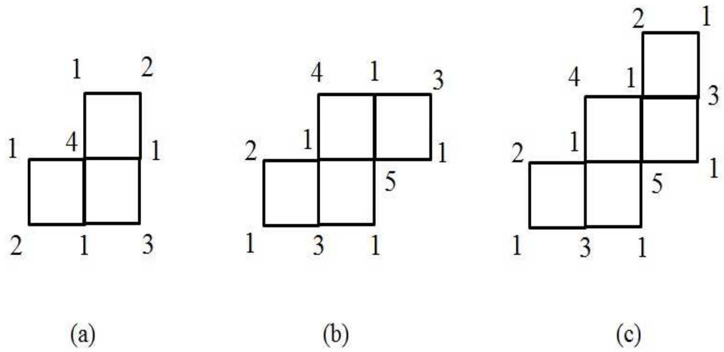

Figure 11. SL_4, SL_5, and SL_6.

By inspection we can see that $\chi_r(SL_4') = 5$. Since SL_{10} is a 2-connected graph and is the union of two copies of SL_4' plus two additional vertices we have that $\chi_r(SL_{10}) \geq 7$ by Lemma 3. Let $j \geq 2$ and suppose that the claim holds for j. Then $\chi_r(SL_{2^{j+1}+2^j-2}) \geq \chi_r(SL_{2^j+2^{j-1}-2}) + 2$ by Lemma 18. So $\chi_r(SL_{2^j+2^{j-1}-2}) \geq \chi_r(SL_{2^j+2^{j-1}-2}) + 2 = \chi_r(L_{2^j+2^{j-1}-1})$. By Lemma 20, $\chi_r(SL_{2^{j+1}+2^j-2}) = \chi_r(L_{2^j+2^{j-1}-1})$ and the claim holds. The result clearly holds for $1 \leq n \leq 3$ since $SL_n = L_n$. We next consider the cases $4 \leq n \leq 6$. It is known that $\chi_r(L_4) = 4$ and $\chi_r(L_5) = 5$ [12]. Since L_4 is a subgraph of SL_4 we have that $\chi_r(SL_4) \geq 4$. The labeling in Figure 11 (a) gives the reverse inequality. Since SL_5 contains the subgraph SL_4' (as shown in Figures 11 (a) and (b)) it follows that $\chi_r(SL_5) \geq 5$. The labeling shown in Figure 11 (c) shows that $\chi_r(SL_6) \leq 5$. Hence $5 = \chi_r(SL_5) = \chi_r(L_5)$.

Let $n \geq 7$. The proof proceeds by induction on j for values of n in the interval $[2^j - 1, 2^{j+1} - 1)$ for $j \geq 3$.

$2^k - 1 \leq n \leq\ 2^k + 2^{k+1} - 3$. Note that $\chi_r(L_n) = \chi_r(L_{n+1})$. By Lemma 20, we have $\chi_r(SL_{2^j-1}) \leq \chi_r(SL_n) \leq \chi_r(L_{n+1})$. By Lemma 21, $\chi_r(SL_{2^j-1}) = \chi_r(L_{2^j-1}) = \chi_r(L_{n+1})$. Hence $\chi_r(SL_n) = \chi_r(L_n)$. If $n = 2^k + 2^{k-1} - 2$ then $\chi_r(SL_{n+1})$ by Lemma 22. If $2^k+2^{k+1}-1 \leq n \leq 2^{k+1}-2$, then $\chi_r(SL_{2^j+2^{j-1}-2}) = \chi_r(L_{2^j+2^{j-1}-1}) \leq \chi_r(SL_n) \leq \chi_r(L_{2^{j+1}-2}) = \chi_r(SL_{2^{j+1}-2})$ by Lemmas 21 and 22. But since $\chi_r(L_{2^j+2^{j-1}-1}) = \chi_r(L_{2^{j+1}-2})$, $\chi_r(L_{2^j+2^{j-1}-2}) = \chi_r(L_{2^{j+1}-2})$. Hence $\chi_r(SL_n) = \chi_r(L_n)$ as desired. This completes the inductive step. ■

5. Ladders with Multiple Bends

With ladders with a single bend, the direction of the bend is not important, as they will result in isomorphic graphs. However for ladders with multiple bends both the directions and the locations of the bends can have an impact on the rank number. We will use the notation BL_n^m to denote a bent ladder of length n with m bends.

It was shown by Novotny, Ortiz, and Narayan [12] that a ladder can be optimally ranked so that there is an pattern of alternating ones (see Figure 11(a)). In some cases the labelling pattern from a ladder graph can be adapted to fit a bent ladder, as is the case in Figure 11 (b). However if the bends are in a different direction and in different places, the rank number can increase. We will define a 'bad bend' when the alternating labeling is forced to label a vertex of degree 4 with a 1. A bend is defined to be a 'good bend' otherwise. We describe this in the next example.

Example 23. We start with the ladder $P_2 \times P_{10}$. It was shown by Novotny, Ortiz, and Narayan [12] that $\chi_r(P_2 \times P_{10}) = 6$.

The labeling shows that the rank number of the graphs in Figures 12 (a) and (b) is less than or equal to 6. However the graph shown in Figure 12 (c) has a bad bend and a rank number of at least 7. To see this note that a 6-ranking would force the two circled vertices to be labeled 5 and 6 and labeling the remaining vertices with 1, 2, 3, and 4 does not permit a ranking.

The following two lemmas can serve as the base cases for generalizing the upper bound for the rank number of a ladder with an arbitrary number of bends. In Lemma 24 we consider a ladder with only good bends, and in Lemma 25 we consider a ladder with one bad bend.

Lemma 24. *Let BL_n^2 be a ladder with two good bends. Then $\chi_r\left(BL_n^2\right) \leq \chi_r(L_n)$ for all n.*

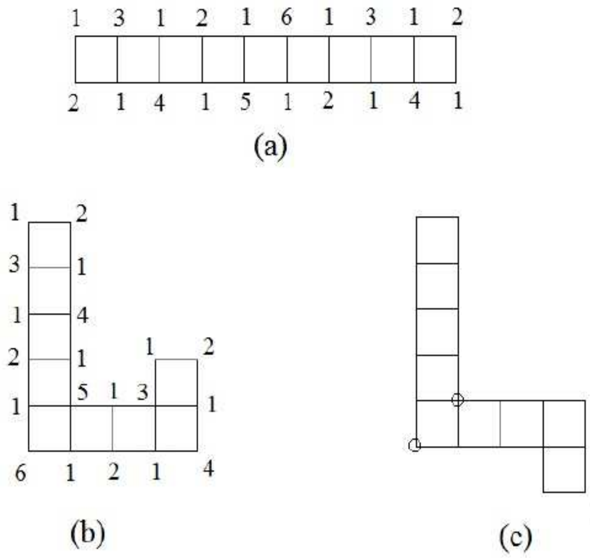

Figure 12. One ladder with two bent ladders.

Proof. Consider a χ_r-ranking of L_{n+1} labeled with alternating ones. We label the vertices in L_a as they are in the first a rungs of L_{n+1}. Label the vertices of L_b as they are in the rungs from $a+3 \leq m \leq a+2+b$ and label the vertices in L_c as they are in the last c rungs of L_{n+1}. The remaining vertices can be labeled as shown in Figure 13. The first graph shows the case when the length of the middle ladder is odd and the second graph shows where the length of the middle ladder is even. Let c and f be the vertices on the ends of L_b that are not labeled 1.

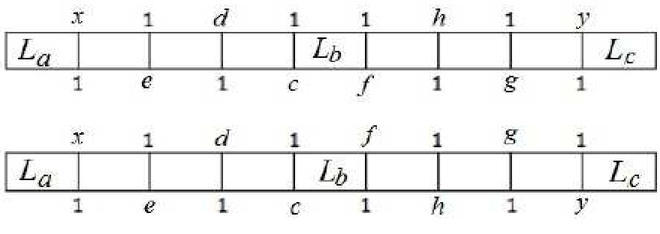

Figure 13. Ladder graphs with the central ladder having an odd or even length.

The two configurations of a ladder with two bends are shown below in Figure 14. We consider two vertices x and y where $f(x) = f(y)$. If x and y are both in L_a, L_b, and L_c then the ranking condition must be met. If $x \in V(L_a)$ and $y \in V(L_b)$, then the paths between them must go through either d or e. If $x \in V(L_a)$ and $y \in V(L_c)$ then the path must go through d or e and g or h. Finally, if $x \in V(L_b)$

and $y \in V(L_c)$ then the path between them must go through g or h. ■

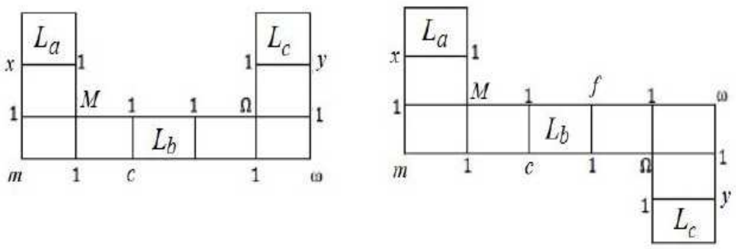

Figure 14. A double bent ladder with the central ladder having odd or even length. Here $M = \max(d, e)$ and $m = \min(d, e)$.

Lemma 25. *Let BL_n^2 be a ladder with one good bend and one bad bend. Then $\chi_r\left(BL_n^2\right) \leq \chi_r\left(L_{n+1}\right)$ for all n.*

Proof. Consider a χ_r-ranking of L_{n+1} labeled with alternating ones. We label the vertices in L_a as they are in the first a rungs of L_{n+1}. Label the vertices of L_b as they are in the rungs from $a + 3 \leq m \leq a + 2 + b$ and label the vertices in L_c as they are in the last c rungs of L_{n+1}. The remaining vertices can be labeled as shown in Figure 15. The first graph shows the case when the length of the middle ladder is odd and the second graph shows where the length of the middle ladder is even. Let c and f be the vertices on the ends of L_b that are not labeled 1. Without loss of generality, we assume the bend between L_b and L_c is the bad bend.

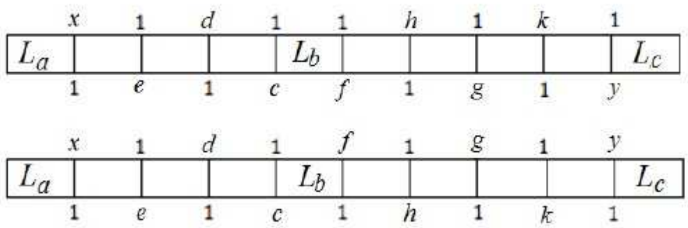

Figure 15. Ladder graphs with the central ladder having an odd or even length.

The two configurations of a ladder with two bends are shown in Figure 16. We consider two vertices x and y where $f(x) = f(y)$. If x and y are both in L_a, L_b, and L_c then the ranking condition must be met. if $x \in V(L_a)$ and $y \in V(L_b)$, then the paths between them must go through either d or e. If $x \in V(L_a)$ and $y \in V(L_c)$ then the path must go through d or e, and g or h and k. Finally, if

$x \in V(L_b)$ and $y \in V(L_c)$ then the path between them must go through g or h and k. ■

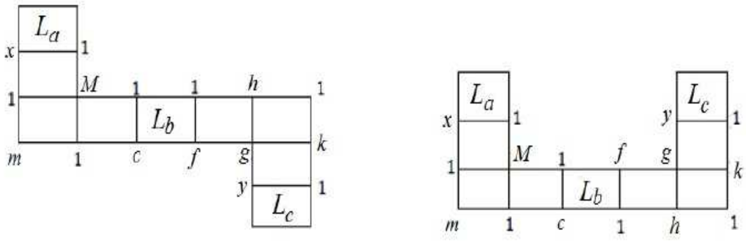

Figure 16. A double bent ladder with the central ladder having odd or even length. Here $M = \max(d, e)$ and $m = \min(d, e)$.

We next explore the ladders with $t > 2$ bends. We note that after the first bend there are two choices for each additional bend so number of cases to consider grows exponentially. However present some general results when all of the bends are good bends.

Lemma 26. *Let BL_n^t be a ladder with t good bends. Then $\chi_r\left(BL_n^t\right) \leq \chi_r\left(L_n\right)$ for all n.*

Proof. We will proceed by induction on t. The base case of $t = 1$ is trivial. Assume the hypothesis is true $t = j$. Consider a χ_r-ranking of BL_n^j. Suppose that $\chi_r\left(BL_n^j\right) \leq \chi_r\left(L_n\right)$. Label the vertices in L_a as they are in the first a rungs of BL_n^j and label the vertices of $\chi_r\left(BL_b^j\right)$ as they are in the last b rungs of BL_n. The remaining vertices are labeled as shown in Figure 17 (a). Consider the labeling of BL_b^{t+1} shown in Figure 17 (b) where $M = \max(r, s)$ and $m = \min(r, s)$. To see that this is labeling a ranking consider two vertices x and y where $f(x) = f(y)$. If x and y are both in L_a or BL_b^j then the ranking condition holds. If $x \in L_a$ and $y \in BL_b^j$ then any path between them must pass through either m or M, and hence is a ranking. Then $\chi_r\left(BL_n^{t+1}\right) \leq \chi_r\left(L_n\right)$. The proof then follows by induction. ■

Lemma 27. *Let BL_n^t be a ladder with t good bends. Then $\chi_r\left(L_{n-t}\right) \leq \chi_r\left(BL_n^t\right) \leq \chi_r\left(L_n\right)$ for all n.*

Proof. Consider a χ_r-ranking of BL_n^t. By Lemma 10, $\chi_r\left(BL_n^1\right) \geq \chi_r\left(L_{n-1}\right)$. Assume the formula holds for $\chi_r\left(BL_n^t\right)$. Then we construct BL_n^{t+1} as shown in Figure 18. Let $M = \max\{x, y, z\}$. Then the labeling is a ranking of BL_{n-1}^t. Thus $\chi_r\left(BL_n^{t+1}\right) \geq \chi_r\left(BL_{n-1}^t\right)$, and since $\chi_r\left(BL_{n-1}^t\right) \geq \chi_r\left(L_{n-1-t}\right)$, $\chi_r\left(BL_n^{t+1}\right) \geq \chi_r\left(L_{n-1-t}\right)$. The proof then follows by induction on t. ■

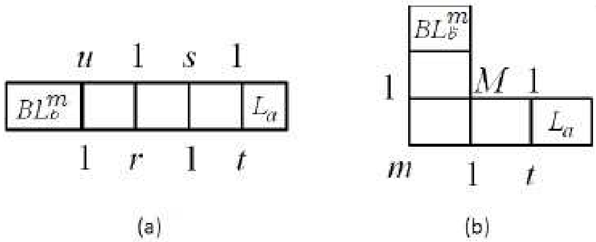

Figure 17. Adding an additional good bend to a ladder with multiple bends.

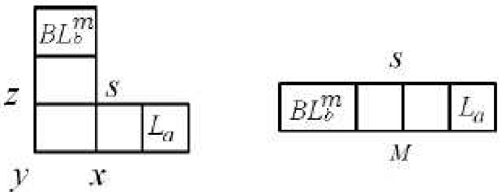

Figure 18. Unbending a ladder.

We can apply the result involving the rank number of a ladder, to obtain new results for some ladders with multiple good bends.

Theorem 28. *Let $t \leq 2^{j-1}-1$. Then*

$$\chi_r\left(BL_n^t\right) = \begin{cases} 2j & when \quad 2^j+t-1 \leq n \leq 2^j+2^{j-1}-2, \\ 2j+1 & when \quad 2^j+2^{j-1}+t-1 \leq n \leq 2^{j+1}-2. \end{cases}$$

Proof. Let $2^j+t-1 \leq n \leq 2^j+2^{j-1}-2$.

It was shown in [12] we have that $\chi_r(L_n) = 2j$ whenever $2^j-1 \leq n \leq 2^j+2^{j-1}-2$. Then as long as $2^j+t-1 \leq 2^j+2^{j-1}-2$ we have $\chi_r(L_{n-t}) = \chi_r(L_n)$.

$\chi_r(L_{2^j-1}) = \chi_r(L_{2^j+2^{j-1}-2}) = 2j$. However we need to stay in this range when we subtract t. Hence we have $t \leq 2^{j-1}-1$. This inequality insures that our upper bounds are at least as big as our lower bounds. For the sake of completeness, we include the details.

Note that $t \leq 2^{j-1}-1$
$\Leftrightarrow 2^{j-1}+t \leq 2^{j-1}+2^{j-1}-1 = 2\cdot 2^{j-1}-1 = 2^j-1$
$\Leftrightarrow 2^j+2^{j-1}+t \leq 2^{j+1}-1$
$\Leftrightarrow 2^j+2^{j-1}+t-1 \leq 2^{j+1}-2$.
For the second set of bounds, $t \leq 2^{j-1}-1$
$\Leftrightarrow t+1 \leq 2^{j-1}$
$\Leftrightarrow 2t+2 \leq 2^j$
$\Leftrightarrow 2^{j+1}+2t+2 \leq 2^{j+1}+2^j$

$\Leftrightarrow 2^j + t + 1 \leq 2^j + 2^{j-1}$
$\Leftrightarrow 2^j + t - 1 \leq 2^j + 2^{j-1} - 2.$ ■

We next consider the inclusion of bad bends in a ladder.

Lemma 29. *Let q be the number of bad bends. Then $\chi_r(BL_n^m) \leq \chi_r(L_{n+q})$ for all n and m.*

Proof. Consider a χ_r-ranking of BL_n^m labeled with alternating ones. There are two possible ways to extend BL_n^m to BL_n^{m+1}, where the $(m+1)$-st bend is either good or bad. If the bend is good then label the vertices in L_a as they appear in the first a rungs of BL_n^m. Then label the vertices of BL_b^m as they appear in the last b rungs of BL_n^m. If the bend is bad then label the vertices in L_a as they appear in the first a rungs of BL_{n+1}^m. Then label the vertices of BL_b^m as they appear in the last b rungs of BL_{n+1}^m. For both cases the remaining vertices can be labeled as shown in Figure 19.

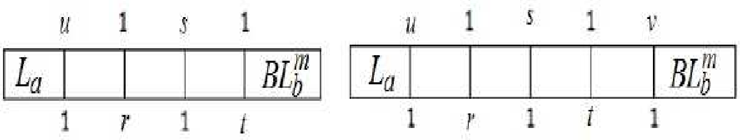

Figure 19. Labelings for a ladder graph with multiple bends.

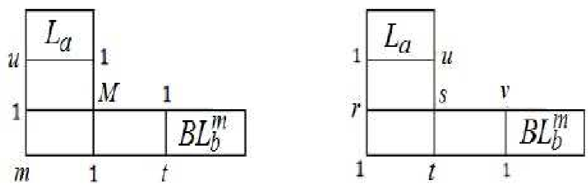

Figure 20. Adding an additional bend to a ladder with multiple bends, where $M = \max(d, e)$ and $m = \min(d, e)$.

We illustrate the bending of the ladder in Figure 20. We consider two vertices x and y where $f(x) = f(y)$. If x and y are both in $V(L_a)$ or both in $V(BL_b^m)$ then it is clear that the labeling is a ranking. If $x \in V(L_a)$ and $y \in V(BL_b^m)$, then the paths between them must go through r or s as before, meeting the ranking condition. We will have one of two cases (i) $\chi_r\left(BL_n^{m+1}\right) \leq \chi_r(BL_n^m) \leq \chi_r(L_{n+q})$ or (ii) $\chi_r\left(BL_n^{m+1}\right) \leq \chi_r(BL_n^m) \leq \chi_r(L_{n+q+1})$. In either case we have $\chi_r\left(BL_n^{m+1}\right) \leq \chi_r\left(L_{n+q'}\right)$ where q' is the number of bad bends that are added.

■

Lemma 30. *$\chi_r\left(BL_n^m\right) \leq \chi_r\left(L_{n+\left\lfloor\frac{m}{2}\right\rfloor}\right)$ for all m and n.*

Proof. The number of bad bends in a ladder with m bends is bounded by $\left\lfloor\frac{m}{2}\right\rfloor$. Since for every $x \geq y$, $\chi_r\left(L_x\right) \geq \chi_r\left(L_y\right)$ we have that $\chi_r\left(BL_n^m\right) \leq \chi_r\left(L_{n+q}\right) \leq \chi_r\left(L_{n+\left\lfloor\frac{m}{2}\right\rfloor}\right)$ for all n and m. ∎

It turns out that bending the ladder has a relatively small impact on the number. We show that the rank number can increase by at most one.

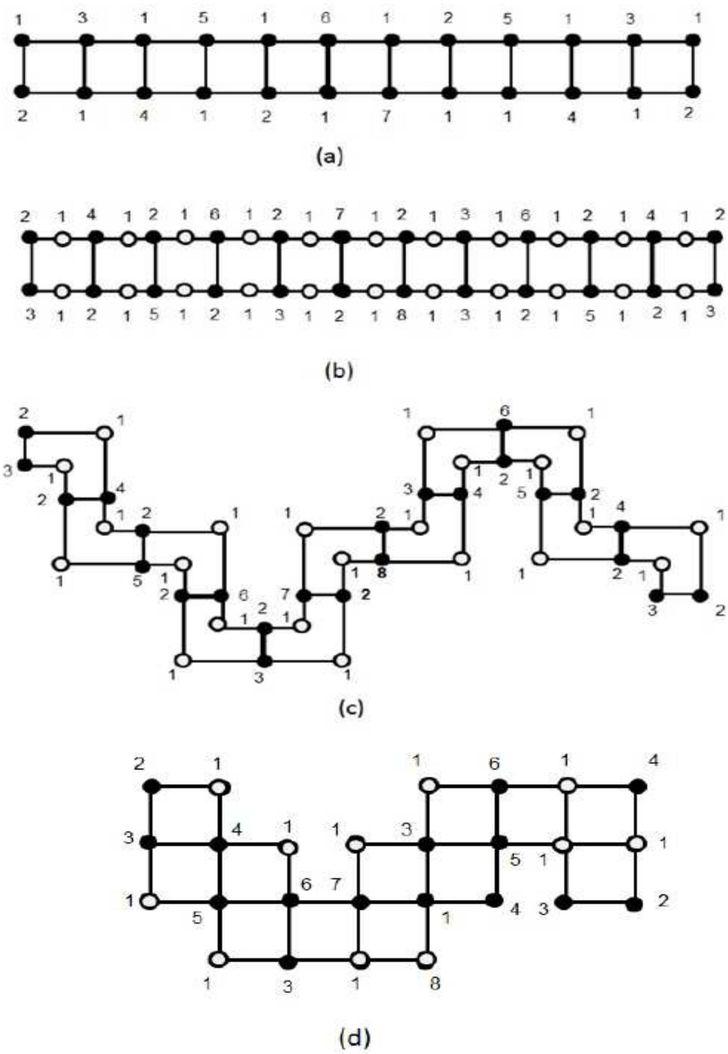

Figure 21.

Theorem 31. *For any ladder with multiple bends, the rank number is either* $\chi_r(L_n)$ *or* $\chi_r(L_n)+1$.

Proof. Let f be a χ_r-ranking L_n. Let G be the graph obtained by subdividing each horizontal edge of the ladder L_n. Then we construct another ranking f' of G by letting $f'(v) = 1$ for all of the new vertices v, and let $f'(v) = f(v)+1$ for all vertices that appear in L_n and G. We can make bends by drawing G on a grid and "making turns" using the new vertices. The edges along the inside corners can be contracted keeping the largest label to obtain a ranking of the desired bent ladder. These steps are illustrated in Figure 21 (a)–(d). ■

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