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Note

A NOTE ON FACE COLORING ENTIRE WEIGHTINGS OF PLANE GRAPHS[1]

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Abstract

Given a weighting of all elements of a 2-connected plane graph $G = (V, E, F)$, let $f(\alpha)$ denote the sum of the weights of the edges and vertices incident with the face α and also the weight of α. Such an entire weighting is a proper face colouring provided that $f(\alpha) \neq f(\beta)$ for every two faces α and β sharing an edge. We show that for every 2-connected plane graph there is a proper face-colouring entire weighting with weights 1 through 4. For some families we improved 4 to 3.

Keywords: entire weighting, plane graph, face colouring.

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1. Introduction

In the last years several papers appeared that study various colourings defined by weightings (labellings) of elements of the graph. First such a colouring was introduced by Karoński, Łuczak and Thomason [14]. Let G be a graph. Given a weighting of the edge set of G, let $f(v)$ denote the sum of the weights of the edges incident to v for each $v \in V(G)$. A weighting is *irregular* if the resulting vertex weighting f is injective, and the minimum k such that this can be done with weights 1 to k is the *irregularity strength* of the graph, see [9, 11]. A weaker condition is to require $f(u) \neq f(v)$ only when u and v are adjacent; we call such a weighting a *proper vertex-colouring edge-weighting*, since the resulting f is a proper vertex colouring.

Karoński *et al.* posed the following conjecture.

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Conjecture 1 ([14], 2004). *Every connected graph with at least three vertices has a proper vertex-colouring edge-weighting from* $\{1, 2, 3\}$.

Conjecture 1 is true for 3-colourable graphs [14]. Regardless of chromatic number, there is a fixed bound k such that colours 1 to k always suffice.

In [1] it was shown that $k = 30$ suffices. This was reduced to 16 in [2] and to 13 in [16]. Currently, the best known result is $k = 5$ by Kalkowski, Karoński and Pfender [13].

If each vertex is also given a weight forming total weighting, the sum at a vertex includes the weight of the vertex, and the vertex weighting f is injective then we obtain the total vertex irregular weighting first introduced by Bača, Jendrol', Miller and Ryan [5] in 2007. The minimum k such that this can be done with weights 1 to k is the *total vertex irregularity strength*. A weaker condition, to require $f(u) \neq f(v)$ only when u and v are adjacent, leads to a *proper vertex-colouring total-weighting*. Using this definition and motivated by the above mentioned papers, Przybyło and Woźniak [15] posed the following $1, 2$-conjecture.

Conjecture 2 ([15], 2010). *Every connected graph has a proper vertex-colouring total-weighting from* $\{1, 2\}$.

Przybyło and Woźniak [15] showed that 1,2-conjecture is true for 3-colourable graphs; that colours 1 throught 11 always suffice for a proper total vertex irregularity weighting and that colours 1 through $1 + \lfloor \chi(G)/2 \rfloor$ suffice. The breakthrough by Kalkowski [12] is that every graph has a proper vertex-colouring total-weighting with vertex weights in $\{1, 2\}$ and the edge weights in $\{1, 2, 3\}$.

Motivated by the above mentioned conjectures, papers [7, 8] and mainly by the paper of Wang and Zhu [17] we introduce in this note a concept of the entire weighting for 2-connected plane graphs. If each element of a plane graph $G = (V, E, F)$ is given a weight forming entire weighting, then let $f(\alpha)$ of the face $\alpha \in F(G)$ denote the sum of the weights of the edges and the weights of the vertices incident with α and the weight of α. A weighting is the *face irregular entire weighting* if the resulting face-weighting f is injective, and the minimum k such that this can be done with weights 1 through k is the *entire face irregularity strength*, see Bača *et al.* [4]. A weaker requirement is that $f(\alpha) \neq f(\beta)$ only when faces α and β share an edge (i.e. are adjacent). Call such a weighting the *proper face-colouring entire k-weighting* provided that it is done with weights 1 to k.

In this note we discuss the problem of finding the minimum k such that for every 2-connected plane graph G there exists a proper face-colouring entire k-weighting. We show that $k \leq 4$ in general and that for some families of 2-connected plane graphs $k \leq 3$. At the end we state a conjecture concerning this minimum k.

2. Results

Let $G = (V, E, F)$ be a 2-connected plane graph with $V = V(G)$, $E = E(G)$ and $F = F(G)$ denoting the vertex set, the edge set and the face set, respectively. For a face α let $V(\alpha)$ and $E(\alpha)$ be the set of vertices and the set of edges incident with the face α. For an integer k let $w : V(G) \cup E(G) \cup F(G) \to \{1, 2, \ldots, k\}$ be an integer weighting. Let $f(\alpha) = w(\alpha) + \sum_{uv \in E(\alpha)} w(uv) + \sum_{v \in V(\alpha)} w(v)$ be the colour of the face α. The weighting w is called a *proper face-colouring entire k-weighting*, if $f(\alpha) \neq f(\beta)$ for adjacent faces α, β.

Let $G^* = (F^*, E^*, V^*)$ be the dual of a 2-connected plane graph G. One of main results of this note is the following theorem.

Theorem 3. *For every 2-connected plane graph $G = (V, E, F)$ there is a proper face-colouring entire χ^*-weighting, where $\chi(G^*) = \chi^*$ denotes the chromatic number of the dual G^* of G.*

Proof. It is easy to see that there exists a proper face colouring $\varphi : F(G) \to \{1, 2, \ldots, \chi^*\}$. Because of the Four Colour Theorem, $\chi^* \leq 4$, see [3]. Now we associate the following weighting w with elements of G: put $w(v) = 2$ for every vertex $v \in V(G)$, $w(e) = 2$ for every edge $e \in E(G)$ and $w(\alpha) = \varphi(\alpha)$ for every face $\alpha \in F(G)$.

Next we have to show that for every two faces α and β sharing an edge $f(\alpha) \neq f(\beta)$. To this end suppose that α is an i-gon and β is a j-gon, $j \geq i \geq 2$. If $i < j$, then $f(\alpha) = \varphi(\alpha) + 4i \leq 4(i+1) < 4j + \varphi(\beta) = f(\beta)$. If $i = j$, because $\varphi(\alpha) \neq \varphi(\beta)$, we immediately have $f(\alpha) \neq f(\beta)$. ∎

Corollary 4. *Every 2-connected plane graph has a proper face-colouring entire 4-weighting.*

Grötzsch [10] (see also [6]) proved that every triangle-free planar graph is 3-colorable. This implies:

Theorem 5. *Every 2-connected plane graph G whose dual G^* is triangle-free has a proper face-colouring entire 3-weighting.*

Theorem 6. *Every 2-connected plane graph all faces of which are m-gons, $m \in \{3, 4, 5\}$, has a proper face-colouring entire 3-weighting.*

Proof. Let $G = (V, E, F)$ be a 2-connected plane graph and let $G^* = (F^*, E^*, V^*)$ be the dual of G. By Kalkowski [12] there is a proper vertex-colouring total-weighting from $\{1, 2, 3\}$. Let w^* be this weighting and let $f^*(\alpha^*) = w^*(\alpha^*) + \sum_{e \in E(\alpha)} w^*(e^*)$.

Define an entire weighting w of G from $\{1, 2, 3\}$ as follows: $w(v) = 2$ for every $v \in V(G)$, $w(\alpha) = w^*(\alpha^*)$ for every face $\alpha \in F(G)$ and $w(e) = w^*(e^*)$ for every edge $e \in E(G)$. Then the colour $f(\alpha)$ of the face $\alpha \in F(G)$ is defined as $f(\alpha) = w(\alpha) + \sum_{e \in E(G)} w(e) + \sum_{v \in V(G)} w(v) = w^*(\alpha^*) + \sum_{e \in E(G)} w^*(e^*) + 2m$ $= f^*(\alpha^*) + 2m$. But $f^*(\alpha^*) \neq f^*(\beta^*)$ if $\alpha^*\beta^*$ is an edge of G^*. This implies $f(\alpha) \neq f(\beta)$ for adjacent faces α and β because in this case $f^*(\alpha^*) \neq f^*(\beta^*)$. ■

An Eulerian plane graph G is a connected one each vertex of which has an even degree. It is well known that chromatic number $\chi(G^*) = 2$. Using Theorem 3 and this fact we obtain:

Theorem 7. *Every* 2*-connected Eulerian plane graph has a proper face-colouring entire* 2*-weighting.*

We expect that any 2-connected plane graph has a proper face-colouring entire 3-weighting. But unfortunately at this moment, we are not able to prove it. We can prove the following:

Theorem 8. *Every* 2*-connected cubic plane graph has a proper face-colouring entire* 3*-weighting.*

Proof. Let $G = (V, E, F)$ be a 2-connected cubic plane graph. Proof consists of two main parts. In the first part we associate each face α with colours $f(\alpha) = w(\alpha) + \sum_{e \in E(\alpha)} w(e) + \sum_{v \in V(\alpha)} w(v)$ using weighting w as in the proof of Theorem 3. This weighting w uses labels from $\{1, 2, 3, 4\}$ and has property that $f(\alpha) \neq f(\beta)$ whenever α and β share an edge in common. Note that the weights 4 are used only on some faces.

In the second part this weighting will be locally changed keeping the colours of faces fixed. The main aim is to delete (lowered) label 4 from faces of G. We proceed as follows: For every face α which $w(\alpha) = 4$ we choose a vertex $z \in V(\alpha)$ and two edges e_1 and e_2 incident with α and with z. Next we locally change the weighting w to the new weighting $\widetilde{w}$ so that $\widetilde{w}(\alpha) = w(\alpha) - 1 = 3$, $\widetilde{w}(z) = w(z) - 1 = 1$, $\widetilde{w}(e_i) = w(e_i) + 1 = 3$, $i = 1, 2$, for all quadruples α, z, e_1, e_2 with $w(\alpha) = 4$. For all other elements x of G we put $\widetilde{w}(x) = w(x)$. It is easy to see that $\widetilde{w}(y) \leq 3$ for all elements y of G and that the colours of all faces of G are not changed. ■

We even strongly believe that the following is true.

Conjecture 9. *Every* 2*-connected plane graph has a proper face-colouring entire* 2*-weighting.*

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