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HEAVY SUBGRAPH PAIRS FOR TRACEABILITY OF BLOCK-CHAINS

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Abstract

A graph is called traceable if it contains a Hamilton path, i.e., a path containing all its vertices. Let G be a graph on n vertices. We say that an induced subgraph of G is o_{-1}-heavy if it contains two nonadjacent vertices which satisfy an Ore-type degree condition for traceability, i.e., with degree sum at least $n-1$ in G. A block-chain is a graph whose block graph is a path, i.e., it is either a P_1, P_2, or a 2-connected graph, or a graph with at least one cut vertex and exactly two end-blocks. Obviously, every traceable graph is a block-chain, but the reverse does not hold. In this paper we characterize all the pairs of connected o_{-1}-heavy graphs that guarantee traceability of block-chains. Our main result is a common extension of earlier work on degree sum conditions, forbidden subgraph conditions and heavy subgraph conditions for traceability.

Keywords: o_{-1}-heavy subgraph, block-chain traceable graph, Ore-type condition, forbidden subgraph.

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1. Introduction

We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph. If a subgraph G' of G contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an *induced subgraph* of G. For a given graph H, we say that G is *H-free* if G does not contain an induced subgraph isomorphic to H. For a family $\mathcal{H}$ of graphs, G is called $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. Note that if H_1 is an induced subgraph of H_2, then an H_1-free graph is also H_2-free.

The graph $K_{1,3}$ is called a *claw*; its only vertex with degree 3 is called the *center* of the claw, and the other vertices are called the *end-vertices* of the claw.

If a graph is P_2-free, then it is an empty graph (contains no edges). To avoid the discussion of this trivial case, in the following, we throughout assume that our forbidden subgraphs have at least three vertices.

Some graphs that we will use in this paper are shown in Figure 1.

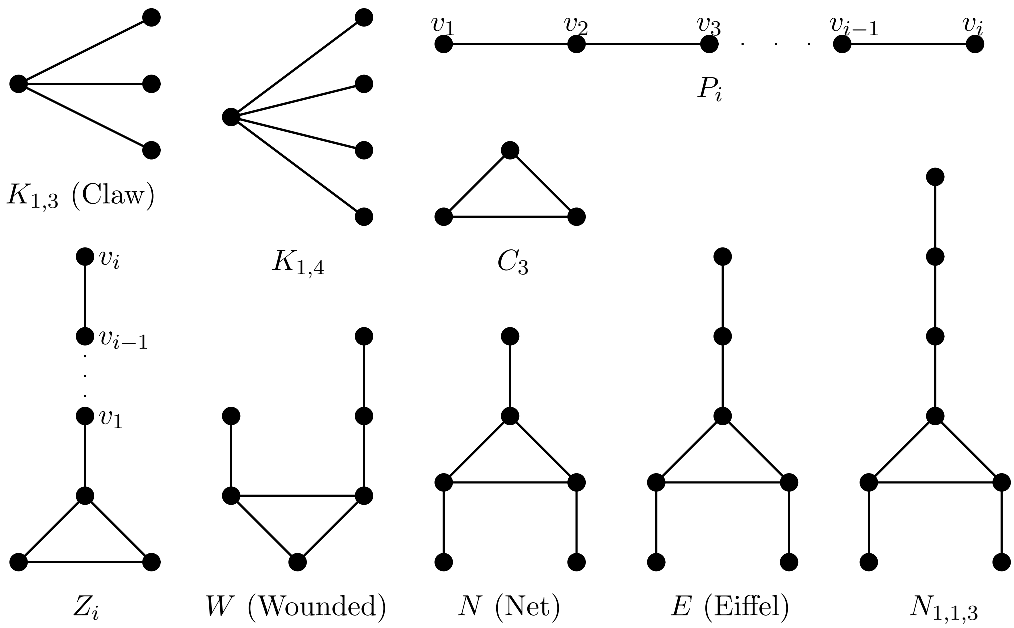

Figure 1. Graphs $K_{1,3}, K_{1,4}, P_i, C_3, Z_i, W, N, E$ and $N_{1,1,3}$.

A graph is called *traceable* if it contains a *Hamilton path*, i.e., a path containing all its vertices. If a graph is connected and P_3-free, then it is a complete graph and it is trivially traceable. In fact, it is not difficult to show that P_3 is the only single subgraph H such that every connected H-free graph is traceable. Moving to the more interesting case of pairs of subgraphs, the following theorem on forbidden pairs for traceability is well-known.

Theorem 1 (Duffus, Gould and Jacobson [6]). *If G is a connected $\{K_{1,3}, N\}$-free graph, then G is traceable.*

Obviously, if H is an induced subgraph of N, then the pair $\{K_{1,3}, H\}$ is also a forbidden pair that guarantees the traceability of every connected graph. In fact, Faudree and Gould proved that these are precisely all the forbidden pairs with this property.

Theorem 2 (Faudree and Gould [7]). *The only connected graph S such that every connected S-free graph is traceable is P_3. Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of N.*

Forbidding pairs of graphs as induced subgraphs might impose such a strong condition on the graphs under consideration that hamiltonian properties are almost trivially obtained. As an example, consider the graph Z_1 obtained from a claw by adding one edge between two end vertices of the claw. Then one easily shows that, apart from paths and cycles, connected $\{K_{1,3}, Z_1\}$-free graphs are only a matching away from complete graphs, i.e., their complements consist of isolated vertices and isolated edges. This is one of the motivations to relax forbidden subgraph conditions to conditions in which the subgraphs are allowed, but what additional conditions are imposed on these subgraphs if they are not forbidden. Early examples of this approach in the context of hamiltonicity and pancyclicity date back to the early 1990s [1, 4]. The idea to put a minimum degree bound on one or two of the end-vertices of an induced claw has been explored in [3]. Here we follow the ideas and terminology of [5] by putting an Ore-type degree sum condition on at least one pair of nonadjacent vertices in certain induced subgraphs. These degree sum conditions arise from one of the earliest papers in this area, in which Ore proved that a graph G on $n \geq 3$ vertices is hamiltonian if the degree sum of any two nonadjacent vertices of G is at least n. Ore's result implies that a graph on n vertices is traceable if the degree sum of any two nonadjacent vertices is at least $n-1$. A natural way to find common extensions of such degree sum conditions and forbidden subgraph conditions for traceability is to impose that certain pairs of vertices of induced subgraphs have degree sum at least $n-1$. This motivates the following concepts and terminology.

Let G be a graph on n vertices and let G' be an induced subgraph of G. We say that G' is *o_{-1}-heavy* if there are (at least) two nonadjacent vertices of G' with degree sum at least $n-1$ in G. For a given graph H, the graph G is *H-o_{-1}-heavy* if every induced subgraph of G isomorphic to H is o_{-1}-heavy. For a family $\mathcal{H}$

of graphs, G is called *$\mathcal{H}$-o_{-1}-heavy* if G is H-o_{-1}-heavy for every $H \in \mathcal{H}$. Note that an H-free graph is also H-o_{-1}-heavy; and if H_1 is an induced subgraph of H_2, then an H_1-o_{-1}-heavy graph is also H_2-o_{-1}-heavy. Hence the family of H-o_{-1}-heavy graphs is richer than the family of H-free graphs if H contains a pair of nonadjacent vertices, and the family of H'-o_{-1}-heavy graphs is richer than the family of H-o_{-1}-heavy graphs if H' properly contains H as an induced subgraph.

In this paper, instead of $K_{1,3}$-free and $K_{1,3}$-o_{-1}-heavy, we use the terms *claw-free* and *claw-o_{-1}-heavy*, respectively.

For connected $\mathcal{H}$-o_{-1}-heavy graphs, unfortunately only a small graph and a pair of small graphs can guarantee their traceability, as was shown recently in [11].

Theorem 3 (Li and Zhang [11]). *The only connected graph S such that every connected S-o_{-1}-heavy graph is traceable is P_3. Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$-o_{-1}-heavy implies G is traceable if and only if* (*up to symmetry*) *$R = K_{1,3}$ and $S = C_3$.*

Note that, since C_3 is a clique, a graph is C_3-o_{-1}-heavy if and only if it is C_3-free. For claw-o_{-1}-heavy and H-free graphs, in [11] the following stronger statement has been proved.

Theorem 4 (Li and Zhang [11]). *Let $H \neq P_3$ be a connected graph and let G be a connected claw-o_{-1}-heavy graph. Then G being H-free implies G is traceable if and only if $H = C_3$, Z_1 or P_4.*

In this paper, we are going to improve the above results by excluding graphs that are more or less trivially non-traceable. Therefore, we focus on graphs that satisfy a simple and easy to verify necessary condition for traceability. Adopting the terminology of [8], we say that a graph is a *block-chain* if it is nonseparable (2-connected or P_1 or P_2) or it has at least one cut vertex and has exactly two end-blocks. Note that every traceable graph is necessarily a block-chain, but that the reverse does not hold. Also note that it is easy to check by a polynomial algorithm whether a given graph is a block-chain or not.

In the 'only-if' part of the proof of Theorem 3 many graphs are used that are not block-chains (and are therefore trivially non-traceable). A natural extension is to consider forbidden subgraph and o_{-1}-heavy subgraph conditions for a block-chain to be traceable. Very recently, in [9] we characterized all the pairs of forbidden subgraphs with this property.

Theorem 5 (Li, Broersma and Zhang [9]). *The only connected graph S such that every S-free block-chain is traceable is P_3. Let R and S be connected graphs with $R, S \neq P_3$ and let G be a block-chain. Then G being $\{R, S\}$-free implies G is traceable if and only if* (*up to symmetry*) *$R = K_{1,3}$ and S is an induced subgraph of $N_{1,1,3}$, or $R = K_{1,4}$ and $S = P_4$.*

In this paper we characterize the pairs of connected graphs R and S other than P_3 guaranteeing that every $\{R,S\}$-o_{-1}-heavy block-chain is traceable. First note that we can easily obtain that the statement 'every H-o_{-1}-heavy block-chain is traceable' only holds if $H = P_3$. This can be deduced from Theorems 3 and 5. For o_{-1}-heavy pairs of subgraphs, we will prove the following common extension of Theorems 2 and 3.

Theorem 6. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a block-chain. Then G being $\{R,S\}$-o_{-1}-heavy implies G is traceable, if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of W or N.*

In Section 2, we prove the 'only if' part of Theorem 6. For the 'if' part of Theorem 6, it suffices to prove the following two statements.

Theorem 7. *If G is a $\{K_{1,3}, W\}$-o_{-1}-heavy block-chain, then G is traceable.*

Theorem 8. *If G is a $\{K_{1,3}, N\}$-o_{-1}-heavy block-chain, then G is traceable.*

We prove Theorems 7 and 8 in Sections 4 and 5, respectively.

2. The 'only if' Part of Theorem 6

Let R and S be two graphs other than P_3 such that every $\{R,S\}$-o_{-1}-heavy block-chain is traceable. By Theorem 5, we have that (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $N_{1,1,3}$, or $R = K_{1,4}$ and $S = P_4$.

In Figure 2, we sketched some families of block-chains that are not traceable. All members of these families have exactly two cut vertices, two end-blocks consisting of K_2s, and one 2-connected non-end-block, so all these graphs are obviously block-chains. Since all the graphs of these families have exactly two vertices with degree 1, it is easy to verify that they do not admit a Hamilton path (between these two vertices, because all the other vertices ought to be internal vertices of any Hamilton path). We leave the details for the reader.

Noting that members of $\mathcal{G}_4$ are $\{K_{1,4}, P_4\}$-o_{-1}-heavy, we get that $\{R,S\} \neq \{K_{1,4}, P_4\}$. Thus $R = K_{1,3}$ and S is an induced subgraph of $N_{1,1,3}$.

Note that all members of $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are claw-o_{-1}-heavy. So S must be a common induced subgraph of all members of $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ that is not o_{-1}-heavy. Note that all members of $\mathcal{G}_1$ are P_6-o_{-1}-heavy, all members of $\mathcal{G}_2$ are Z_3-o_{-1}-heavy, and all members of $\mathcal{G}_3$ are E-o_{-1}-heavy. The only remaining possibility is that S is an induced subgraph of W or N. This completes the proof of the 'only if' part of the statement of Theorem 6.

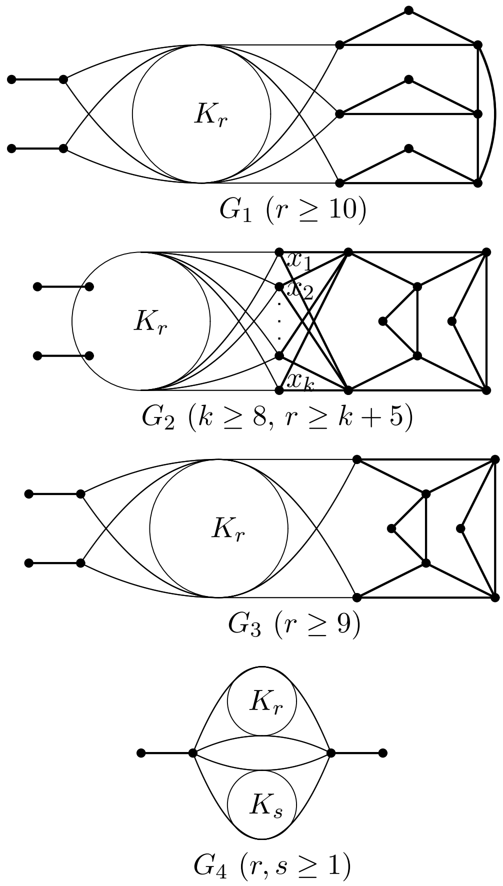

Figure 2. Some families of block-chains that are not traceable.

3. Preliminaries

In the next two sections we will prove Theorems 7 and 8, respectively. Before we do so, in this section we introduce some additional terminology and notation, and we will prove some useful lemmas.

We adopt the following terminology, notation and lemma from [10].

Throughout this paper, k and ℓ will always denote positive integers. If $k \leq \ell$, we use $[x_k, x_\ell]$ to denote the set $\{x_k, x_{k+1}, \ldots, x_\ell\}$. If $[x_k, x_\ell]$ is a nonempty subset of the vertex set of a graph G, we use $G[x_k, x_\ell]$ instead of $G[[x_k, x_\ell]]$, to denote the subgraph induced by $[x_k, x_\ell]$ in G.

Let P be a path and $x, y \in V(P)$. We use $P[x, y]$ to denote the subpath of P from x to y (inclusive).

Let G be a graph and $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be two pairs of vertices in $V(G)$

with $x_1 \neq x_2$ and $y_1 \neq y_2$. We define an $(\{x_1, x_2\}, \{y_1, y_2\})$-*disjoint path pair*, or briefly an (x_1x_2, y_1y_2)-*pair*, as a union of two vertex-disjoint paths P and Q such that

(1) the origins of P and Q are in $\{x_1, x_2\}$, and

(2) the termini of P and Q are in $\{y_1, y_2\}$.

If G is a graph on $n \geq 2$ vertices, $x \in V(G)$, and a graph G' is obtained from G by adding a new vertex y and a pair of edges yx, yz, where $z \neq x$ is an arbitrary vertex of G, then we say that G' is a 1-*extension of* G *at* x *to* y. Similarly, if $x_1, x_2 \in V(G)$, $x_1 \neq x_2$, then the graph G' obtained from G by adding two new vertices y_1, y_2 and the edges y_1x_1, y_2x_2 and y_1y_2 is called the 2-*extension of* G *at* (x_1, x_2) *to* (y_1, y_2). We also call G' a 1-*extension* (*at* x *to* y) or 2-*extension* (*at* (x_1, x_2) *to* (y_1, y_2)) of G if it contains a spanning subgraph that is a 1-extension (at x to y) or 2-extension (at (x_1, x_2) to (y_1, y_2)) of G.

Let G be a graph and let $u, v, w \in V(G)$ be distinct vertices of G. We say that G is (u, v, w)-*composed* (or briefly *composed*) if G has a spanning subgraph D (called the *carrier* of G) such that there is an ordering $v_{-k}, \ldots, v_0, \ldots, v_\ell$ $(k, \ell \geq 1)$ of $V(D)$ $(=V(G))$ and a sequence of graphs $D_1, \ldots, D_r$ $(r \geq 1)$ such that

(1) $u = v_{-k}$, $v = v_0$, $w = v_\ell$,

(2) D_1 is a triangle with $V(D_1) = \{v_{-1}, v_0, v_1\}$,

(3) $V(D_i) = [v_{-k_i}, v_{\ell_i}]$ for some k_i, ℓ_i, $1 \leq k_i \leq k$, $1 \leq \ell_i \leq \ell$, and D_{i+1}, $i = 1, \ldots, r-1$, satisfies one of the following:

   (a) D_{i+1} is a 1-extension of D_i at v_{-k_i} to v_{-k_i-1} or at v_{ℓ_i} to v_{ℓ_i+1}, or

   (b) D_{i+1} is a 2-extension of D_i at (v_{-k_i}, v_{ℓ_i}) to $(v_{-k_i-1}, v_{\ell_i+1})$,

(4) $D_r = D$.

The ordering $v_{-k}, \ldots, v_0, \ldots, v_\ell$ will be called a *canonical ordering* and the sequence $D_1, \ldots, D_r$ a *canonical sequence* of D (and also of G). Note that a composed graph G can have several carriers, canonical orderings and canonical sequences. Clearly, a composed graph G and any of its carriers D are 2-connected, for any canonical ordering; moreover, $P = v_{-k} \cdots v_0 \cdots v_\ell$ is a Hamilton path in D (called a *canonical path*), and if $D_1, \ldots, D_r$ is a canonical sequence, then any D_i is $(v_{-k_i}, v_0, v_{\ell_i})$-composed, $i = 1, \ldots, r$. Note that a (u, v, w)-composed graph is also (w, v, u)-composed.

The following lemma on composed graphs will be needed in our proofs. A proof of the lemma can be found in [10].

Lemma 9. *Let G be a composed graph and let D and $v_{-k}, \ldots, v_0, \ldots, v_\ell$ be a carrier and a canonical ordering of G. Then*

(1) *D has a Hamilton (v_0, v_{-k})-path, and*

(2) *for every* $v_s \in V(G) \setminus \{v_{-k}\}$*, D has a spanning* (v_0v_ℓ, v_sv_{-k})*-pair.*

Let G be a graph on n vertices. A sequence of vertices $v_1v_2 \cdots v_k$ such that for all $i \in [1, k-1]$, either $v_iv_{i+1} \in E(G)$ or $d(v_i) + d(v_{i+1}) \geq n-1$, is called an o_{-1}*-path* of G.

The following useful lemma on o_{-1}-paths is proved in [11], and the reader can find an analogous cycle version of the lemma in [10]. Its elementary proof is similar to the proof of the aforementioned result of Ore, and that forms the basis for the well-known Bondy-Chvátal closure for hamiltonicity.

Lemma 10. *Let G be a graph and let P' be an o_{-1}-path in G. Then there is a path P in G such that $V(P') \subset V(P)$.*

Let G be a graph on n vertices. In the following, we denote $\overline{E}_{-1}(G) = \{uv : uv \in E(G) \text{ or } d(u) + d(v) \geq n-1\}$. Let D be an (x_1x_2, y_1y_2)-pair of G. If $x_1x_2 \in \overline{E}_{-1}(G)$ or $y_1y_2 \in \overline{E}_{-1}(G)$, then using Lemma 10, it is easy to see that G contains a path P with $V(D) \subset V(P)$.

Let G be a graph on n vertices, P be a path of G, $x_1, x, x_2 \in V(P)$ be three distinct vertices appearing in the given order along P, and set $X = V(P[x_1, x_2])$. We say that the pair (x_1, x_2) is x*-good on* P, if for some $j \in \{1, 2\}$ there is a vertex $x' \in X \setminus \{x_j\}$ such that

(1) there is an (x, x_{3-j})-path Q with $V(Q) = X \setminus \{x_j\}$,

(2) there is an $(xx_{3-j}, x'x_j)$-pair D with $V(D) = X$, and

(3) $d(x_j) + d(x') \geq n-1$.

In this case, we say that Q and D are a path and disjoint path pair *associated with* x, respectively. We present and prove one final useful lemma in this section.

Lemma 11. *Let G be a graph, and P be a path of G. Let $x, y \in V(P)$ and let R be an (x, y)-path in G which is internally-disjoint with P. If there are vertices $x_1, x_2, y_1, y_2 \in V(P) \setminus \{x, y\}$ such that*

(1) x_1, x, x_2, y_1, y, y_2 appear in this order along P (where possibly $x_2 = y_1$),

(2) (x_1, x_2) is x-good on P, and

(3) (y_1, y_2) is y-good on P,

then there is a path P' in G such that $V(P) \cup V(R) \subset V(P')$.

Proof. Assume the contrary. Let Q_1 and D_1 be a path and disjoint path pair associated with x, and let Q_2 and D_2 be a path and disjoint path pair associated with y. Let $R' = P[x_2, y_1]$, $R_1 = P[v_1, x_1]$ and $R_2 = P[y_2, v_p]$, where v_1 is the origin and v_p is the terminus of P.

Using the definition of x-good, we distinguish two main cases and a number of subcases.

Case 1. Q_1 is an (x, x_1)-path, D_1 is an $(xx_1, x'x_2)$-pair, and $d(x_2) + d(x') \geq n-1$.

Case 1.1. Q_2 is a (y, y_2)-path, D_2 is a $(yy_2, y'y_1)$-pair, and $d(y_1) + d(y') \geq n-1$. In this subcase the path $T = R_1 \cup R_2 \cup R \cup R' \cup Q_1 \cup D_2$ is a (v_1v_p, x_2y')-pair which contains all the vertices of $V(P) \cup V(R)$, and $T' = R_1 \cup R_2 \cup R \cup R' \cup Q_2 \cup D_1$ is a $(v_1v_p, x'y_1)$-pair which contains all the vertices of $V(P) \cup V(R)$. Thus by Lemma 10, $d(x_2) + d(y') < n-1$ and $d(x') + d(y_1) < n-1$, a contradiction to $d(x_2) + d(x') \geq n-1$ and $d(y_1) + d(y') \geq n-1$.

Case 1.2. Q_2 is a (y, y_1)-path, D_2 is a $(yy_1, y'y_2)$-pair, and $d(y_2) + d(y') \geq n-1$.

Case 1.2.1. The $(xx_1, x'x_2)$-pair D_1 is formed by an (x, x_2)-path and an (x_1, x')-path. In this subcase, the path $T = R_1 \cup R_2 \cup R \cup R' \cup Q_1 \cup Q_2$ is a (v_1v_p, x_2y_2)-pair which contains all the vertices of $V(P) \cup V(R)$, and the path $T' = R_1 \cup R_2 \cup R \cup R' \cup D_1 \cup D_2$ is a $(v_1v_p, x'y')$-pair which contains all the vertices of $V(P) \cup V(R)$. By Lemma 10, $d(x_2) + d(y_2) < n-1$ and $d(x') + d(y') < n-1$, a contradiction.

Case 1.2.2. The $(xx_1, x'x_2)$-pair D_1 is formed by an (x, x')-path and an (x_1, x_2)-path.

Case 1.2.2.1 The $(yy_1, y'y_2)$-pair D_2 is formed by a (y, y_2)-path and a (y_1, y')-path. This subcase can be proved similarly as Case 1.2.1.

Case 1.2.2.2. The $(yy_1, y'y_2)$-pair D_2 is formed by a (y, y')-path and a (y_1, y_2)-path. In this subcase, the path $T = R_1 \cup R_2 \cup R \cup R' \cup Q_1 \cup D_2$ is a (v_1v_p, x_2y')-pair which contains all the vertices of $V(P) \cup V(R)$, and the path $T' = R_1 \cup R_2 \cup R \cup R' \cup D_1 \cup Q_1$ is a $(v_1v_p, x'y_2)$-pair which contains all the vertices of $V(P) \cup V(R)$. By Lemma 10, $d(x_2) + d(y') < n-1$ and $d(x') + d(y_2) < n-1$, a contradiction.

Case 2. Q_1 is an (x, x_2)-path, D_1 is an $(xx_2, x'x_1)$-pair, and $d(x_1) + d(x') \geq n-1$.

Case 2.1. Q_2 is a (y, y_2)-path, D_2 is a $(yy_2, y'y_1)$-pair, and $d(y_1) + d(y') \geq n-1$. This case can be proved similarly as Case 1.2.

Case 2.2. Q_2 is a (y, y_1)-path, D_2 is a $(yy_1, y'y_2)$-pair, and $d(y_2) + d(y') \geq n-1$. In this subcase the path $T = R_1 \cup R_2 \cup R \cup R' \cup D_1 \cup Q_2$ is a (v_1v_p, x_1y')-pair which contains all the vertices of $V(P) \cup V(R)$, and $T' = R_1 \cup R_2 \cup R \cup R' \cup Q_1 \cup D_2$ is a $(v_1v_p, x'y_2)$-pair which contains all the vertices of $V(P) \cup V(R)$. By Lemma 10, $d(x_1) + d(y') < n-1$ and $d(x') + d(y_2) < n-1$, a contradiction.

This completes the proof of Lemma 11. ■

Let G be a graph with at least one cut vertex and exactly two end-blocks, and let P be a path of G. If the two end-vertices of P are inner vertices (not a cut vertex of G) of two distinct end-blocks of G, then we call P a *penetrating path* of G. If G is a nonseparable graph, then every path of G is considered to be a penetrating path. Note that a penetrating path of a block-chain G contains all the cut vertices of G, and that a path of a block-chain G is a penetrating path if and only if for every end-block of G the path contains at least one inner vertex of the end-block.

4. Proof of Theorem 7

Proof. Suppose G is a $\{K_{1,3}, W\}$-o_{-1}-heavy block-chain on n vertices. It suffices to prove that G is traceable. We proceed by contradiction.

Clearly, G contains a penetrating path. Let P be a longest penetrating path of G. We use p to denote the number of vertices of P. Assume that G is not traceable. Then $V(G) \setminus V(P) \neq \emptyset$. Let H be a component of $G - V(P)$. If $N_P(H)$ consists of only one vertex x, then $G[H \cup \{x\}]$ contains an end-block of G, contradicting that P is a penetrating path of G. Thus we assume that H has at least two neighbors on P. Let R be a path with two end-vertices on P, all internal vertices in H, and of length at least 2; subject to this, we choose R as short as possible. Suppose without loss of generality, that $P = v_1v_2 \cdots v_p$ and $R = z_0z_1z_2 \cdots z_{r+1}$, where $z_0 = v_s$ and $z_{r+1} = v_t$, $s < t$.

It is easy to see that $N(v_1) \subset V(P)$ and $N(v_p) \subset V(P)$. Thus we have $2 \leq s < t \leq p - 1$. We are going to prove ten claims in order to reach a contradiction in all cases.

Claim 1. *Let $x \in V(H)$ and $y \in \{v_{s-1}, v_{s+1}, v_{t-1}, v_{t+1}\}$. Then $xy \notin \overline{E}_{-1}(G)$.*

Proof. Without loss of generality, assume $y = v_{s-1}$ and $xy \in \overline{E}_{-1}(G)$. Let Q' be an (x, z_1)-path in H. Then $Q = P[v_1, v_{s-1}]v_{s-1}xQ'z_1v_sP[v_s, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(Q')$. By Lemma 10, there is a path containing all the vertices of $V(P) \cup V(Q')$, which is a longer penetrating path than P, a contradiction. $\square$

Claim 2. $v_{s-1}v_{s+1} \in \overline{E}_{-1}(G)$, $v_{t-1}v_{t+1} \in \overline{E}_{-1}(G)$.

Proof. If $v_{s-1}v_{s+1} \notin E(G)$, then using Claim 1, the graph induced by $\{v_s, z_1, v_{s-1}, v_{s+1}\}$ is a claw, where $d(z_1) + d(v_{s\pm1}) < n-1$. Since G is a claw-o_{-1}-heavy graph, we have that $d(v_{s-1}) + d(v_{s+1}) \geq n - 1$.

The second assertion can be proved similarly. $\square$

Claim 3. $v_{s-1}v_{t-1} \notin \overline{E}_{-1}(G)$, $v_{s+1}v_{t+1} \notin \overline{E}_{-1}(G)$, $v_sv_{t\pm1} \notin \overline{E}_{-1}(G)$ and $v_{s\pm1}v_t \notin \overline{E}_{-1}(G)$.

Proof. If $v_{s-1}v_{t-1} \in \overline{E}_{-1}(G)$, then $Q = P[v_1, v_{s-1}]v_{s-1}v_{t-1}P[v_{t-1}, v_s]v_sRv_tP[v_t, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$. Using Lemma 10, we reach a contradiction.

If $v_sv_{t-1} \in \overline{E}_{-1}(G)$, then $Q = P[v_1, v_{s-1}]v_{s-1}v_{s+1}P[v_{s+1}, v_{t-1}]v_{t-1}v_sRv_tP[v_t, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, again a contradiction.

If $v_sv_{t+1} \in \overline{E}_{-1}(G)$, then $Q = P[v_1, v_{s-1}]v_{s-1}v_{s+1}P[v_{s+1}, v_t]v_tRv_sv_{t+1}P[v_{t+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, again a contradiction.

The other assertions can be proved similarly. □

Claim 4. *Either* $v_{s-1}v_{s+1} \in E(G)$ *or* $v_{t-1}v_{t+1} \in E(G)$.

Proof. Assume the contrary. By Claim 2, we have $d(v_{s-1}) + d(v_{s+1}) \geq n - 1$ and $d(v_{t-1}) + d(v_{t+1}) \geq n - 1$. By Claim 3, we have $d(v_{s-1}) + d(v_{t-1}) < n - 1$ and $d(v_{s+1}) + d(v_{t+1}) < n - 1$, a contradiction. □

Now, we distinguish two cases. We deal with the case that $r = 1$ and $v_sv_t \in E(G)$ later, but first deal with the case that $r \geq 2$, or $r = 1$ and $v_sv_t \notin E(G)$.

Case 1. $r \geq 2$, or $r = 1$ and $v_sv_t \notin E(G)$. By Claim 4, without loss of generality, we assume that $v_{s-1}v_{s+1} \in E(G)$. Thus $G[v_{s-1}, v_{s+1}]$ is (v_{s-1}, v_s, v_{s+1})-composed.

Claim 5. $v_sz_2 \notin \overline{E}_{-1}(G)$.

Proof. By the choice of the path R, we have $v_sz_2 \notin E(G)$. Now we are going to prove that $d(v_s) + d(z_2) < n - 1$. In order to show this, we first prove a number of subclaims.

Claim 5.1. *Every neighbor of* v_s *is in* $V(P) \cup V(H)$, *every neighbor of* z_2 *is in* $V(P) \cup V(H)$.

Proof. Assume the contrary. Let $z' \in V(H')$ be a neighbor of v_s, where H' is a component of $G - V(P)$ other than H. Then we have $z'z_1 \notin E(G)$ and $N_{G-P}(z') \cap N_{G-P}(z_1) = \emptyset$.

By Claim 1, we have $v_{s-1}z_1 \notin \overline{E}_{-1}(G)$, and similarly, $v_{s-1}z' \notin \overline{E}_{-1}(G)$. Thus the graph induced by $\{v_s, v_{s-1}, z_1, z'\}$ is a claw with $d(v_{s-1}) + d(z_1) < n - 1$ and $d(v_{s-1}) + d(z') < n - 1$. Thus we get that $d(z_1) + d(z') \geq n - 1$.

Since $N_{G-P}(z_1) \cap N_{G-P}(z') = \emptyset$, and z, z' are both not adjacent to v_1 and v_p, there exists some i with $2 \leq i \leq p - 2$ such that $z_1v_i, z'v_{i+1} \in E(G)$. Thus $Q = P[v_1, v_i]v_iz_1z'v_{i+1}P[v_{i+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup \{z_1, z'\}$. By Lemma 10, there exists a penetrating path containing all the vertices of $V(P) \cup \{z_1, z'\}$, a contradiction.

If $z_2 = v_t$, the second assertion can be proved similarly; and if $z_2 \neq v_t$, the assertion is obvious. □

Let $h = |V(H)|$, and $k = |N_H(v_s)|$. Then we have $d_H(v_s)+d_H(z_2) \leq h+k$. Since $z_1 \in N_H(v_s)$, we have $k \geq 1$. Let $N_H(v_s) = \{y_1, y_2, \ldots, y_k\}$, where $y_1 = z_1$.

Claim 5.2. *$y_iy_j \in \overline{E}_{-1}(G)$ for all $1 \leq i < j \leq k$.*

Proof. If $y_iy_j \notin E(G)$, then by Claim 1, the graph induced by $\{v_s, v_{s-1}, y_i, y_j\}$ is a claw, where $d(y_i) + d(v_{s-1}) < n-1$ and $d(y_j) + d(v_{s-1}) < n-1$. Thus we have $d(y_i) + d(y_j) \geq n-1$. □

Now, let Q' be the o_{-1}-path $Q' = z_2y_1y_2\cdots y_kv_s$. It is clear that $R[z_2, v_t]$ and Q' are internally-disjoint, and Q' contains at least k vertices of H. In the following, we use P' to denote the path $P[v_1, v_{s-1}]v_{s-1}v_{s+1}P[v_{s+1}, v_p]$ if $z_2 \neq v_t$, and to denote the o_{-1}-path $P[v_1, v_{s-1}]v_{s-1}v_{s+1}P[v_{s+1}, v_{t-1}]v_{t-1}v_{t+1}P[v_{t+1}, v_p]$ if $z_2 = v_t$.

Claim 5.3. *If $v_sv_i \in E(G)$ for some i with $2 \leq i \leq p-1$, then $z_2v_{i-1}, z_2v_{i+1} \notin E(G)$.*

Proof. If $v_sv_i \in E(G)$ for some i with $2 \leq i \leq p-1$ and $z_2v_{i-1} \in E(G)$, then $Q = P'[v_1, v_{i-1}]v_{i-1}z_2Q'v_sv_iP'[v_i, v_p]$ is an o_{-1}-path containing all vertices of $V(P) \cup V(Q')$. By Lemma 10, we have a contradiction.

Similarly, we can prove the assertion for z_2v_{i+1}. □

By Claim 3, we have $v_sv_{t-1} \notin E(G)$. Let v_ℓ be the last vertex in $P[v_{s+1}, v_{t-1}]$ such that $v_sv_\ell \in E(G)$.

Claim 5.4. *$t - \ell \geq k+1$, and for every vertex $v_i \in [v_{\ell+1}, v_{\ell+k}]$, $z_2v_i \notin E(G)$.*

Proof. If $t - \ell \leq k$, then $Q = P[v_1, v_{s-1}]\, v_{s-1}v_{s+1}P[v_{s+1}, v_\ell]\, v_\ell v_s\, Q'\, z_2\, R[z_2, v_t]v_t$ $P[v_t, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(Q') \cup V(R) \setminus [v_{\ell+1}, v_{t-1}]$, which yields a longer penetrating path than P. Using Lemma 10, we have a contradiction.

If $z_2v_i \notin E(G)$ for some $v_i \in [v_{\ell+1}, v_{\ell+k}]$, then $Q = P'[v_1, v_\ell]v_\ell v_sQ'z_2v_iP'[v_i, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(Q') \setminus [v_{\ell+1}, v_{i-1}]$, which again yields a longer penetrating path than P, a contradiction. □

Claim 5.5. $d_{[v_{s+1}, v_{t-1}]}(v_s) + d_{[v_{s+1}, v_{t-1}]}(z_2) \leq t - s - k - 1$, $d_{[v_1, v_{s-1}]}(v_s) + d_{[v_1, v_{s-1}]}(z_2) \leq s - 2$; $d_{[v_{t+1}, v_p]}(v_s) + d_{[v_{t+1}, v_p]}(z_2) \leq p - t - 1$.

Proof. Note that $v_sv_{t-1} \notin E(G)$ and $z_2v_{s+1} \notin E(G)$. If v_s has d neighbors in $[v_{s+1}, v_{t-2}]$, then by Claims 5.3. and 5.4 has at most $t-s-2-d-k+1$ neighbors in $[v_{s+2}, v_{t-1}]$.

Note that $z_2v_{s-1} \notin E(G)$ and $v_sv_1 \notin E(G)$. If v_s has d neighbors in $[v_2, v_{s-1}]$, then by Claim 5.3, z_2 has at most $s-2-d$ neighbors in $[v_1, v_{s-2}]$.

Similarly, note that $v_sv_{t+1} \notin E(G)$ and $z_2v_p \notin E(G)$. If z_2 has d neighbors in $[v_{t+1}, v_{p-1}]$, then by Claim 5.3, v_s has at most $p-t-1-d$ neighbors in $[v_{t+2}, v_p]$. □

Now we can complete the proof of Claim 5. Note that v_s and z_2 are possibly adjacent to v_t, but they cannot be adjacent to v_s. By Claim 5.3, we have $d_P(v_s)+d_P(z_2) \leq p-k-2$. Recall that $d_H(v_s)+d_H(z_2) \leq h+k$. By Claim 5.1, we have that $d(v_s)+d(z_2) \leq p+h-2 < n-1$. □

Recall that $G[v_{s-1}, v_{s+1}]$ is (v_{s-1}, v_s, v_{s+1})-composed. Now we prove the following claims.

Claim 6. *If $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed with canonical path $P[v_{s-k}, v_{s+\ell}]$, then $s-k \geq 2$ and $s+\ell \leq t-3$.*

Proof. Let $D_1, D_2, \ldots, D_r$ be a canonical sequence of $G[v_{s-k}, v_{s+\ell}]$ corresponding to the canonical path $P[v_{s-k}, v_{s+\ell}]$. If $s-k=1$, then by Lemma 9, there is a Hamilton $(v_s, v_{s+\ell})$-path Q' of $G[v_{s-k}, v_{s+\ell}]$. Thus $Q = z_1 v_s Q' v_{s+\ell} P[v_{s+\ell}, v_p]$ is a path containing all the vertices of $V(P) \cup \{z_1\}$, a contradiction.

If $s+\ell \geq t-2$, then consider the graph D_i, where i is the smallest integer such that $v_{t-2} \in V(D_i)$. Let $V(D_i) = [v_{s-k'}, v_{t-2}]$. By Lemma 9, there exists a Hamilton $(v_{s-k'}, v_s)$-path Q' of $G[v_{s-k'}, v_{t-2}]$. Thus $Q = P[v_1, v_{s-k'}] v_{s-k'} Q' v_s R v_t v_{t-1} v_{t+1} P[v_{t+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, yielding another contradiction. □

Claim 7. *If $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed with canonical path $P[v_{s-k}, v_{s+\ell}]$, where $s-k \geq 2$ and $s+\ell \leq t-3$, and any two nonadjacent vertices in $[v_{s-k-1}, v_{s+\ell+1}]$ have degree sum less than $n-1$, then one of the following is true,*

(1) *$G[v_{s-k-1}, v_{s+\ell}]$ is a 1-extension of $G[v_{s-k}, v_{s+\ell}]$ at v_{s-k} to v_{s-k-1};*

(2) *$G[v_{s-k}, v_{s+\ell+1}]$ is a 1-extension of $G[v_{s-k}, v_{s+\ell}]$ at $v_{s+\ell}$ to $v_{s+\ell+1}$; or*

(3) *$G[v_{s-k-1}, v_{s+\ell+1}]$ is a 2-extension of $G[v_{s-k}, v_{s+\ell}]$ at $(v_{s-k}, v_{s+\ell})$ to $(v_{s-k-1}, v_{s+\ell+1})$.*

Thus in all cases we obtain a composed graph larger than $G[v_{s-k}, v_{s+\ell}]$.

Proof. Assume the contrary. This implies that v_{s-k-1} has only one neighbor v_{s-k}, and $v_{s+\ell+1}$ has only one neighbor $v_{s+\ell}$ in $[v_{s-k-1}, v_{s+\ell+1}]$. We prove a number of subclaims in order to reach contradictions in all cases.

Claim 7.1. *Let $i \in [s-k-1, s+\ell+1] \setminus \{s\}$ and $j = 1, 2$. Then $v_i z_j \notin \overline{E}_{-1}(G)$.*

Proof. Without loss of generality, we assume that $i < s$. If $i = s-1$, the assertion is true by Claims 1 and 3. So we assume that $i \in [s-k-1, s-2]$ and $i+1 \in [s-k, s-1]$. By the definition of composed subgraphs, there exists an $i' \in [s+1, s+\ell]$ such that $G[v_i, v_{i'}]$ is $(v_i, v_s, v_{i'})$-composed. By Lemma 9, there exists a Hamilton $(v_s, v_{i'})$-path Q' of $G[v_i, v_{i'}]$.

If $z_j \neq v_t$, then $Q = P[v_1, v_i] v_i z_j R[z_j, v_s] v_s Q' v_{i'} P[v_{i'}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup \{z_j\}$, yielding a contradiction.

If $z_j = v_t$, then $Q = P[v_1, v_i]v_iv_tRv_sQ'v_{i'}P[v_{i'}, v_{t-1}]v_{t-1}v_{t+1}P[v_{t+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, yielding another contradiction. □

Let $G' = G[[v_{s-k-1}, v_{s+\ell}] \cup \{z_1, z_2\}]$ and $G'' = G[[v_{s-k-1}, v_{s+\ell+1}] \cup \{z_1, z_2\}]$.

Claim 7.2. *G'' and G' are $\{K_{1,3}, W\}$-free.*

Proof. By Claims 5 and 7.1, and the condition that any two nonadjacent vertices in $[v_{s-k-1}, v_{s+\ell+1}]$ have degree sum less than $n-1$, we have that any two nonadjacent vertices in G'' have degree sum less than $n-1$. Since G (and hence G'') is $\{K_{1,3}, W\}$-o_{-1}-heavy, we have that G'' is $\{K_{1,3}, W\}$-free. The second assertion follows easily. □

Claim 7.3. *$N_{G'}(v_s) \setminus \{z_1\}$ is a clique.*

Proof. If there are two vertices $x, x' \in N_{G'}(v_s) \setminus \{z_1\}$ such that $xx' \notin E(G')$, then the graph induced by $\{v_s, z_1, x, x'\}$ is a claw, a contradiction. □

Now, we define $N_i = \{x \in V(G') : d_{G'}(x, v_{s-k-1}) = i\}$. Then we have $N_0 = \{v_{s-k-1}\}$, $N_1 = \{v_{s-k}\}$ and $N_2 = N_{G'}(v_{s-k}) \setminus \{v_{s-k-1}\}$.

By the definition of composed subgraphs, we have $|N_2| \geq 2$. If there are two vertices $x, x' \in N_2$ such that $xx' \notin E(G')$, then the graph induced by $\{v_{s-k}, v_{s-k-1}, x, x'\}$ is a claw, a contradiction. Thus we have that N_2 is a clique.

We assume $v_s \in N_j$, where $j \geq 2$. Then $z_1 \in N_{j+1}$ and $z_2 \in N_{j+2}$.

If $|N_i| = 1$ for some $i \in [2, j-1]$, let $N_i = \{x\}$. Then x is a cut vertex of the graph $G[v_{s-k}, v_{s+\ell}]$. By the definition of composed subgraphs, $G[v_{s-k}, v_{s+\ell}]$ is 2-connected. This implies $|N_i| \geq 2$ for every $i \in [2, j-1]$.

Claim 7.4. *For $i \in [1, j]$, N_i is a clique.*

Proof. We prove this claim by induction on i. For $i = 1, 2$, the claim is true by the above arguments. So we assume that $3 \leq i \leq j$, and we have that $N_{i-3}, N_{i-2}, N_{i-1}, N_{i+1}$ and N_{i+2} are nonempty, and that $|N_{i-1}| \geq 2$.

Let x be a vertex in N_i that has a neighbor y in N_{i+1}. We claim that for every $x' \in N_i$, $xx' \in E(G)$. Suppose that $xx' \notin E(G)$. If x and x' have a common neighbor in N_{i-1}, denote it by w; then let v be a neighbor of w in N_{i-2}, and the graph induced by $\{w, v, x, x'\}$ is a claw, a contradiction. Thus we have that x and x' have no common neighbors in N_{i-1}. Now, let w be a neighbor of x in N_{i-1}, and let w' be a neighbor of x' in N_{i-1}. Then $xw', x'w \notin E(G)$. Let v be a neighbor of w in N_{i-2}, and let u be a neighbor of v in N_{i-3}. If $w'v \notin E(G)$, then the graph induced by $\{w, v, w', x\}$ is a claw, a contradiction. Thus we have $w'v \in E(G)$, and then the graph induced by $\{w', v, u, w, x, y\}$ is a W, a contradiction. Thus as we claimed, x is adjacent to every other vertex in N_i.

Now, we claim that for every two distinct vertices x' and x'' in N_i other than x, $x'x'' \in E(G)$. Supposed that $x'x'' \notin E(G)$. If $x'y \in E(G)$, then similarly as before, we can prove that x' is adjacent to any other vertices in N_i; then $x'x'' \in E(G)$. Thus we assume that $x'y \notin E(G)$, and similarly, $x''y \notin E(G)$. Then the graph induced by $\{x, x', x'', y\}$ is a claw, a contradiction. This implies that N_i is a clique. □

If there exists some vertex $x \in N_j$ other than v_s, then we have $v_s x \in E(G)$ by Claim 7.4. Let w be a neighbor of v_s in N_{j-1}, and let v be a neighbor of w in N_{j-2}. Then $wx \in E(G)$ by Claim 7.3. Thus the graph induced by $\{x, w, v, v_s, z_1, z_2\}$ is a W, a contradiction. So we assume N_j consists of only one vertex v_s.

If there exists some vertex $x \in N_{j+1}$ other than z_1, then v_s is a cut vertex of the graph $G[v_{s-k}, v_{s+\ell}]$, a contradiction. So we assume that all vertices in $[u_{-k}, u_\ell]$ are in $\bigcup_{i=1}^{j} N_i$.

Let $v_{s+\ell} \in N_i$, where $i \in [2, j-1]$. If $v_{s+\ell}$ has a neighbor in N_{i+1}, then let y be a neighbor of $v_{s+\ell}$ in N_{i+1}, and let w be a neighbor of $v_{s+\ell}$ in N_{i-1}. Then the graph induced by $\{v_{s+\ell}, w, y, v_{s+\ell+1}\}$ is a claw, a contradiction. So we have that $v_{s+\ell}$ has no neighbors in N_{i+1}.

Let z be a vertex in N_{i+2}, let y be a neighbor of z in N_{i+1}, let x be a neighbor of y in N_i, and let w be a neighbor of x in N_{i-1}. Thus $x \neq v_{s+\ell}$. If $wv_{s+\ell} \notin E(G)$, then the graph induced by $\{x, w, v_{s+\ell}, y\}$ is a claw, a contradiction. So we have that $wv_{s+\ell} \in E(G)$ and the graph induced by $\{w, v_{s+\ell}, v_{s+\ell+1}, x, y, z\}$ is a W, a contradiction. This final contradiction completes the proof of Claim 7. □

Using Claim 7, we can consider a largest composed subgraph, in the following sense. We choose k, ℓ such that:

(1) $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed with canonical path $P[v_{s-k}, v_{s+\ell}]$,

(2) any two nonadjacent vertices in $[v_{s-k}, v_{s+\ell}]$ have degree sum less than $n-1$ and

(3) $k + \ell$ is as large as possible.

Claim 8. *$(v_{s-k-1}, v_{s+\ell})$ or $(v_{s-k}, v_{s+\ell+1})$ or $(v_{s-k-1}, v_{s+\ell+1})$ is v_s-good on P.*

Proof. By Claim 7, we have that there exists a vertex $v_i \in [v_{s-k+1}, v_{s+\ell}]$ such that $d(v_{s-k-1}) + d(v_i) \geq n-1$, or there exists a vertex $v_i \in [v_{s-k}, v_{s+\ell-1}]$ such that $d(v_{s+\ell+1}) + d(v_i) \geq n-1$, or $d(v_{s-k-1}) + d(v_{s+\ell+1}) \geq n-1$.

Suppose first that there exists a vertex $v_i \in [v_{s-k+1}, v_{s+\ell}]$ such that $d(v_{s-k-1}) + d(v_i) \geq n-1$. Since $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed, by Lemma 9, there exists a $(v_s, v_{s+\ell})$-path Q such that $V(Q) = [v_{s-k}, v_{s+\ell}]$, and there exists a $(v_s v_{s+\ell}, v_i v_{s-k})$-pair D' such that $V(D') = [v_{s-k}, v_{s+\ell}]$, and $D = D' \cup \{v_{s-k} v_{s-k-1}\}$ is a $(v_s v_{s+\ell}, v_i v_{s-k-1})$-pair such that $V(D) = [v_{s-k-1}, v_{s+\ell}]$. Thus $(v_{s-k-1}, v_{s+\ell})$ is v_s-good on P.

If there exists a vertex $v_i \in [v_{s-k}, v_{s+\ell-1}]$ such that $d(v_{s+\ell+1}) + d(v_i) \geq n-1$, we can prove the result similarly.

Now suppose that $d(v_{s-k-1}) + d(v_{s+\ell+1}) \geq n-1$. Since $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed, by Lemma 9, there exists a $(v_s, v_{s+\ell})$-path Q' such that $V(Q') = [v_{s-k}, v_{s+\ell}]$, and there exists a (v_s, v_{s-k})-path Q'' such that $V(Q'') = [v_{s-k}, v_{s+\ell}]$. Then $Q = Q'v_{s+\ell}v_{s+\ell+1}$ is a $(v_s, v_{s+\ell+1})$-path such that $V(Q) = [v_{s-k}, v_{s+\ell+1}]$, and the two path $Q''\, v_{s-k}\, v_{s-k-1}$ and $v_{s+\ell+1}$ form a $(v_s\, v_{s-k-1}, v_{s+\ell+1}v_{s-k-1})$-pair such that $V(D) = [v_{s+\ell+1}, v_{s-k-1}]$. Thus $(v_{s+\ell+1}, v_{s-k-1})$ is v_s-good on P. □

Claim 9. *There exist some k' and ℓ' such that $(v_{t-k'}, v_{t+\ell'})$ is v_t-good on P, where $s+\ell+1 \leq t-k'$ and $t+\ell' \leq p$.*

Proof. By Claim 6, we have $s+\ell \geq t-3$.

If $v_{t-1}v_{t+1} \notin E(G)$, then by Claim 2, $d(v_{t-1}) + d(v_{t+1}) \geq n-1$. Then $Q = v_tv_{t-1}$ is a (v_t, v_{t-1})-path, and the two paths v_tv_{t+1} and v_{t-1} form a $(v_tv_{t-1}, v_{t-1}v_{t+1})$-pair. Thus we have that (v_{t-1}, v_{t+1}) is v_t-good on P.

Now we assume that $v_{t-1}v_{t+1} \in E(G)$, and then $G[v_{t-1}, v_{t+1}]$ is (v_{t-1}, v_t, v_{t+1})-composed.

Claim 9.1. *If $G[v_{t-k'}, v_{t+\ell'}]$ is $(v_{t-k'}, v_t, v_{t+\ell'})$-composed with canonical path $P[v_{t-k'}, v_{t+\ell'}]$, then $t-k' \geq s+\ell+2$ and $t+\ell' \leq p-1$.*

Proof. Let $D_1, D_2, \ldots, D_r$ be a canonical sequence of $G[v_{t-k'}, v_{t+\ell'}]$ corresponding to the canonical path $P[v_{t-k'}, v_{t+\ell'}]$. Similarly as in the proof of Claim 6, we have that $t+\ell' \leq p-1$. Suppose now that $t-k' \leq s+\ell+1$. Consider the graph D_i, where i is the smallest integer such that $v_{s+\ell+1} \in V(D_i)$. Let $V(D_i) = [v_{s+\ell+1}, v_{t+\ell''}]$. By Lemma 9, there exists a Hamilton (v_s, v_{s-k})-path Q' of $G[v_{s-k}, v_{s+\ell}]$ and there exists a Hamilton path Q'' of $G[v_{s+\ell+1}, v_{t+\ell''}]$. Thus $Q = P[v_1, v_{s-k}]v_{s-k}Q'v_sRv_tQ''v_{t+\ell''}P[v_{t+\ell''}, v_p]$ is a path containing all the vertices of $V(P) \cup V(R)$, a contradiction. □

Similar to Claim 7, we have another claim that provides a tool for considering a largest composed subgraph.

Claim 9.2. *If $G[v_{t-k'}, v_{t+\ell'}]$ is $(v_{t-k'}, v_t, v_{t+\ell'})$-composed with canonical path $P[v_{t-k'}, v_{t+\ell'}]$, where $t-k' \geq s+\ell+2$ and $t+\ell \leq p-1$, and any two nonadjacent vertices in $[v_{t-k'-1}, v_{t+\ell'+1}]$ have degree sum less than $n-1$, then one of the following is true:*

(1) *$G[v_{t-k'-1}, v_{t+\ell'}]$ is a 1-extension of $G[v_{t-k'}, v_{t+\ell'}]$ at $v_{t-k'}$ to $v_{t-k'-1}$,*

(2) *$G[v_{t-k'}, v_{t+\ell'+1}]$ is a 1-extension of $G[v_{t-k'}, v_{t+\ell'}]$ at $v(t+\ell)]$ to $v_{t+\ell+1}$ or*

(3) *$G[v_{t-k'-1}, v_{t+\ell'+1}]$ is a 2-extension of $G[v_{t-k'}, v_{t+\ell'}]$ at $(v_{t-k'}, v_{t+\ell'})$ to $(v_{t-k'-1}, v_{t+\ell'+1})$.*

Hence in all cases we obtain a composed graph larger than $G[v_{t-k'}, v_{t+\ell'}]$.

Now we choose k', ℓ' such that:

(1) $G[v_{t-k'}, v_{t+\ell'}]$ is $(v_{t-k'}, v_t, v_{t+\ell'})$-composed with canonical path $P[v_{t-k'}, v_{t+\ell'}]$;

(2) any two nonadjacent vertices in $[v_{t-k'}, v_{t+\ell'}]$ have degree sum less than $n-1$; and

(3) $k' + \ell'$ is as large as possible.

Similar to Claim 8, we have that $(v_{t-k'-1}, v_{t+\ell'})$ or $(v_{t-k'}, v_{t+\ell'+1})$ or $(v_{t-k'-1}, v_{t+\ell'+1})$ is v_t-good on P. □

Using Claims 8 and 9, by Lemma 11, we get that there exists a path containing all the vertices of $V(P) \cup V(R)$, a contradiction. This completes the proof for Case 1.

Case 2. $r = 1$ and $v_s v_t \in E(G)$. Recall that $v_s v_{s+1} \in E(G)$ and $v_s v_{t-1} \notin E(G)$. Let v_{s+k} be the first vertex in $[v_{s+1}, v_{t-1}]$ such that $v_s v_{s+k} \notin E(G)$. Then $s+2 \le s+k \le t-1$.

Claim 10. Let $v_i \in [v_{s+1}, v_{s+k}]$ and $x \in \{z_1, v_t, v_{t+1}\}$. Then $v_i x \notin \overline{E}_{-1}(G)$.

Proof. By Claims 1 and 3, we have that $v_{s+1}z_1, v_{s+1}v_t, v_{s+1}v_{t+1} \notin \overline{E}_{-1}(G)$. Thus we assume that $v_i \in [v_{s+2}, v_{s+k}]$. Then $v_s v_i \in E(G)$. If $v_i z_1 \in \overline{E}(G)$, then $Q = P[v_1, v_{s-1}]v_{s-1}\, v_{s+1}P[v_{s+1}, v_{i-1}]v_{i-1}v_s z_1 v_i P[v_i, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, yielding a contradiction using Lemma 10. If $v_i v_t \in \overline{E}(G)$, then $Q = P[v_1, v_{s-1}]v_{s-1}v_{s+1}\; P[v_{s+1}, v_{i-1}]v_{i-1}v_s z_1 v_t v_i P[v_i, v_{t-1}]v_{t-1}v_{t+1}$ $P[v_{t+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, yielding another contradiction. If $v_i v_{t+1} \in \overline{E}(G)$, then $Q = P[v_1, v_{s-1}]v_{s-1}v_{s+1}P[v_{s+1}, v_{i-1}]$ $v_{i-1}v_s z_1 v_t P[v_t, v_i]v_i v_{t+1}P[v_{t+1}, v_p]$ is an o_{-1}-path containing all the vertices of $V(P) \cup V(R)$, yielding the last contradiction. □

Using Claims 1, 3 and 10, the subgraph induced by $\{z_1, v_t, v_{t+1}, v_s, v_{s+k-1}, v_{s+k}\}$ is a W that is not o_{-1}-heavy, our final contradiction, completing the proof of Theorem 7. ■

5. Proof of Theorem 8

The proof is modelled along the same lines as the proof of Theorem 7.

Proof. Suppose G is a $\{K_{1,3}, N\}$-o_{-1}-heavy block-chain on n vertices. It suffices to prove that G is traceable. We proceed by contradiction.

Clearly, G contains a penetrating path. As in the previous section, we choose a longest penetrating path $P = v_1 v_2 \cdots v_p$, a component H of $G - V(P)$, and a path $R = z_0z_1z_2 \cdots z_{r+1}$, where $z_0 = v_s$ and $z_{r+1} = v_t$, $s < t$ with two end-vertices on P and all internal vertices in H, and of length at least 2, but as short as possible subject to these conditions.

Similarly as in Section 4, we get the following claims. We omit the details.

Claim 1. *Let $x \in V(H)$ and $y \in \{v_{s-1}, v_{s+1}, v_{t-1}, v_{t+1}\}$. Then $xy \notin \overline{E}_{-1}(G)$.*

Claim 2. $v_{s-1}v_{s+1} \in \overline{E}_{-1}(G)$, $v_{t-1}v_{t+1} \in \overline{E}_{-1}(G)$.

Claim 3. $v_{s-1}v_{t-1} \notin \overline{E}_{-1}(G)$, $v_{s+1}v_{t+1} \notin \overline{E}_{-1}(G)$, $v_sv_{t\pm1} \notin \overline{E}_{-1}(G)$ and $v_{s\pm1}v_t \notin \overline{E}_{-1}(G)$.

Claim 4. *Either $v_{s-1}v_{s+1} \in E(G)$ or $v_{t-1}v_{t+1} \in E(G)$.*

By Claim 4, without loss of generality, we assume that $v_{s-1}v_{s+1} \in E(G)$. Thus $G[v_{s-1}, v_{s+1}]$ is (v_{s-1}, v_s, v_{s+1})-composed.

Claim 5. *If $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed with canonical path $P[v_{s-k}, v_{s+\ell}]$, then $s-k \geq 2$ and $s+\ell \leq t-3$.*

The proof of Claim 5 is similar to that of Claim 6 in Section 4.

Now we prove the following claim.

Claim 6. *If $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed with canonical path $P[v_{s-k}, v_{s+\ell}]$, where $s-k \geq 2$ and $s+\ell \leq t-3$, and any two nonadjacent vertices in $[v_{s-k-1}, v_{s+\ell+1}]$ have degree sum less than $n-1$, then one of the following is true:*

(1) *$G[v_{s-k-1}, v_{s+\ell}]$ is a 1-extension of $G[v_{s-k}, v_{s+\ell}]$ at v_{s-k} to v_{s-k-1};*

(2) *$G[v_{s-k}, v_{s+\ell+1}]$ is a 1-extension of $G[v_{s-k}, v_{s+\ell}]$ at $v_{s+\ell}$ to $v_{s+\ell+1}$; or*

(3) *$G[v_{s-k-1}, v_{s+\ell+1}]$ is a 2-extension of $G[v_{s-k}, v_{s+\ell}]$ at $(v_{s-k}, v_{s+\ell})$ to $(v_{s-k-1}, v_{s+\ell+1})$.*

Thus in all cases we obtain a composed graph larger than $G[v_{s-k}, v_{s+\ell}]$.

Proof. Assume the contrary. This implies that v_{s-k-1} has only one neighbor v_{s-k}, and $v_{s+\ell+1}$ has only one neighbor $v_{s+\ell}$, in $[v_{s-k-1}, v_{s+\ell+1}]$. We need a number of subclaims.

Claim 6.1. *For $i \in [s-k-1, s+\ell+1] \setminus \{s\}$, $v_iz_1 \notin \overline{E}_{-1}(G)$.*

This claim can be proved in a similar way as Claim 7.1 in Section 4. We omit the details.

Let $G' = G[[v_{s-k-1}, v_{s+\ell}] \cup \{z_1\}]$ and $G'' = G[[v_{s-k-1}, v_{s+\ell+1}] \cup \{z_1\}]$.

Similar to Claims 7.2 and 7.3 in Section 4, we obtain the following statements.

Claim 6.2. *G'' and G' are $\{K_{1,3}, N\}$-free.*

Claim 6.3. *$N_{G'}(v_s) \setminus \{z_1\}$ is a clique.*

Now, we define $N_i = \{x \in V(G') : d_{G'}(x, v_{s-k-1}) = i\}$. Then we have $N_0 = \{v_{s-k-1}\}$, $N_1 = \{v_{s-k}\}$ and $N_2 = N_{G'}(v_{s-k}) \setminus \{v_{s-k-1}\}$.

By the definition of composed graphs, we have $|N_2| \geq 2$. If there are two vertices $x, x' \in N_2$ such that $xx' \notin E(G')$, then the graph induced by $\{v_{s-k}, v_{s-k-1}, x, x'\}$ is a claw, a contradiction. Thus we have that N_2 is a clique.

We assume $v_s \in N_j$, where $j \geq 2$. Then $z_1 \in N_{j+1}$.

If $|N_i| = 1$ for some $i \in [2, j-1]$, then let $N_i = \{x\}$; then x is a cut vertex of the graph $G[v_{s-k}, v_{s+\ell}]$. By the definition of composed graphs, $G[v_{s-k}, v_{s+\ell}]$ is 2-connected. This implies $|N_i| \geq 2$ for every $i \in [2, j-1]$.

Claim 6.4. *For $i \in [1, j]$, N_i is a clique.*

Proof. We prove this claim by induction on i. For $i = 1, 2$, the claim is true by the above arguments. So we assume that $3 \leq i \leq j$, and we have that $N_{i-3}, N_{i-2}, N_{i-1}$ and N_{i+1} are nonempty, and that $|N_{i-1}| \geq 2$.

Let x and x' be two distinct vertices in N_i. We claim that $xx' \in E(G)$. Suppose that $xx' \notin E(G)$. If x and x' have a common neighbor in N_{i-1}, denote it by w; then let v be a neighbor of w in N_{i-2}, and the graph induced by $\{w, v, x, x'\}$ is a claw, a contradiction. Thus we have that x and x' have no common neighbors in N_{i-1}. Now, let w be a neighbor of x in N_{i-1}, and let w' be a neighbor of x' in N_{i-1}. Then $xw', x'w \notin E(G)$. Let v be a neighbor of w in N_{i-2}, and let u be a neighbor of v in N_{i-3}. If $w'v \notin E(G)$, then the graph induced by $\{w, v, w', x\}$ is a claw, a contradiction. Thus we have $w'v \in E(G)$, and then the graph induced by $\{v, u, w, x, w', x'\}$ is an N, a contradiction. This implies that N_i is a clique.

□

If there exists some vertex $y \in N_{j+1}$ other than z_1, then we have $yv_s \notin E(G)$ by Claim 6.3. Let x be a neighbor of y in N_j, let w be a neighbor of v_s in N_{j-1}, and let v be a neighbor of w in N_{j-2}. Then $xv_s \in E(G)$ by Claim 6.4, and $xw \in E(G)$ by Claim 6.3. Thus the graph induced by $\{w, v, x, y, v_s, z_1\}$ is an N, a contradiction. So we assume that all vertices in $[v_{s-k}, v_{s+\ell}]$ are in $\bigcup_{i=1}^{j} N_i$.

If $v_{s+\ell} \in N_j$, then let w be a neighbor of v_s in N_{j-1}, and let v be a neighbor of w in N_{j-2}. Then the graph induced by $\{w, v, v_{s+\ell}, v_{s+\ell+1}, v_s, z_1\}$ is an N, a contradiction. Thus we have that $v_{s+\ell} \notin N_j$ and thus $j \geq 3$.

Let $v_{s+\ell} \in N_i$, where $i \in [2, j-1]$. If $v_{s+\ell}$ has a neighbor in N_{i+1}, then let y be a neighbor of $v_{s+\ell}$ in N_{i+1}, and let w be a neighbor of $v_{s+\ell}$ in N_{i-1}. Then the graph induced by $\{v_{s+\ell}, w, y, v_{s+\ell+1}\}$ is a claw, a contradiction. Thus we have that $v_{s+\ell}$ has no neighbors in N_{i+1}.

Let y be a vertex in N_{i+1}, and let x be a neighbor of y in N_i. Then $x \neq v_{s+\ell}$. Let w be a neighbor of x in N_{i-1}, and let v be a neighbor of w in N_{i-2}. If $wv_{s+\ell} \notin E(G)$, then the graph induced by $\{x, w, v_{s+\ell}, y\}$ is a claw, a contradiction. So we have that $wv_{s+\ell} \in E(G)$ and the graph induced by $\{w, v, v_{s+\ell}, v_{s+\ell+1}, x, y\}$ is an N, a contradiction.

This completes the proof of Claim 6. □

Using Claim 6, we consider a largest composed subgraph, in the following sense. We choose k, ℓ such that:

(1) $G[v_{s-k}, v_{s+\ell}]$ is $(v_{s-k}, v_s, v_{s+\ell})$-composed with canonical path $P[v_{s-k}, v_{s+\ell}]$,

(2) any two nonadjacent vertices in $[v_{s-k}, v_{s+\ell}]$ have degree sum less than $n-1$ and

(3) $k+\ell$ is as large as possible.

Similar to Claims 8 and 9 in Section 4, we obtain the following claims. We omit the details.

Claim 7. *$(v_{s-k-1}, v_{s+\ell})$ or $(v_{s-k}, v_{s+\ell+1})$ or $(v_{s-k-1}, v_{s+\ell+1})$ is v_s-good on P.*

Claim 8. *There exist some k' and ℓ' such that $(v_{t-k'}, v_{t+\ell'})$ is v_t-good on P, where $s+\ell+1 \leq t-k'$ and $t+\ell' \leq p$.*

Using Claims 7 and 8, Lemma 11 implies that there exists a path containing all the vertices of $V(P) \cup V(R)$, our final contradiction. ■

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