Discussiones Mathematicae
Graph Theory 34 (2014) 263–278
doi:10.7151/dmgt.1733

THE IRREGULARITY OF GRAPHS UNDER GRAPH OPERATIONS

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Abstract

The *irregularity* of a simple undirected graph G was defined by Albertson [5] as $\operatorname{irr}(G) = \sum_{uv\in E(G)} |d_G(u) - d_G(v)|$, where $d_G(u)$ denotes the degree of a vertex $u \in V(G)$. In this paper we consider the irregularity of graphs under several graph operations including join, Cartesian product, direct product, strong product, corona product, lexicographic product, disjunction and symmetric difference. We give exact expressions or (sharp) upper bounds on the irregularity of graphs under the above mentioned operations.

Keywords: irregularity of graphs, total irregularity of graphs, graph operations, Zagreb indices.

2010 Mathematics Subject Classification: 05C30, 05C76, 05C90.

1. Introduction

Let G be a simple undirected graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The *degree* of a vertex v in G is the number of edges incident with v and it is denoted by $d_G(v)$. A graph G is *regular* if all its vertices have the same degree, otherwise it is *irregular*. However, in many applications and problems it is of big importance to know how irregular a given graph is. Several graph topological indices have been proposed for that purpose. Among the most investigated ones are: the irregularity of a graph introduced by Albertson [5], the variance of vertex degrees [7], and Collatz-Sinogowitz index [12].

The *imbalance* of an edge $e = uv \in E$, defined as $\operatorname{imb}(e) = |d_G(u) - d_G(v)|$, appeares implicitly in the context of Ramsey problems with repeat degrees [6], and later in the work of Chen, Erdős, Rousseau, and Schlep [11], where 2-colorings of edges of a complete graph were considered. In [5], Albertson defined the *irregularity* of G as

$$\mathrm{irr}(G) = \sum\nolimits_{e \in E(G)} \mathrm{imb}(e). \tag{1}$$

It is shown in [5] that for a graph G, $\mathrm{irr}(G) < 4n^3/27$ and that this bound can be approached arbitrary closely. Albertson also presented upper bounds on irregularity for bipartite graphs, triangle-free graphs and a sharp upper bound for trees. Some claims about bipartite graphs given in [5] have been formally proved in [21]. Related to Albertson is the work of Hansen and Mélot [20], who characterized the graphs with n vertices and m edges with maximal irregularity. The irregularity measure irr also is related to the *first Zagreb index* $M_1(G)$ and the *second Zagreb index* $M_2(G)$, one of the oldest and most investigated topological graph indices, defined as follows:

$$M_1(G) = \sum\nolimits_{v \in V(G)} d_G^2(v) \quad \text{and} \quad M_2(G) = \sum\nolimits_{uv \in E(G)} d_G(u) d_G(v).$$

Alternatively the first Zagreb index can be expressed as

$$M_1(G) = \sum\nolimits_{uv \in E(G)} (d_G(u) + d_G(v)). \tag{2}$$

Fath-Tabar [16] established new bounds on the first and the second Zagreb indices that depend on the irregularity of graphs as defined in (1). In line with the standard terminology of chemical graph theory, and the obvious connection with the first and the second Zagreb indices, Fath-Tabar named the sum in (1) the *third Zagreb index* and denoted it by $M_3(G)$. However, in the rest of the paper, we will use its older name and call it the irregularity of a graph. Two other most frequently used graph topological indices, that measure how irregular a graph is, are the *variance of degrees* and the *Collatz-Sinogowitz index* [12]. Let G be a graph with n vertices and m edges, and λ_1 be the index or largest eigenvalue of the adjacency matrix $A = (a_{ij})$ (with $a_{ij} = 1$ if vertices i and j are joined by an edge and 0 otherwise). Let n_i denotes the number of vertices of degree i for $i = 1, 2, \ldots, n-1$. Then, the variance of degrees and the Collatz-Sinogowitz index are respectively defined as

$$\mathrm{Var}(G) = \frac{1}{n} \sum\nolimits_{i=1}^{n-1} n_i \left(i - \frac{2m}{n} \right)^2 \qquad \text{and} \qquad \mathrm{CS}(G) = \lambda_1 - \frac{2m}{n}. \tag{3}$$

Results of comparing irr, CS and Var are presented in [7, 13, 18]. There have been other attempts to determine how irregular graph is [2, 3, 4, 8, 9, 10, 22], but heretofore this has not been captured by a single parameter as it was done by the irregularity measure by Albertson. The graph operation, especially graph products, plays significant role not only in pure and applied mathematics, but also in computer science. For example, the Cartesian product provide an important model for linking computers. In order to synchronize the work of the whole system it is necessary to search for Hamiltonian paths and cycles in the network. Thus, some results on Hamiltonian paths and cycles in Cartesian product of graphs can be applied in computer network design [27]. Many of the problems can be easily handled if the related graphs are regular or close to regular.

The aim of this paper is to investigate the irregularity measure by Albertson under several graph operations including join, Cartesian product, direct product, strong product, corona product, lexicographic product, disjunction and symmetric difference. Detailed exposition on some graph operations one can find in [19].

In the sequel we will introduce additional necessary notation and results that will be used in the rest of the paper. For details of the mathematical theory and chemical applications of the Zagreb indices see surveys [14, 17, 24, 25] and papers [15, 28, 29].

A vertex is *isolated* if its degree is zero. An *independent set* is a set of vertices in a graph, no two of which are adjacent. The vertices from an independent set are *independent vertices*. For two graphs G_1 and G_2, with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and disjoint edge sets $E(G_1)$ and $E(G_2)$, the *disjoint union* of G_1 and G_2 is the graph $G = G_1 \cup G_2$, with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. Obviously, $\mathrm{irr}(G \cup H) = \mathrm{irr}(G) + \mathrm{irr}(H)$.

To obtain some of the bounds in the next section, we will use a recently introduced irregularity measure of a graph, so-called the *total irregularity* of a graph [1], defined as

$$\mathrm{irr}_t(G) = \frac{1}{2} \sum\nolimits_{u,v \in V(G)} |d_G(u) - d_G(v)| . \tag{4}$$

Theorem 1 [1]. *For a simple undirected graph G with n vertices, it holds that*

$$\mathrm{irr}_t(G) \leq \begin{cases} \frac{1}{12}\big(2n^3 - 3n^2 - 2n\big) & n \text{ even}, \\ \\ \frac{1}{12}\big(2n^3 - 3n^2 - 2n + 3\big) & n \text{ odd}. \end{cases}$$

Moreover, the bounds are sharp.

2. RESULTS

2.1. Join

The *join* $G+H$ of simple undirected graphs G and H is the graph with the vertex set $V(G+H) = V(G) \cup V(H)$ and the edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G),\ v \in V(H)\}$.

Theorem 2. *Let G and H be simple undirected graphs with $|V(G)| = n_1$ and $|V(H)| = n_2$ such that $n_1 \geq n_2$. Then*

$$\operatorname{irr}(G+H) \leq \operatorname{irr}(G) + \operatorname{irr}(H) + n_2(n_1-1)(n_1-2).$$

Moreover, the bound is sharp.

Proof. By the definition, $|V(G+H)| = |V(G)| + |V(H)| = n_1 + n_2$ and $|E(G+H)| = |E(G)| + |E(H)| + n_1 n_2$. The irregularity of $G+H$ is

$$\begin{aligned}\operatorname{irr}(G+H) &= \textstyle\sum_{uv\in E(G+H)} |d_{G+H}(u) - d_{G+H}(v)| \\ &= \textstyle\sum_{xy\in E(G)} |d_{G+H}(x) - d_{G+H}(y)| \\ &+ \textstyle\sum_{zt\in E(H)} |d_{G+H}(z) - d_{G+H}(t)| \\ &+ \textstyle\sum_{u\in V(G)} \sum_{v\in V(H)} |d_{G+H}(u) - d_{G+H}(v)|.\end{aligned}$$

For a vertex $u \in V(G)$, it holds that $d_{G+H}(u) = d_G(u) + n_2$, and for a vertex $v \in V(H)$, it holds that $d_{G+H}(v) = d_H(v) + n_1$. Thus, further we have

$$\begin{aligned}\operatorname{irr}(G+H) &= \textstyle\sum_{xy\in E(G)} |d_G(x) - d_G(y)| + \sum_{zt\in E(H)} |d_H(z) - d_H(t)| \\ &+ \textstyle\sum_{u\in V(G)} \sum_{v\in V(H)} |d_G(u) - d_H(v) + n_2 - n_1| \\ &= \operatorname{irr}(G) + \operatorname{irr}(H) + \textstyle\sum_{u\in V(G)} \sum_{v\in V(H)} |n_1 - n_2 + d_H(v) - d_G(u)|.\end{aligned}$$

Under the constrains $n_1 \geq n_2$, $d_G(u) \leq n_1 - 1$, and $d_H(v) \leq n_2 - 1$, the double sum $\sum_{u\in V(G)} \sum_{v\in V(H)} |n_1 - n_2 + d_H(v) - d_G(u)|$ is maximal when H is a graph with maximal sum of vertex degrees, i.e., H is the complete graph K_{n_2}, and G is a graph with minimal sum of vertex degrees, i.e., G is a tree on n_1 vertices T_{n_1}. Thus,

$$\begin{aligned}\sum_{\substack{u\in V(G)\\ v\in V(H)}} |n_1 - n_2 + d_H(v) - d_G(u)| &\leq \sum_{\substack{u\in V(T_{n_1})\\ v\in V(K_{n_2})}} \left|n_1 - n_2 + d_{K_{n_2}}(v) - d_{T_{n_1}}(u)\right| \\ &= \textstyle\sum_{u\in V(T_{n_1})} \sum_{v\in V(K_{n_2})} \left|n_1 - 1 - d_{T_{n_1}}(u)\right| \\ &= n_2 \textstyle\sum_{u\in V(T_{n_1})} \left(n_1 - 1 - d_{T_{n_1}}(u)\right) \\ &= n_2 n_1 (n_1 - 1) - 2n_2(n_1 - 1) \\ &= n_2(n_1-1)(n_1-2),\end{aligned}$$

and

$$\operatorname{irr}(G+H) \leq \operatorname{irr}(G) + \operatorname{irr}(H) + n_2(n_1-1)(n_1-2). \tag{5}$$

When $n_1 \leq 2$, $\operatorname{irr}(G) = \operatorname{irr}(H) = \operatorname{irr}(G+H) = 0$, and the claim of the theorem is fulfilled. From the derivation, it follows that (5) is equality when H is complete graph on n_2 vertices and G is any tree on n_1 vertices. ∎

Example 3. Let denote by H_i a graph with $|V(H_i)| = i$ isolated vertices. Then, the *bipartite graph* $K_{i,j}$ is a join of H_i and H_j. Analogously, the *complete k-partite graph* $G = K_{n_1,\ldots,n_k}$ is join of $H_{n_1}, \ldots,$ and H_{n_k}. We have that $\operatorname{irr}(K_{n_i,n_j}) = n_i n_j \left| n_j - n_i \right|$. For the irregularity of $K_{n_1,\ldots,n_k}$ we obtain

$$\begin{aligned}\operatorname{irr}(K_{n_1,\ldots,n_k}) &= \textstyle\sum_{uv\in E(K_{n_1,\ldots,n_k})} \left| d_G(u) - d_G(v) \right| \\ &= \textstyle\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \sum_{uv\in E(K_{n_i,n_j})} \left| d_G(u) - d_G(v) \right| \\ &= \textstyle\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} n_i n_j \left| n_j - n_i \right| = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \operatorname{irr}(K_{n_i,n_j}).\end{aligned}$$

2.2. Cartesian product

The *Cartesian product* $G \,\square\, H$ of two simple undirected graphs G and H is the graph with the vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and the edge set $E(G \,\square\, H) = \{(u_i, v_k)(u_j, v_l) : [(u_i u_j \in E(G)) \wedge (v_k = v_l)] \vee [(v_k v_l \in E(H)) \wedge (u_i = u_j)]\}$.

Theorem 4. *Let G and H be simple undirected graphs with $|V(G)| = n_1$ and $|V(H)| = n_2$. Then*

$$\operatorname{irr}(G \,\square\, H) = n_2 \operatorname{irr}(G) + n_1 \operatorname{irr}(H).$$

Proof. From the definiton of the Cartesian product, it follows $|V(G \,\square\, H)| = |V(G)|\,|V(H)|$, $|E(G \,\square\, H)| = n_2 \,|E(G)| + n_1 \,|E(H)|$, and $d_{G \,\square\, H}(u_i, v_k) = d_G(u_i) + d_H(v_k)$. Hence,

$$\begin{aligned}\operatorname{irr}(G \,\square\, H) &= \textstyle\sum_{(u_i,v_k)(u_j,v_l)\in E(G \,\square\, H)} \left| d_{G \,\square\, H}(u_i, v_k) - d_{G \,\square\, H}(u_j, v_l) \right| \\ &= \textstyle\sum_{u_i u_j \in E(G)} \sum_{v\in V(H)} \left| d_{G \,\square\, H}(u_i, v) - d_{G \,\square\, H}(u_j, v) \right| \\ &+ \textstyle\sum_{v_k v_l \in E(H)} \sum_{u\in V(G)} \left| d_{G \,\square\, H}(u, v_k) - d_{G \,\square\, H}(u, v_l) \right| \\ &= \textstyle\sum_{u_i u_j \in E(G)} \sum_{v\in V(H)} \left| (d_G(u_i) + d_H(v)) - (d_G(u_j) + d_H(v)) \right| \\ &+ \textstyle\sum_{v_k v_l \in E(H)} \sum_{u\in V(G)} \left| (d_G(u) + d_H(v_k)) - (d_G(u) + d_H(v_l)) \right| \\ &= n_2 \operatorname{irr}(G) + n_1 \operatorname{irr}(H).\end{aligned}$$

∎

2.3. Direct product

The *direct product* $G \times H$ (also known as *tensor product*, *Kronecker product* [26], *categorical product* [23] and *conjunctive product*) of simple undirected graphs G

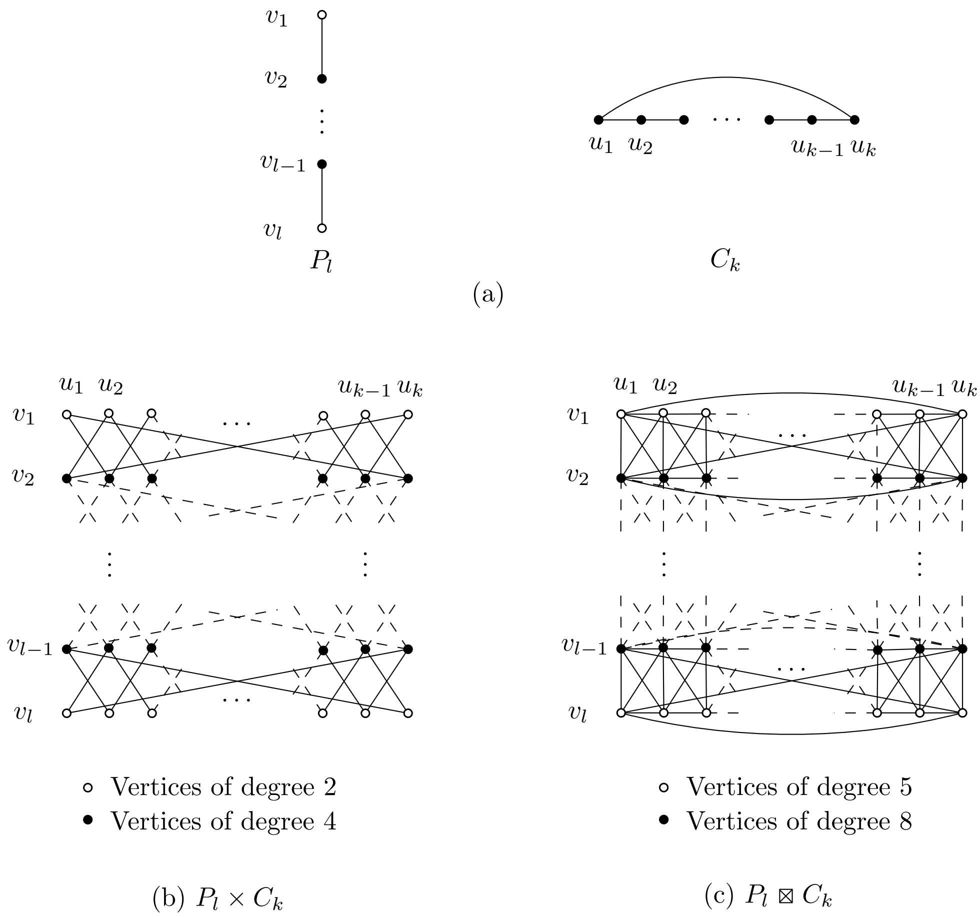

Figure 1. (a) Path graph on l vertices P_l, and cycle graph on k vertices C_k,
(b) direct product graph $P_l \times C_k$, and
(c) strong product graph $P_l \boxtimes C_k$.

and H is the graph with the vertex set $V(G \times H) = V(G) \times V(H)$, and the edge set $E(G \times H) = \{(u_i, v_k)(u_j, v_l) : (u_i, u_j) \in E(G) \wedge (v_k, v_l) \in E(H)\}$.

Theorem 5. *Let G and H be simple undirected graphs. Then*

$$\mathrm{irr}(G \times H) \leq \mathrm{irr}(G)\, M_1(H) + \mathrm{irr}(H)\, M_1(G).$$

Moreover, the bound is sharp for infinitely many graphs.

Proof. From the definition of the direct product, it follows $|V(G \times H)| = |V(G)|\,|V(H)|$, $|E(G \times H)| = 2\,|E(G)|\,|E(H)|$, and $d_{G\times H}(u_i, v_k) = d_G(u_i) d_H(v_k)$. The irregularity of the direct product $G \times H$ is

$$\begin{aligned}
\operatorname{irr}(G \times H) &= \textstyle\sum_{(u_i,v_k)(u_j,v_l)\in E(G\times H)} |d_{G\times H}(u_i,v_k) - d_{G\times H}(u_j,v_l)| \\
&= 2\textstyle\sum_{u_iu_j\in E(G)}\sum_{v_kv_l\in E(H)} |d_G(u_i)d_H(v_k) - d_G(u_j)d_H(v_l)| \\
&= \textstyle\sum_{u_iu_j\in E(G)}\sum_{v_kv_l\in E(H)} |(d_G(u_i) - d_G(u_j))(d_H(v_k) + d_H(v_l)) \\
&\quad + (d_G(u_i) + d_G(u_j))(d_H(v_k) - d_H(v_l))| \,.
\end{aligned}$$

Applying the triangle inequality in the above expression and by equation (2), we obtain

$$\begin{aligned}
\operatorname{irr}(G \times H) &\le \textstyle\sum_{u_iu_j\in E(G)}\sum_{v_kv_l\in E(H)} [\, |d_G(u_i) - d_G(u_j)|\ (d_H(v_k) + d_H(v_l)) \\
&\quad + |d_H(v_k) - d_H(v_l)|\ (d_G(u_i) + d_G(u_j))\ ] \\
&= \operatorname{irr}(G)\, M_1(H) + \operatorname{irr}(H)\, M_1(G).
\end{aligned}$$

To prove that the presented bound is the best possible, consider the direct product $P_l \times C_k$, $l \ge 1, k \ge 3$ (an illustration is given in Figure 1(b)). Straightforward calculations gives that $M_1(P_l) = 4l - 6, M_1(C_k) = 4k, \operatorname{irr}(P_l) = 2, \operatorname{irr}(C_k) = 0$. The graph $P_l \times C_k$ is comprised of $2k$ vertices of degree 2, and $k(l-2)$ vertices of degree 4. Each vertex of degree 2 is adjacent with exactly two vertices with degree 4. Hence, $\operatorname{irr}(P_l \times C_k) = 8k$. On the other hand, the bound obtain here is $\operatorname{irr}(P_l \times C_k) \le \operatorname{irr}(P_l)\, M_1(C_k) + \operatorname{irr}(C_k)\, M_1(P_l) = 8k$. ∎

2.4. Strong product

The *strong product* $G \boxtimes H$ of two simple undirected graphs G and H is the graph with the vertex set $V(G \boxtimes H = V(G) \times V(H)$ and the edge set $E(G \boxtimes H) = \{(u_i,v_k)(u_j,v_l) : [(u_iu_j \in E(G))\wedge\ (v_k = v_l)] \vee [(v_kv_l \in E(H)) \wedge (u_i = u_j)] \vee [(u_iu_j \in E(G)) \wedge (v_kv_l \in E(H))]\}$.

Theorem 6. *Let G and H be simple undirected graphs with $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$ and $|E(H)| = m_2$. Then*

$$\operatorname{irr}(G \boxtimes H) \le (n_2 + 4\,m_2 + M_1(H))\operatorname{irr}(G) + (n_1 + 4\,m_1 + M_1(G))\operatorname{irr}(H).$$

Moreover, the bound is sharp for infinitely many graphs.

Proof. For the strong product $V(G \boxtimes H)$, it holds that $|V(G \boxtimes H)| = |V(G)|\,|V(H)|$, $|E(G \boxtimes H)| = |V(H)|\,|E(G)| + |V(G)|\,|E(H)| + 2\,|E(G)|\,|E(H)|$, and $d_{G\boxtimes H}(u_i,v_k) = d_G(u_i) + d_H(v_k) + d_G(u_i)d_H(v_k)$. The irregularity of $G \boxtimes H$ is

$$\begin{aligned}
\operatorname{irr}(G \boxtimes H) &= \sum_{(u_i,v_k)(u_j,v_l)\in E(G\boxtimes H)} |d_{G\boxtimes H}(u_i,v_k) - d_{G\boxtimes H}(u_j,v_l)| \\
&= \sum_{v_k=v_l\in V(H)}\sum_{u_iu_j\in E(G)} |d_{G\boxtimes H}(u_i,v_k) - d_{G\boxtimes H}(u_j,v_l)| \\
&\quad + \sum_{u_i=u_j\in V(G)}\sum_{v_kv_l\in E(H)} |d_{G\boxtimes H}(u_i,v_k) - d_{G\boxtimes H}(u_j,v_l)| \\
&\quad + 2\sum_{u_iu_j\in E(G)}\sum_{v_kv_l\in E(H)} |d_{G\boxtimes H}(u_i,v_k) - d_{G\boxtimes H}(u_j,v_l)| \,.
\end{aligned} \tag{6}$$

By an algebraic transformation and applying the triangle inequality, we obtain

$$
\begin{aligned}
|d_{G\boxtimes H}(u_i,v_k) - d_{G\boxtimes H}(u_j,v_l)| &= |d_G(u_i) + d_H(v_k) + d_G(u_i)d_H(v_k) \\
&\quad - d_G(u_j) - d_H(v_l) - d_G(u_j)d_H(v_l)| \\
&\le |d_G(u_i) - d_G(u_j)| + |d_H(v_k) - d_H(v_l)| \\
&\quad + \frac{1}{2}(d_G(u_i) + d_G(u_j))\,|d_H(v_l) - d_H(v_k)| \\
&\quad + \frac{1}{2}(d_H(v_l) + d_H(v_k))\,|d_G(u_i) - d_G(u_j)| \\
&= f_1(i,j,k,l).
\end{aligned} \tag{7}
$$

Substituting (7) in (6), we obtain

$$
\begin{aligned}
\operatorname{irr}(G\boxtimes H) \le &\sum_{\substack{v_k=v_l\in V(H)\\ u_iu_j\in E(G)}} f_1(i,j,k,l) + \sum_{\substack{u_i=u_j\in V(G)\\ v_kv_l\in E(H)}} f_1(i,j,k,l) \\
&+ 2\sum_{\substack{u_iu_j\in E(G)\\ v_kv_l\in E(H)}} f_1(i,j,k,l).
\end{aligned} \tag{8}
$$

We have

$$
\begin{aligned}
\sum_{\substack{v_k=v_l\in V(H)\\ u_iu_j\in E(G)}} f_1(i,j,k,l) &= \sum_{v\in V(H)}\sum_{u_iu_j\in E(G)} |d_G(u_i) - d_G(u_j)| \\
&\quad + \sum_{v\in V(H)}\sum_{u_iu_j\in E(G)} |d_H(v) - d_H(v)| \\
&\quad + \frac{1}{2}\sum_{v\in V(H)}\sum_{u_iu_j\in E(G)} (d_G(u_i) + d_G(u_j))\,|d_H(v) - d_H(v)| \\
&\quad + \frac{1}{2}\sum_{v\in V(H)}\sum_{u_iu_j\in E(G)} (d_H(v) + d_H(v))\,|d_G(u_i) - d_G(u_j)| \\
&= (n_2 + 2m_2)\operatorname{irr}(G).
\end{aligned} \tag{9}
$$

Similarly,

$$
\begin{aligned}
\sum_{\substack{u_i=u_j\in V(G)\\ v_kv_l\in E(H)}} f_1(i,j,k,l) &= \sum_{u\in V(G)}\sum_{v_kv_l\in E(H)} |d_G(u) - d_G(u)| \\
&\quad + \sum_{u\in V(G)}\sum_{v_kv_l\in E(H)} |d_H(v_k) - d_H(v_l)| \\
&\quad + \frac{1}{2}\sum_{u\in V(G)}\sum_{v_kv_l\in E(H)} (d_G(u) + d_G(u))\,|d_H(v_k) - d_H(v_l)| \\
&\quad + \frac{1}{2}\sum_{u\in V(G)}\sum_{v_kv_l\in E(H)} (d_H(v_k) + d_H(v_l))\,|d_G(u) - d_G(u)| \\
&= (n_1 + 2\,m_1)\operatorname{irr}(H).
\end{aligned} \tag{10}
$$

Finally,

$$
\begin{aligned}
\sum_{\substack{u_iu_j\in E(G)\\ v_kv_l\in E(H)}} f_1(i,j,k,l) &= \sum_{\substack{u_iu_j\in E(G)\\ v_kv_l\in E(H)}} |d_G(u_i)-d_G(u_j)| \\
&+ \sum_{\substack{u_iu_j\in E(G)\\ v_kv_l\in E(H)}} |d_H(v_k)-d_H(v_l)| \\
&+ \frac{1}{2}\sum_{\substack{u_iu_j\in E(G)\\ v_kv_l\in E(H)}} (d_G(u_i)+d_G(u_j))\,|d_H(v_k)-d_H(v_l)| \\
&+ \frac{1}{2}\sum_{\substack{u_iu_j\in E(G)\\ v_kv_l\in E(H)}} (d_H(v_k)+d_H(v_l))\,|d_G(u_i)-d_G(u_j)| \\
&= (m_2+\frac{1}{2}M_1(H))\mathrm{irr}(G)+(m_1+\frac{1}{2}M_1(G))\mathrm{irr}(H)
\end{aligned}
\tag{11}
$$

From (8), (9), (10) and (11), we have

$$\mathrm{irr}(G\boxtimes H) \quad \le (n_2+4\,m_2+M_1(H))\,\mathrm{irr}(G)+(n_1+4\,m_1+M_1(G))\,\mathrm{irr}(H).$$

To prove that the presented bound is the best possible, consider the strong product $P_l \boxtimes C_k$, $l \ge 1, k \ge 3$ illustrated in Figure 1(c). Simple calculations gives that $M_1(P_l) = 4l-6, M_1(C_k) = 4k, \mathrm{irr}(P_l) = 2, \mathrm{irr}(C_k) = 0$. The graph $P_l \boxtimes C_k$ is comprised of $2k$ vertices of degree 5, and $k(l-2)$ vertices of degree 8. Each vertex of degree 5 is adjacent with exactly three vertices with degree 8. Hence, $\mathrm{irr}(P_l \boxtimes C_k) = 18k$. On the other hand, the bound obtain here is $\mathrm{irr}(P_l\boxtimes C_k) \le (k+4k+M_1(C_k))\,\mathrm{irr}(P_l)+(l+4(l-1)+M_1(P_l))\,\mathrm{irr}(C_k) = 18k$. ■

2.5. Corona product

The *corona* product $G\odot H$ of simple undirected graphs G and H with $|V(G)| = n_1$ and $|V(H)| = n_2$, is defined as the graph who is obtained by taking the disjoint union of G and n_1 copies of H and for each i, $1 \le i \le n_1$, inserting edges between the ith vertex of G and each vertex of the ith copy of H. Thus, the corona graph $G \odot H$ is the graph with the vertex set $V(G\odot H) = V(G)\bigcup_{i=1,\ldots,n_1} V(H_i)$ and the edge set $E(G\odot H) = E(G)\bigcup_{i=1,\ldots,n_1} E(H_i)\cup\{u_iv_j : u_i \in V(G), v_j \in V(H_i)\}$, where H_i is the ith copy of the graph H.

Theorem 7. *Let G and H be simple undirected graphs with $|V(G)| = n_1$ and $|V(H)| = n_2$. Then*

$$\mathrm{irr}(G\odot H) \le \mathrm{irr}(G)+n_1\,\mathrm{irr}(H)+n_1\left(n_2^2+n_1n_2-4n_2+2\right).$$

Moreover, the bound is sharp.

Proof. By the definition of $G\odot H$, $|V(G\odot H)| = |V(G)| + n_1\,|V(H)| = n_1 + n_1 n_2$, and $|E(G\odot H)| = |E(G)| + n_1\,|E(H)| + n_1 n_2$. The irregularity of $G\odot H$ is

$$\begin{aligned} \mathrm{irr}(G\odot H) &= \textstyle\sum_{uv\in E(G\odot H)} |d_{G\odot H}(u) - d_{G\odot H}(v)| \\ &= \textstyle\sum_{xy\in E(G)} |d_{G\odot H}(x) - d_{G\odot H}(y)| \\ &\quad + \textstyle\sum_{i=1}^{n_1}\sum_{zt\in E(H_i)} |d_{G\odot H}(z) - d_{G\odot H}(t)| \\ &\quad + \textstyle\sum_{u\in V(G)}\sum_{v\in V(H)} |d_{G\odot H}(u) - d_{G\odot H}(v)|\,. \end{aligned}$$

For a vertex $u\in V(G)$, it holds that $d_{G\odot H}(u) = d_G(u) + n_2$ and for a vertex $v\in V(H_i)$, $1\le i\le n_2$, we have $d_{G\odot H}(v) = d_H(v) + 1$. Thus,

$$\begin{aligned} \mathrm{irr}(G\odot H) &= \sum_{xy\in E(G)} |d_G(x) + n_2 - d_G(y) - n_2| \\ &+ \sum_{i=1}^{n_1}\sum_{zt\in E(H)} |d_H(z) + 1 - d_H(t) - 1| \\ &+ \sum_{u\in V(G)}\sum_{v\in V(H)} |d_G(u) + n_2 - d_H(v) - 1| \\ &= \sum_{xy\in E(G)} |d_G(x) - d_G(y)| + n_1\sum_{zt\in E(H)} |d_H(z) - d_H(t)| \\ &+ \sum_{u\in V(G)}\sum_{v\in V(H)} |d_G(u) - d_H(v) + n_2 - 1| \\ &= \mathrm{irr}(G) + n_1\mathrm{irr}(H) + \sum_{u\in V(G)}\sum_{v\in V(H)} |d_G(u) - d_H(v) + n_2 - 1|\,. \end{aligned} \tag{12}$$

Since $n_1\ge n_2$, the double sum in (12) is maximal when $\sum_{u\in V(G)} d_G(u)$ is maximal, i.e., G is the complete graph K_{n_1}, and $\sum_{v\in V(H)} d_H(v)$ is minimal, i.e., H is a tree on n_2 vertices T_{n_2}. Thus,

$$\begin{aligned} \sum_{\substack{u\in V(G)\\ v\in V(H)}} f_1(i,j,k,l) &= \sum_{u\in V(G)}\sum_{v\in V(H)} |d_G(u) - d_H(v) + n_2 - 1| \\ &\le \sum_{u\in V(K_{n_1})}\sum_{v\in V(T_{n_2})} \left|d_{K_{n_1}}(u) - d_{T_{n_2}}(v) + n_2 - 1\right| \\ &= \sum_{u\in V(K_{n_1})}\sum_{v\in V(T_{n_2})} \left|n_1 - 1 - d_{T_{n_2}}(v) + n_2 - 1\right| \\ &= n_1\sum_{v\in V(T_{n_2})} \left(n_1 + n_2 - 2 - d_{T_{n_2}}(v)\right) \\ &= n_1 n_2(n_1 + n_2 - 2) - 2n_1(n_2 - 1) \\ &= n_1(n_2^2 + n_1 n_2 - 4n_2 + 2). \end{aligned} \tag{13}$$

Substituting (13) into (12), we obtain

$$\mathrm{irr}(G\odot H) \;\le\; \mathrm{irr}(G) + n_1\,\mathrm{irr}(H) + n_1\left(n_2^2 + n_1 n_2 - 4n_2 + 2\right). \tag{14}$$

From the derivation of the bound (14), it follows that the sharp bound is obtained when G is compete graph on n_1 vertices and H is any tree on n_2 vertices. ■

2.6. Lexicographic product

The *lexicographic product* $G \circ H$ (also known as the *graph composition*) of simple undirected graphs G and H is the graph with the vertex set $V(G \circ H) = V(G) \times V(H)$ and the edge set $E(G \circ H) = \{(u_i, v_k)(u_j, v_l) : [u_i u_j \in E(G)] \vee [(v_k v_l \in E(H)) \wedge (u_i = u_j)]\}$.

Theorem 8. *Let G and H be simple undirected graphs with $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$ and $|E(H)| = m_2$. Then*

$$\mathrm{irr}(G \circ H) \leq n_2^3 \,\mathrm{irr}(G) + n_1 \,\mathrm{irr}(H) + \frac{1}{6} m_1 \left(2n_2^3 - 3n_2^2 - 2n_2 + 3\right).$$

Proof. By the definition of $G \circ H$, it follows that $|V(G \circ H)| = n_1 n_2$, $|E(G \circ H)| = n_1 m_2 + n_2^2 m_1$ and $d_{G \circ H}(u_i, v_j) = n_2 d_G(u_i) + d_H(v_j)$ for all $1 \leq i \leq n_1, 1 \leq j \leq n_2$. Applying those relations, we obtain

$$\begin{aligned}
\mathrm{irr}(G \circ H) &= \textstyle\sum_{(u_i,v_k)(u_j,v_l) \in E(G \circ H)} |d_{G \circ H}(u_i, v_k) - d_{G \circ H}(u_j, v_l)| \\
&= \textstyle\sum_{u_i \in V(G)} \sum_{v_k v_l \in E(H)} |(n_2 d_G(u_i) + d_H(v_k)) - (n_2 d_G(u_i) + d_H(v_l))| \\
&+ \textstyle\sum_{v_k, v_l \in V(H)} \sum_{u_i u_j \in E(G)} |n_2 d_G(u_i) + d_H(v_k) - n_2 d_G(u_j) - d_H(v_l)| \\
&= \textstyle\sum_{u_i \in V(G)} \sum_{v_k v_l \in E(H)} |d_H(v_k) - d_H(v_l)| \\
&+ \textstyle\sum_{v_k, v_l \in V(H)} \sum_{u_i u_j \in E(G)} |n_2 (d_G(u_i) - d_G(u_j)) + (d_H(v_k) - d_H(v_l))| \\
&= n_1 \mathrm{irr}(H) \\
&+ \textstyle\sum_{v_k, v_l \in V(H)} \sum_{u_i u_j \in E(G)} |n_2 (d_G(u_i) - d_G(u_j)) + d_H(v_k) - d_H(v_l)|.
\end{aligned}$$

By applying the triangle inequality in the last expression, we have

$$\mathrm{irr}(G \circ H) \leq n_1 \,\mathrm{irr}(H) + n_2^3 \,\mathrm{irr}(G) + |E(G)| \textstyle\sum_{v_k, v_l \in V(H)} |d_H(v_k) - d_H(v_l)|.$$

Finally, by Theorem 1, we obtain

$$\mathrm{irr}(G \circ H) \leq n_2^3 \,\mathrm{irr}(G) + n_1 \,\mathrm{irr}(H) + \tfrac{1}{6} m_1 \left(2n_2^3 - 3n_2^2 - 2n_2 + 3\right). \qquad \blacksquare$$

2.7. Disjunction

The *disjunction* $G \vee H$ of simple undirected graphs G and H with $|V(G)| = n_1$ and $|V(H)| = n_2$ is the graph with the vertex set $V(G \vee H) = V(G) \times V(H)$ and the edge set $E(G \vee H) = \{(u_i, v_k)(u_j, v_l) : u_i u_j \in E(G) \vee v_k v_l \in E(H)\}$.

Theorem 9. *Let G and H be simple undirected graphs with $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$ and $|E(H)| = m_2$. Then*

$$\begin{aligned}
\mathrm{irr}(G \vee H) &\leq (n_2^3 - M_1(H)) \,\mathrm{irr}(G) + (2\, n_2 m_2 + M_1(H)) \,\mathrm{irr}_t(G) \\
&+ (n_1^3 - M_1(G)) \,\mathrm{irr}(H) + (2\, n_1 m_1 + M_1(G)) \,\mathrm{irr}_t(H).
\end{aligned}$$

Proof. By definition of $G \vee H$, it holds that $|V(G \vee H)| = n_1 n_2$, $|E(G \vee H)| = n_2^2 m_1 + n_1^2 m_2 - 2m_1 m_2$ and $d_{(G\vee H)}(u_i, v_k) = n_2 d_G(u_i) + n_1 d_H(v_k) - d_G(u_i) d_H(v_k)$ for all $1 \le i \le n_1, 1 \le j \le n_2$. Applying those relations, we obtain

$$\begin{aligned}
\operatorname{irr}(G \vee H) &= \sum\nolimits_{(u_i,v_k)(u_j,v_l)\in E(G\vee H)} |d_{G\vee H}(u_i,v_k) - d_{G\vee H}(u_j,v_l)| \\
&= \sum\nolimits_{v_k,v_l\in V(H)} \sum\nolimits_{u_iu_j\in E(G)} |d_{G\vee H}(u_i,v_k) - d_{G\vee H}(u_j,v_l)| \\
&+ \sum\nolimits_{u_i,u_j\in V(G)} \sum\nolimits_{v_kv_l\in E(H)} |d_{G\vee H}(u_i,v_k) - d_{G\vee H}(u_j,v_l)| \\
&- 2\sum\nolimits_{u_iu_j\in E(G)} \sum\nolimits_{v_kv_l\in E(H)} |d_{G\vee H}(u_i,v_k) - d_{G\vee H}(u_j,v_l)|\,.
\end{aligned} \tag{15}$$

After an algebraic transformation and the triangle inequality, we have

$$\begin{aligned}
|d_{G\vee H}(u_i,v_k) - d_{G\vee H}(u_j,v_l)| &= |n_2 d_G(u_i) + n_1 d_H(v_k) - d_G(u_i) d_H(v_k) \\
&\quad -n_2 d_G(u_j) - n_1 d_H(v_l) + d_G(u_j) d_H(v_l)| \\
&\le n_2 |d_G(u_i) - d_G(u_j)| + n_1 |d_H(v_k) - d_H(v_l)| \\
&+ \frac{1}{2} (d_G(u_i) + d_G(u_j)) |d_H(v_k) - d_H(v_l)| \\
&+ \frac{1}{2} (d_H(v_k) + d_H(v_l)) |d_G(u_i) - d_G(u_j)| \\
&= f_2(i,j,k,l).
\end{aligned} \tag{16}$$

Substituting (16) in (15), we obtain

$$\begin{aligned}
\operatorname{irr}(G \vee H) &\le \sum\nolimits_{v_k,v_l\in V(H)} \sum\nolimits_{u_iu_j\in E(G)} f_2(i,j,k,l) \\
&+ \sum\nolimits_{u_i,u_j\in V(G)} \sum\nolimits_{v_kv_l\in E(H)} f_2(i,j,k,l) \\
&- 2\sum\nolimits_{u_iu_j\in E(G)} \sum\nolimits_{v_kv_l\in E(H)} f_2(i,j,k,l).
\end{aligned} \tag{17}$$

Applying (1), (2) and (4), we have

$$\begin{aligned}
\sum\nolimits_{\substack{v_k,v_l\in V(H)\\ u_iu_j\in E(G)}} f_2(i,j,k,l) &= \sum\nolimits_{\substack{v_k,v_l\in V(H)\\ u_iu_j\in E(G)}} n_2 |d_G(u_i) - d_G(u_j)| \\
&+ \sum\nolimits_{\substack{v_k,v_l\in V(H)\\ u_iu_j\in E(G)}} n_1 |d_H(v_k) - d_H(v_l)| \\
&+ \frac{1}{2} \sum\nolimits_{\substack{v_k,v_l\in V(H)\\ u_iu_j\in E(G)}} (d_G(u_i) + d_G(u_j)) |d_H(v_k) - d_H(v_l)| \\
&+ \frac{1}{2} \sum\nolimits_{\substack{v_k,v_l\in V(H)\\ u_iu_j\in E(G)}} (d_H(v_k) + d_H(v_l)) |d_G(u_i) - d_G(u_j)| \\
&= n_2(n_2^2 + 2m_2)\operatorname{irr}(G) + (M_1(G) + 2n_1 m_1)\operatorname{irr}_t(H).
\end{aligned} \tag{18}$$

Similarly,

$$\begin{aligned}\sum\nolimits_{\substack{u_i,u_j\in V(G)\\ v_kv_l\in E(H)}} f_2(i,j,k,l) &= \sum\nolimits_{\substack{u_i,u_j\in V(G)\\ v_kv_l\in E(H)}} n_2\,|d_G(u_i)-d_G(u_j)|\,r\\ &+\sum\nolimits_{\substack{u_i,u_j\in V(G)\\ v_kv_l\in E(H)}} n_1\,|d_H(v_k)-d_H(v_l)|\\ &+\frac{1}{2}\sum\nolimits_{\substack{u_i,u_j\in V(G)\\ v_kv_l\in E(H)}}(d_G(u_i)+d_G(u_j))\,|d_H(v_k)-d_H(v_l)|\\ &+\frac{1}{2}\sum\nolimits_{\substack{u_i,u_j\in V(G)\\ v_kv_l\in E(H)}}(d_H(v_k)+d_H(v_l))\,|d_G(u_i)-d_G(u_j)|\\ &= n_1(n_1^2+2m_1)\mathrm{irr}(H)+(M_1(H)+2n_2m_2)\mathrm{irr}_t(G).\end{aligned} \tag{19}$$

Finally,

$$\begin{aligned}\sum\nolimits_{\substack{u_iu_j\in E(G)\\ v_kv_l\in E(H)}} f_2(i,j,k,l) &= \sum\nolimits_{\substack{u_iu_j\in E(G)\\ v_kv_l\in E(H)}} n_2\,|d_G(u_i)-d_G(u_j)|\\ &+\sum\nolimits_{\substack{u_iu_j\in E(G)\\ v_kv_l\in E(H)}} n_1\,|d_H(v_k)-d_H(v_l)|\\ &+\frac{1}{2}\sum\nolimits_{\substack{u_iu_j\in E(G)\\ v_kv_l\in E(H)}}(d_G(u_i)+d_G(u_j))\,|d_H(v_k)-d_H(v_l)|\\ &+\frac{1}{2}\sum\nolimits_{\substack{u_iu_j\in E(G)\\ v_kv_l\in E(H)}}(d_H(v_k)+d_H(v_l))\,|d_G(u_i)-d_G(u_j)|\\ &= (n_2m_2+\frac{1}{2}M_1(H))\mathrm{irr}(G)+(n_1m_1+\frac{1}{2}M_1(G))\mathrm{irr}(H).\end{aligned} \tag{20}$$

Combining (17), (18), (19) and (20), we finally obtain

$$\begin{aligned}\mathrm{irr}(G\vee H) \le{}& (n_2^3-M_1(H))\mathrm{irr}(G)+(2\,n_2m_2+M_1(H))\mathrm{irr}_t(G)\\ &+(n_1^3-M_1(G))\mathrm{irr}(H)+(2\,n_1m_1+M_1(G))\mathrm{irr}_t(H).\end{aligned} \tag{21}$$

■

2.8. Symmetric difference

The *symmetric difference* $G\oplus H$ of simple undirected graphs G and H with $|V(G)|=n_1$ and $|V(H)|=n_2$ is the graph with the vertex set $V(G\oplus H)=V(G)\times V(H)$ and the edge set $E(G\oplus H)=\{(u_i,v_k)(u_j,v_l):\text{ either } u_iu_j\in E(G) \text{ or } v_kv_l \in E(H)\}$. It holds that $|V(G\oplus H)|=n_1n_2$, $|E(G\oplus H)|=n_1^2m_2+n_2^2m_1-4m_1m_2$ and $d_{(G\oplus H)}(u_i,v_j)=n_2d_G(u_i)+n_1d_H(v_j)-2d_G(u_i)d_H(v_j)$ for all $1\le i\le n_1, 1\le j\le n_2$. Much as in the previous case, we present only the bound on the irregularity of symmetric difference of two graphs.

Theorem 10. *Let G and H be simple undirected graphs with $|V(G)|=n_1$, $|E(G)|=m_1$, $|V(H)|=n_2$ and $|E(H)|=m_2$. Then*

$$\begin{aligned}\mathrm{irr}(G\oplus H) &\le (n_2^3-4M_1(H))\,\mathrm{irr}(G)+2\,(n_2m_2+M_1(H))\,\mathrm{irr}_t(G)\\ &+(n_1^3-4M_1(G))\,\mathrm{irr}(H)+2\,(n_1m_1+M_1(G))\,\mathrm{irr}_t(H).\end{aligned}$$

3. Conclusion

In this paper we consider the irregularity of simple undirected graphs, as defined by Albertson [5], under several graph operations. We presented the exact expression for Cartesian product, and sharp upper bounds for join, corona product, direct product and strong product. It is an open problem if the presented upper bounds on the irregularity of lexicographic product, disjunction and symmetric difference are the best possible.

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Received 3 September 2012
Revised 22 January 2013
Accepted 11 March 2013