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Note

# THE DOMINATION NUMBER OF $\boldsymbol{K}_{\boldsymbol{n}}^{3}$ 

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#### Abstract

Let $K_{n}^{3}$ denote the Cartesian product $K_{n} \square K_{n} \square K_{n}$, where $K_{n}$ is the complete graph on $n$ vertices. We show that the domination number of $K_{n}^{3}$ is $\left\lceil\frac{n^{2}}{2}\right\rceil$.


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## 1. Introduction

Let $G_{1}$ and $G_{2}$ be two graphs. Per the notation of West [14], the Cartesian product of $G_{1}$ and $G_{2}$ is the graph $G_{1} \square G_{2}$ with vertex set $V\left(G_{1} \square G_{2}\right)=V\left(G_{1}\right) \times$ $V\left(G_{2}\right)$ and edge set containing $\left(\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)\right)$ if and only if either $x_{1}=x_{2}$ and $y_{1}$ is adjacent to $y_{2}$, or $y_{1}=y_{2}$ and $x_{1}$ is adjacent to $x_{2}$. To isomorphism, Cartesian product is a binary operator that is both commutative and associative.

Let $G$ be a graph. Then a dominating set of $G$ is a subset $D$ of $V(G)$ such that for every vertex $v$ in $V(G), v$ is equal or adjacent to some vertex in $D$. The domination number of $G$, denoted $\gamma(G)$, is the cardinality of the smallest
dominating set of $G$. (See the text of Haynes et al. [7] for further study of domination.)

Denoting $K_{n} \square K_{n} \square K_{n}$ by $K_{n}^{3}$, we show $\gamma\left(K_{n}^{3}\right)=\left\lceil\frac{n^{2}}{2}\right\rceil$.
Research on the domination number of Cartesian products of graphs has been driven in large part by the open conjecture of Vizing $[12,13]$ that posits the domination number of a Cartesian product to be bounded from below by the product of the domination numbers of the factors. Products of graphs in special classes have received particular attention. Following the work of Jacobson and Kinch [8] and Chang [1, 2] on products of paths, Gonçalves et al. [5] have determined $\lambda\left(P_{n} \square P_{m}\right)$ for arbitrarily large $m$ and $n$. Considering the Cartesian product of cycles, Klavžar and Seifter [9] determined $\gamma\left(C_{k} \square C_{n}\right)$ for $k=3,4$ and 5. El-Zahar and Shaheen $[3,4,11]$ have subsequently obtained results for additional $k, n$. The hypercube $Q_{n}$, too, has been studied. In [10], Pai and Chiu reviewed existing results in [6] on $\gamma\left(Q_{n}\right)$ for the purpose of analysing the power domination number of $Q_{n}$, a variant of $\gamma\left(Q_{n}\right)$.

We point out that because the Hamming graph $H(d, n)$ is isomorphic to the Cartesian product of $d$ copies of $K_{n}$, we herein establish $\gamma(H(3, n))$. The domination numbers of $H(1, n)$ and $H(2, n)$ are well known.

## 2. Proof

Since the claim is clearly true for $n=1$, we henceforth assume $n \geq 2$. The vertices of $K_{n}^{3}$ shall be denoted in the usual way as lattice points $(x, y, z)$ in 3 -space, $1 \leq x, y, z \leq n$, where $x, y$ and $z$ specify a row, column, and level, respectively. For a given subset $S$ of $V\left(K_{n}^{3}\right)$, the cross-section of $S$ at row $x$ (resp. column $y$, level $z$ ) shall refer to the set of vertices in $S$ that are in row $x$ (resp. column $y$, level $z$ ). For a dominating set $D$ of $K_{n}^{3}, m_{D}$ will denote the smallest integer $i$ such that some cross-section of $K_{n}^{3}$ contains precisely $i$ vertices in $D$.

Our strategy is outlined as follows:
(1) We show that there exists a dominating set of $K_{n}^{3}$ of cardinality $\left\lfloor\frac{n}{2}\right\rfloor^{2}+(n-$ $\left.\left\lfloor\frac{n}{2}\right\rfloor\right)^{2}$;
(2) We show that if $D$ is a dominating set of $K_{n}^{3}$ of minimum cardinality $\gamma\left(K_{n}^{3}\right)$, then $m_{D} \leq\left\lfloor\frac{n}{2}\right\rfloor$ and $\gamma\left(K_{n}^{3}\right) \geq m_{D}^{2}+\left(n-m_{D}\right)^{2}$;
(3) We observe that the quadratic $f(x)=x^{2}+(n-x)^{2}$ on the non-negative integers is minimized at $x=\left\lfloor\frac{n}{2}\right\rfloor$, implying by (1) and (2) that $m_{D}=\left\lfloor\frac{n}{2}\right\rfloor$ and hence $\gamma\left(K_{n}^{3}\right)=\left\lfloor\frac{n}{2}\right\rfloor^{2}+\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right)^{2}=\left\lceil\frac{n^{2}}{2}\right\rceil$.

To show (1), we let $n_{*}$ denote $\left\lfloor\frac{n}{2}\right\rfloor$ for notational convenience and we form a partition of $V\left(K_{n}^{3}\right)$ consisting of the following eight sets:

$$
\begin{aligned}
& A_{1}=\left\{(x, y, z) \mid 1 \leq x, y, z \leq n_{*}\right\}, \\
& A_{2}=\left\{(x, y, z) \mid 1 \leq x, y \leq n_{*} \text { and } n_{*}+1 \leq z \leq n\right\}, \\
& A_{3}=\left\{(x, y, z) \mid n_{*}+1 \leq x \leq n \text { and } 1 \leq y, z \leq n_{*}\right\}, \\
& A_{4}=\left\{(x, y, z) \mid n_{*}+1 \leq y \leq n \text { and } 1 \leq x, z \leq n_{*}\right\}, \\
& B_{1}=\left\{(x, y, z) \mid n_{*}+1 \leq x, y, z \leq n\right\}, \\
& B_{2}=\left\{(x, y, z) \mid n_{*}+1 \leq x, y \leq n \text { and } 1 \leq z \leq n_{*}\right\}, \\
& B_{3}=\left\{(x, y, z) \mid 1 \leq x \leq n_{*} \text { and } n_{*}+1 \leq y, z \leq n\right\}, \\
& B_{4}=\left\{(x, y, z) \mid 1 \leq y \leq n_{*} \text { and } n_{*}+1 \leq x, z \leq n\right\} .
\end{aligned}
$$

We observe that there exists a subset $S_{A_{1}}$ of $A_{1}$ of cardinality $n_{*}^{2}$ such that every vertex in $\bigcup_{i=1}^{4} A_{i}$ shares a row, column, or level with some vertex in $S_{A_{1}}$. (Form an $n_{*} \times n_{*}$ Latin square in which the cell entries are taken from $\left\{1,2, \ldots, n_{*}\right\}$. Let $S_{A_{1}}$ contain $(x, y, z)$ if and only if the entry at row $x$ and column $y$ of the Latin square is z.) Similarly, there exists a subset $S_{B_{1}}$ of $B_{1}$ of cardinality $\left(n-n_{*}\right)^{2}$ such that every vertex in $\bigcup_{i=1}^{4} B_{i}$ shares a row, column, or level with some vertex in $S_{B_{1}}$. This implies that $S_{A_{1}} \cup S_{B_{1}}$ is a dominating set of $K_{n}^{3}$. Since $S_{A_{1}}$ and $S_{B_{1}}$ are disjoint, there exists a dominating set of $K_{n}^{3}$ of cardinality $n_{*}^{2}+\left(n-n_{*}\right)^{2}=\left\lceil\frac{n^{2}}{2}\right\rceil$.

We now show (2). Let $D$ denote a dominating set of $K_{n}^{3}$ of minimum cardinality $\gamma\left(K_{n}^{3}\right)$. Since $\gamma\left(K_{n}^{3}\right) \leq\left\lceil\frac{n^{2}}{2}\right\rceil$ by (1), we obtain $n m_{D} \leq\left\lceil\frac{n^{2}}{2}\right\rceil$, implying $m_{D} \leq\left\lfloor\frac{n}{2}\right\rfloor$.

With no loss of generality, we assume that the cross-section of $V\left(K_{n}^{3}\right)$ at level $z=1$ contains precisely $m_{D}$ vertices of $D$, and we denote the set of vertices in $D$ that are on level 1 by $D_{1}$. Let $c_{1}$ denote the number of columns at level 1 that contain no vertex in $D_{1}$ and let $r_{1}$ denote the number of rows at level 1 that contain no vertex in $D_{1}$. Since $c_{1} \geq n-m_{D}$ and $r_{1} \geq n-m_{D}$, we may find a set $R_{1}$ of $n-m_{D}$ rows at level 1 and a set $C_{1}$ of $n-m_{D}$ columns at level 1 that contain no vertices in $D_{1}$. Accordingly, at the intersections of these rows and columns we find $\left(n-m_{D}\right)^{2}$ vertices at level 1 that are not adjacent to any vertex in $D_{1}$. Denoting the set of those vertices by $S$, it follows that each vertex $(x, y, 1)$ in $S$ is adjacent to some vertex $(x, y, z)$ in $D$ where $z \geq 2$. Therefore $D$ contains $\left(n-m_{D}\right)^{2}$ distinct vertices (the set of which we denote by $S_{1}$ ) that are particularly adjacent to the $\left(n-m_{D}\right)^{2}$ vertices in $S$. Moreover, there exist $m_{D}$ rows on level 1, none of which is in $R_{1}$. Hence the set $S_{2}$ of vertices in $D$ that are in the cross-section at one of these rows does not intersect $S_{1}$. Since each of these $m_{D}$ cross-sections contains at least $m_{D}$ elements of $D$, we have that $D$ contains at least $m_{D}^{2}+\left(n-m_{D}\right)^{2}$ vertices, thus establishing (2).

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