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Note

THE DOMINATION NUMBER OF K_n^3

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Abstract

Let K_n^3 denote the Cartesian product $K_n \square K_n \square K_n$, where K_n is the complete graph on n vertices. We show that the domination number of K_n^3 is $\left\lceil \frac{n^2}{2} \right\rceil$.

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1. Introduction

Let G_1 and G_2 be two graphs. Per the notation of West [14], the Cartesian product of G_1 and G_2 is the graph $G_1 \square G_2$ with vertex set $V(G_1 \square G_2) = V(G_1) \times V(G_2)$ and edge set containing $((x_1, y_1), (x_2, y_2))$ if and only if either $x_1 = x_2$ and y_1 is adjacent to y_2, or $y_1 = y_2$ and x_1 is adjacent to x_2. To isomorphism, Cartesian product is a binary operator that is both commutative and associative.

Let G be a graph. Then a dominating set of G is a subset D of $V(G)$ such that for every vertex v in $V(G)$, v is equal or adjacent to some vertex in D. The domination number of G, denoted $\gamma(G)$, is the cardinality of the smallest dominating set of G. (See the text of Haynes *et al.* [7] for further study of domination.)

Denoting $K_n \square K_n \square K_n$ by K_n^3, we show $\gamma(K_n^3) = \left\lceil \frac{n^2}{2} \right\rceil$.

Research on the domination number of Cartesian products of graphs has been driven in large part by the open conjecture of Vizing [12, 13] that posits the domination number of a Cartesian product to be bounded from below by the product of the domination numbers of the factors. Products of graphs in special classes have received particular attention. Following the work of Jacobson and Kinch [8] and Chang [1, 2] on products of paths, Gonçalves *et al.* [5] have determined $\lambda(P_n \square P_m)$ for arbitrarily large m and n. Considering the Cartesian product of cycles, Klavžar and Seifter [9] determined $\gamma(C_k \square C_n)$ for $k = 3, 4$ and 5. El-Zahar and Shaheen [3, 4, 11] have subsequently obtained results for additional k, n. The hypercube Q_n, too, has been studied. In [10], Pai and Chiu reviewed existing results in [6] on $\gamma(Q_n)$ for the purpose of analysing the power domination number of Q_n, a variant of $\gamma(Q_n)$.

We point out that because the Hamming graph $H(d, n)$ is isomorphic to the Cartesian product of d copies of K_n, we herein establish $\gamma(H(3, n))$. The domination numbers of $H(1, n)$ and $H(2, n)$ are well known.

2. Proof

Since the claim is clearly true for $n = 1$, we henceforth assume $n \geq 2$. The vertices of K_n^3 shall be denoted in the usual way as lattice points (x, y, z) in 3-space, $1 \leq x, y, z \leq n$, where x, y and z specify a row, column, and level, respectively. For a given subset S of $V(K_n^3)$, the cross-section of S at row x (resp. column y, level z) shall refer to the set of vertices in S that are in row x (resp. column y, level z). For a dominating set D of K_n^3, m_D will denote the smallest integer i such that some cross-section of K_n^3 contains precisely i vertices in D.

Our strategy is outlined as follows:

(1) We show that there exists a dominating set of K_n^3 of cardinality $\left\lfloor \frac{n}{2} \right\rfloor^2 + (n - \left\lfloor \frac{n}{2} \right\rfloor)^2$;

(2) We show that if D is a dominating set of K_n^3 of minimum cardinality $\gamma(K_n^3)$, then $m_D \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $\gamma(K_n^3) \geq m_D^2 + (n - m_D)^2$;

(3) We observe that the quadratic $f(x) = x^2 + (n - x)^2$ on the non-negative integers is minimized at $x = \left\lfloor \frac{n}{2} \right\rfloor$, implying by (1) and (2) that $m_D = \left\lfloor \frac{n}{2} \right\rfloor$ and hence $\gamma(K_n^3) = \left\lfloor \frac{n}{2} \right\rfloor^2 + (n - \left\lfloor \frac{n}{2} \right\rfloor)^2 = \left\lceil \frac{n^2}{2} \right\rceil$.

To show (1), we let n_* denote $\left\lfloor \frac{n}{2} \right\rfloor$ for notational convenience and we form a partition of $V(K_n^3)$ consisting of the following eight sets:

$A_1 = \{(x, y, z) \mid 1 \le x, y, z \le n_*\}$,
$A_2 = \{(x, y, z) \mid 1 \le x, y \le n_* \text{ and } n_* + 1 \le z \le n\}$,
$A_3 = \{(x, y, z) \mid n_* + 1 \le x \le n \text{ and } 1 \le y, z \le n_*\}$,
$A_4 = \{(x, y, z) \mid n_* + 1 \le y \le n \text{ and } 1 \le x, z \le n_*\}$,
$B_1 = \{(x, y, z) \mid n_* + 1 \le x, y, z \le n\}$,
$B_2 = \{(x, y, z) \mid n_* + 1 \le x, y \le n \text{ and } 1 \le z \le n_*\}$,
$B_3 = \{(x, y, z) \mid 1 \le x \le n_* \text{ and } n_* + 1 \le y, z \le n\}$,
$B_4 = \{(x, y, z) \mid 1 \le y \le n_* \text{ and } n_* + 1 \le x, z \le n\}$.

We observe that there exists a subset S_{A_1} of A_1 of cardinality n_*^2 such that every vertex in $\bigcup_{i=1}^4 A_i$ shares a row, column, or level with some vertex in S_{A_1}. (Form an $n_* \times n_*$ Latin square in which the cell entries are taken from $\{1, 2, \ldots, n_*\}$. Let S_{A_1} contain (x, y, z) if and only if the entry at row x and column y of the Latin square is z.) Similarly, there exists a subset S_{B_1} of B_1 of cardinality $(n - n_*)^2$ such that every vertex in $\bigcup_{i=1}^4 B_i$ shares a row, column, or level with some vertex in S_{B_1}. This implies that $S_{A_1} \bigcup S_{B_1}$ is a dominating set of K_n^3. Since S_{A_1} and S_{B_1} are disjoint, there exists a dominating set of K_n^3 of cardinality $n_*^2 + (n - n_*)^2 = \left\lceil \frac{n^2}{2} \right\rceil$.

We now show (2). Let D denote a dominating set of K_n^3 of minimum cardinality $\gamma(K_n^3)$. Since $\gamma(K_n^3) \le \left\lceil \frac{n^2}{2} \right\rceil$ by (1), we obtain $nm_D \le \left\lceil \frac{n^2}{2} \right\rceil$, implying $m_D \le \left\lfloor \frac{n}{2} \right\rfloor$.

With no loss of generality, we assume that the cross-section of $V(K_n^3)$ at level $z = 1$ contains precisely m_D vertices of D, and we denote the set of vertices in D that are on level 1 by D_1. Let c_1 denote the number of columns at level 1 that contain no vertex in D_1 and let r_1 denote the number of rows at level 1 that contain no vertex in D_1. Since $c_1 \ge n - m_D$ and $r_1 \ge n - m_D$, we may find a set R_1 of $n - m_D$ rows at level 1 and a set C_1 of $n - m_D$ columns at level 1 that contain no vertices in D_1. Accordingly, at the intersections of these rows and columns we find $(n - m_D)^2$ vertices at level 1 that are not adjacent to any vertex in D_1. Denoting the set of those vertices by S, it follows that each vertex $(x, y, 1)$ in S is adjacent to some vertex (x, y, z) in D where $z \ge 2$. Therefore D contains $(n - m_D)^2$ distinct vertices (the set of which we denote by S_1) that are particularly adjacent to the $(n - m_D)^2$ vertices in S. Moreover, there exist m_D rows on level 1, none of which is in R_1. Hence the set S_2 of vertices in D that are in the cross-section at one of these rows does not intersect S_1. Since each of these m_D cross-sections contains at least m_D elements of D, we have that D contains at least $m_D^2 + (n - m_D)^2$ vertices, thus establishing (2).

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