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A RAMSEY-TYPE THEOREM FOR MULTIPLE DISJOINT COPIES OF INDUCED SUBGRAPHS

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Abstract

Let k and ℓ be positive integers with $\ell \leq k-2$. It is proved that there exists a positive integer c depending on k and ℓ such that every graph of order $(2k-1-\ell/k)n+c$ contains n vertex disjoint induced subgraphs, where these subgraphs are isomorphic to each other and they are isomorphic to one of four graphs: (1) a clique of order k, (2) an independent set of order k, (3) the join of a clique of order ℓ and an independent set of order $k-\ell$, or (4) the union of an independent set of order ℓ and a clique of order $k-\ell$.

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1. Introduction

Let G and H denote finite undirected graphs without multiple edges and loops. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices of G and the set of edges of G. For a subset $S \subset V(G)$, the subgraph of G induced by S is denoted by $G[S]$.

For two graphs G and H, let us define $N(G,H)$ as the maximum integer n such that there exists a vertex partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_n$ satisfying $G[V_i] \cong H$ for $1 \leq i \leq n$. For a family of graphs $\mathcal{H}$, let us define $N(G,\mathcal{H})$ as the maximum of $N(G,H)$ over $H \in \mathcal{H}$. Furthermore, for a positive integer n, we define an

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integer valued function $f(n,\mathcal{H})$ as the minimum integer s such that $N(G,\mathcal{H}) \geq n$ for every graph G with $|V(G)| \geq s$. By the definition, $f(1,\{K_k,\overline{K_\ell}\})$ is the classical Ramsey number of 2-edge colored graphs, where $\overline{K_\ell}$ is the complement of K_ℓ.

We remark that if $\mathcal{H}$ does not contain K_k or $\overline{K_k}$ for all $k \geq 1$, then $f(n,\mathcal{H})$ is not determined as a finite value, because we have $N(K_s,\mathcal{H}) = 0$ or $N(\overline{K_s},\mathcal{H}) = 0$ for $s \geq 1$. Hence, in the following, we always assume that $\{K_k,\overline{K_\ell}\} \subset \mathcal{H}$ for some k and ℓ.

Our aim is to study $f(n,\mathcal{H})$ for some family of graphs $\mathcal{H}$ with n sufficiently large. In order to explain related results, let us introduce a few more notations. For two graphs G_1 and G_2, the *union* $G_1 \cup G_2$ is the graph such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The *join* $G_1 + G_2$ is the graph such that $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = (V(G_1) \times V(G_2)) \cup E(G_1) \cup E(G_2)$.

Let $\mathcal{G}_k$ be the family of all graphs with k vertices. It is not difficult to see that $f(n,\mathcal{G}_2) = 3n-1$ for $n \geq 1$. Indeed, the inequality $3n-1 \leq f(n,\mathcal{G}_2)$ is followed by the fact $N(K_{2n-1} \cup \overline{K_{n-1}},\mathcal{G}_2) \leq n-1$. The following result is a classical one in the graph Ramsey theory.

Theorem 1 [3]**.** *Let $n \geq 2$. Then $f(n,\{K_3,\overline{K_3}\}) = 5n$.*

The above result is extended for complete graphs with any number of vertices.

Theorem 2 [1, 2]**.** *Let $k,\ell \geq 2$. Then*
$f(n,\{K_k,\overline{K_\ell}\}) = (k+l-1)n + f(1,\{K_{k-1},\overline{K_{\ell-1}}\}) - 2$ for n sufficiently large.

Let $\mathcal{A}_k = \{K_k,\overline{K_k},K_{1,k-1},\overline{K_{1,k-1}}\}$ for $k \geq 3$. Recently, the author proved the following result.

Theorem 3 [5]**.** *Let $k \geq 3$. Then $f(n,\mathcal{A}_k) = \left(2k-1-\frac{1}{k}\right)n + O(1)$.*

Since $\mathcal{G}_3 = \mathcal{A}_3$, we have an immediate consequence of Theorem 3.

Corollary 4. $f(n,\mathcal{G}_3) = \dfrac{14}{3}n + O(1)$.

We will discuss shortly $f(n,\mathcal{G}_4)$ in Section 4.

2. Main Results

For $1 \leq \ell \leq k-2$, let $\mathcal{B}_{k,\ell} = \{K_k,\overline{K_k},K_\ell + \overline{K_{k-\ell}},\overline{K_\ell} \cup K_{k-\ell}\}$. The main result of the paper is as follows.

Theorem 5. *Let k and ℓ be positive integers with $2 \leq \ell \leq k-2$. Then $f(n,\mathcal{B}_{k,\ell}) = \left(2k-1-\frac{\ell}{k}\right)n + O(1)$.*

The proof of Theorem 5 will be given in Section 3. Since $\mathcal{B}_{k,1} = \mathcal{A}_k$ for $k \geq 3$, by combining Theorem 3 and Theorem 5, we have $f(n, \mathcal{B}_{k,\ell}) = (2k-1-\ell/k)n+O(1)$ for $1 \leq \ell \leq k-2$.

In this problem, $\mathcal{B}_{k,\ell}$ is in a special position.

Proposition 6. *Let $k \geq 3$. Let $\mathcal{H}$ be a family of graphs having k vertices such that $\mathcal{H} \cap \mathcal{B}_{k,\ell} = \{K_k, \overline{K_k}\}$ for $1 \leq \ell \leq k-2$. Then we have $f(n, \mathcal{H}) = (2k-1)n+O(1)$.*

In particular, for a graph H with k vertices such that $H \notin \mathcal{B}_{k,\ell}$ for $1 \leq \ell \leq k-2$, we have $f(n, \{K_k, \overline{K_k}, H, \overline{H}\}) = (2k-1)n + O(1)$.

Proof. It suffices to prove the claim in the first half. For a lower bound, let $G = K_{(k-1)n-1} + \overline{K_{kn-1}}$. Then we have $N(G, K_k) = N(G, \overline{K_k}) = n-1$ and $N(G, H) = N(G, \overline{H}) = 0$ for $H \in \mathcal{H} \setminus \{K_k, \overline{K_k}\}$. Hence, we have $f(n, \mathcal{H}) > |V(G)| = (2k-1)n-2$. For an upper bound, by Theorem 2, we have $f(n, \mathcal{H}) \leq f(n, \{K_k, \overline{K_k}\}) = (2k-1)n + f(1, \{K_{k-1}, \overline{K_{k-1}}\}) - 2$ for n sufficiently large. ■

3. Proof of Theorem 5

Proof. **Lower bound.** Let $G = K_m + (K_{(k-\ell)n-1} \cup \overline{K_{(k-1)n-1}})$, where $m = \lfloor (\ell - \ell/k)n \rfloor$.

Claim. $N(G, \mathcal{B}_{k,\ell}) < n$.

Proof. Let $V(G) = V_1 \cup V_2 \cup V_3$ such that $|V_1| = m$, $|V_2| = (k-\ell)n-1$, $|V_3| = (k-1)n-1$ and $E(G) = \binom{V_1}{2} \cup \binom{V_2}{2} \cup (V_1 \times V_2) \cup (V_1 \times V_3)$.

Firstly, we have $N(G, \overline{K_n}) < n$. Indeed, each $\overline{K_k}$ of G contains at least $k-1$ vertices of V_3. Hence, we have $N(G, \overline{K_k}) \leq \lfloor |V_3|/(k-1) \rfloor = n-1$.

In the same manner, we have $N(G, \overline{K_\ell} \cup K_{k-\ell}) < n$. Indeed, each $\overline{K_\ell} \cup K_{k-\ell}$ of G contains at least $k-\ell$ vertices of V_2. Hence, we have $N(G, \overline{K_\ell} \cup K_{k-\ell}) \leq \lfloor |V_2|/(k-\ell) \rfloor = n-1$.
Next, we show that $N(G, K_\ell + \overline{K_{k-\ell}}) < n$. Indeed, each $K_\ell + \overline{K_{k-\ell}}$ of G contains at least ℓ vertices of V_1. Hence, we have $N(G, K_\ell + \overline{K_{k-\ell}}) \leq \lfloor |V_1|/\ell \rfloor < n$.

Lastly, we show that $N(G, K_k) < n$. For $v \in V(G)$, let us assign a weight $w(v)$ such that $w(v) = 1/(k-1)$ for $v \in V_1$, $w(v) = 1/k$ for $v \in V_2$, and $w(v) = 0$ for $v \in V_3$. Furthermore, for $S \subset V(G)$, let $w(S) = \sum_{v \in S} w(v)$. Then we have $w(S) \geq 1$ for any $S \subset V(G)$ such that $G[S] \cong K_k$. On the other hand, the total weight is calculated as

$$\begin{aligned} w(V(G)) &= \frac{|V_1|}{k-1} + \frac{|V_2|}{k} \\ &\leq \frac{1}{k-1}\left(\ell - \frac{\ell}{k}\right)n + \frac{1}{k}((k-\ell)n-1) = n - \frac{1}{k}. \end{aligned}$$

Hence, we have $N(G, K_k) < n$. Therefore, we have $N(G, \mathcal{B}_{k,\ell}) < n$. □

By the claim, we have $f(n, \mathcal{B}_{k,\ell}) > |V(G)| > (2k-1-\ell/k)n - 3$.

Upper bound. Before we start the proof, let us show its outline. The main idea of the proof is a variant of a "bow tie argument", which is originated from the proof of Theorem 1([3], see also [4]). A bow tie is a graph with 5 vertices containing both K_3 and $\overline{K_3}$. Let us summarize how to prove $f(n, \{K_3, \overline{K_3}\}) \leq 5n$ by a bow tie argument. Let G be an underlying graph with $5n$ vertices. What we want to show is that $N(G, \{K_3, \overline{K_3}\}) \geq n$. If G contains no bow tie, it turns out that the structure of G becomes very simple, and we can easily show that $N(G, \{K_3, \overline{K_3}\}) \geq n$. Otherwise, let S be a bow tie of G. We partition G into two graphs $G[S]$ and $G' = G[V(G) \setminus S]$. Since $|V(G')| = 5(n-1)$, by inductive hypothesis, we have $N(G', \{K_3, \overline{K_3}\}) \geq n-1$. Then with an additional K_3 or $\overline{K_3}$ in $G[S]$, we have $N(G, \{K_3, \overline{K_3}\}) \geq n$, as required.

Now, we go back to the proof of Theorem 5. We will show that for $n \geq 1$, there exists a positive constant $c = c(k, \ell)$ depending on k and ℓ, such that $f(n, \mathcal{B}_{k,\ell}) \leq (2k-1-\ell/k)n + c$. We will define the value of c just after Lemma 8. Suppose to a contradiction that G is a counterexample with the smallest number of vertices. We assume $|V(G)| \geq (2k-1-\ell/k)n + c$ and $N(G, \mathcal{B}_{k,\ell}) < n$.

Let us introduce a family of graphs, *$\mathcal{B}_{k,\ell}$-good* graphs, which is considered as a variant of a bow tie. We call a graph G_0 *$\mathcal{B}_{k,\ell}$-good* if there exists a positive integer n_0 such that (1) $|V(G_0)| = (2k-1-\ell/k)n_0$ and (2) $N(G_0, H) \geq n_0$ for all $H \in \mathcal{B}_{k,\ell}$. Then a crucial observation is that a smallest counterexample G contains no $\mathcal{B}_{k,\ell}$-good graph as an induced subgraph. Indeed, if G contains a $\mathcal{B}_{k,\ell}$-good induced subgraph G_0 with $|V(G_0)| = (2k-1-\ell/k)n_0$, let us partition G into two graphs G_0 and $G_1 = G[V(G) \setminus V(G_0)]$. Then we have $|V(G_1)| \geq (2k-1-\ell/k)(n-n_0)+c$. Furthermore, since $N(G_0, H) \geq n_0$ and $N(G, H) < n$ for all $H \in \mathcal{B}_{k,\ell}$, we have $N(G_1, \mathcal{B}_{k,\ell}) < n - n_0$. Hence, G_1 is also a counterexample, a contradiction to the minimality of G.

The following lemma is a key for the proof.

Lemma 7. *Let $n_0 = k^2(k-1)\ell(k-\ell)$. Let G_0 be a graph with $(2k-1-\ell/k)n_0$ vertices. Suppose that there exists a vertex partition $V(G_0) = V_1 \cup V_2 \cup V_3 \cup V_4$ such that $|V_1| = (\ell-1)n_0$, $|V_2| = \ell n_0$, $|V_3| = (k-\ell)n_0$, $|V_4| = (k-\ell-\ell/k)n_0$, and $E(G_0) \supset \binom{V_1}{2} \cup (V_1 \times V_2) \cup \binom{V_3}{2} \cup (V_3 \times V_4)$, $E(\overline{G_0}) \supset \binom{V_2}{2} \cup (V_2 \times V_3) \cup \binom{V_4}{2}$. Then both G_0 and $\overline{G_0}$ are $\mathcal{B}_{k,\ell}$-good.*

Proof. Since $\mathcal{B}_{k,\ell}$-goodness is symmetric for a graph and its complement, it suffices to show that $N(G_0, H) \geq n_0$ for all $H \in \mathcal{B}_{k,\ell}$.

For $H = K_k$, we have

$$N(G_0, K_k) \geq \frac{|V_1|}{k-1} + \frac{|V_3|}{k-1} = \frac{\ell-1}{k-1}n_0 + \frac{k-\ell}{k-1}n_0 = n_0.$$

For $H = \overline{K_k}$, we have

$$N(G_0, \overline{K_k}) \geq \frac{|V_2|}{k-1} + \frac{|V_4|}{k} = \frac{\ell}{k-1}n_0 + \frac{k-\ell-\ell/k}{k}n_0 > n_0.$$

For $H = K_\ell + \overline{K_{k-\ell}}$, we have

$$\begin{aligned}
&N(G_0, K_\ell + \overline{K_{k-\ell}}) \\
&\quad \geq \min\left\{\frac{|V_1|}{\ell}, \frac{|V_2|}{k-\ell}\right\} + \min\left\{\frac{|V_3|}{\ell}, \frac{|V_4|}{k-\ell}\right\} \\
&\quad = \min\left\{\frac{|V_1|}{\ell} + \frac{|V_3|}{\ell}, \frac{|V_1|}{\ell} + \frac{|V_4|}{k-\ell}, \frac{|V_2|}{k-\ell} + \frac{|V_3|}{\ell}, \frac{|V_2|}{k-\ell} + \frac{|V_4|}{k-\ell}\right\} \\
&\quad \geq \min\left\{\frac{k-1}{\ell}, \frac{\ell-1}{\ell} + \frac{k-\ell-1}{k-\ell}, \frac{\ell}{k-\ell} + \frac{k-\ell}{\ell}, \frac{k-1}{k-\ell}\right\} n_0 \geq n_0.
\end{aligned}$$

Finally, for $H = \overline{K_\ell} \cup K_{k-\ell}$, we have

$$N(G_0, \overline{K_\ell} \cup K_{k-\ell}) \geq \tfrac{|V_2|}{\ell} = n_0.$$

Hence, we have $N(G_0, \mathcal{B}_{k,\ell}) \geq n_0$. $\square$

We also use the following basic facts on graph Ramsey theory. (For example, see [4].)

Fact 1. Let $k \geq 1$. There exists a positive integer N_1 depending on k such that for any $n \geq N_1$, every graph with n vertices contains K_k or $\overline{K_k}$ as a subgraph.

Fact 2. Let $k \geq 1$. There exists a positive integer N_2 depending on k such that for any $n \geq N_2$, every bipartite graph $G = G(A, B)$ with $|A| = |B| = n$, where A and B are bipartitions of G, contains two sets of vertices $A' \subset A$ and $B' \subset B$ with $|A'| = |B'| = k$ satisfying $A' \times B' \subset E(G)$ or $A' \times B' \subset E(\overline{G})$.

By $R_1(k)$ or $R_2(k)$ we denote, respectively, the minimum integers N_1 in Fact 1 and N_2 in Fact 2.

Lemma 8. *Let k, k_1, k_2, s, s_0, w be positive integers such that* $\max\{k_1, k_2\} \leq k$, *$R_2(k) - k_1 \leq s_0$ and $s - s_0 = k_1 w$. Let G be a graph. Let $S \subset V(G)$ with $|S| = s$. Then there exists a positive integer N_3 depending on k and s such that for any $t \geq N_3$ and for any $T \subset V(G)$ with $S \cap T = \emptyset$, $|T| = t$, we have partitions $S = S_0 \cup S_1 \cup \cdots \cup S_w$ and $T = T_0 \cup T_1 \cup \cdots \cup T_w$ satisfying that*

(1) *$|S_0| = s_0$,*

(2) *$|S_i| = k_1$, $|T_i| = k_2$ for $1 \leq i \leq w$, and*

(3) *$S_i \times T_i \subset E(G)$ or $S_i \times T_i \subset E(\overline{G})$ for $1 \leq i \leq w$.*

Proof. Let N_3 be a positive integer such that $N_3 \geq k(s+k)$. Then, for $1 \leq i \leq w$, we have

$$|S| - (i-1)k_1 \geq s - (w-1)k_1 \geq R_2(k),$$

and

$$|T| - (i-1)k_2 \geq t - (w-1)k_2 = t - \left(\frac{s-s_0}{k_1} - 1\right) k_2$$
$$\geq t - (s-s_0)k \geq (s_0+k)k \geq R_2(k).$$

Hence, by using Fact 2 w times, we can take subsets $S_i \subset S$ and $T_i \subset T$ one by one such that $|S_i| = k_1$, $|T_i| = k_2$, and $S_i \times T_i \subset E(G)$ or $S_i \times T_i \subset E(\overline{G})$ for $1 \leq i \leq w$. □

By $R_3(k,s)$ we denote the minimum integer N_3 in Lemma 8. In the proof of Theorem 5, we use the existence of $R_1(k)$, $R_2(k)$ and $R_3(k,s)$, but we will not need their exact values.

Let $n_0 = k^2(k-1)\ell(k-\ell)$, which is appeared in Lemma 7. Let us define positive integers ε, α_1, α_2, α_3, β_1, β_2, and γ satisfying the following conditions:

- $\varepsilon = kn_0$,
- α_1, α_2, α_3, β_1, β_2, γ are multiples of $\varepsilon k(2k-1-\ell/k)$,
- $\dfrac{\alpha_1}{\ell} = \dfrac{\alpha_2}{k-1} = \dfrac{\alpha_3}{k-\ell-\ell/k}$,
- $\alpha_i \geq R_2(\varepsilon)$ for $1 \leq i \leq 3$,
- $\dfrac{\ell}{k-\ell}\alpha_3 \geq R_2(\varepsilon k)$,
- $\alpha_2 - \dfrac{\alpha_3 \ell}{k-\ell}$ is a multiple of $\varepsilon\ell$,
- $\dfrac{\beta_1}{\max\{k/2,\ell\}} = \dfrac{\beta_2}{\min\{k/2,k-\ell\}}$,
- $\dfrac{\ell}{k-\ell}\beta_2 \geq R_2(\alpha_2 k)$,
- $\beta_2 \geq R_2(\max\{\alpha_2, R_3(\varepsilon k, \alpha_2)\})$,
- $\beta_1 - \dfrac{\beta_2 \ell}{k-\ell}$ is a multiple of $\alpha_2\ell$,
- $\gamma \geq R_2(\beta_1)$,
- $\gamma \geq R_3(\alpha_2 k, \beta_1)$.

Finally, we define a positive integer c as $c = R_1(\gamma) + \gamma$. Next, we define a family of subsets of the vertices.

A subset $S \subset V(G)$ is called of type A_+ if there exists a partition $S = S_1 \cup S_2 \cup S_3$ such that $|S_i| = \alpha_i$ for $1 \leq i \leq 3$ and $\binom{S_1}{2} \cup (S_1 \times S_2) \cup \binom{S_3}{2} \subset E(G)$, $\binom{S_2}{2} \cup (S_2 \times S_3) \subset E(\overline{G})$.

A subset $S \subset V(G)$ is called of type B_+ if there exists a partition $S = S_1 \cup S_2$ such that $|S_i| = \beta_i$ for $1 \le i \le 2$ and $\binom{S_1}{2} \cup (S_1 \times S_2) \subset E(G)$, $\binom{S_2}{2} \subset E(\overline{G})$.

A subset $S \subset V(G)$ is called of type C_+ if $G[S] \cong K_\gamma$.

Furthermore, a subset $S \subset V(G)$ is called of type A_-, B_-, C_-, if in the complement $\overline{G}$, S is of type A_+, B_+, C_+, respectively.

Let us consider a vertex partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_m$ such that

(P1) for $1 \le i \le m$, V_i is one of the six types A_+, A_-, B_+, B_-, C_+, C_-,

(P2) V_0 contains no subset S of these six types,

(P3) $n(A_+) + n(A_-)$ is maximum with respect to (P1) and (P2),

(P4) $n(B_+) + n(B_-)$ is maximum with respect to (P1), (P2) and (P3),

where for a type X, $n(X)$ denotes the number of indices i with $1 \le i \le m$ such that V_i is of type X.

We call a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_m$ satisfying the above properties from (P1) to (P4) a *standard partition*.

Firstly, we remark that if there exists a partition $V(G) = V_0' \cup V_1 \cup \cdots \cup V_s$ such that V_i is one of the six types for $1 \le i \le s$, then we can extend the partition to $V(G) = V_0 \cup V_1 \cup \cdots \cup V_s \cup V_{s+1} \cup \cdots \cup V_m$ satisfying (P1) and (P2), by taking suitable subsets greedily from V_0'. In particular, starting from $s = 0$, any graph admits at least one standard partition.

We also remark that for a standard partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_m$, we have $|V_0| < c$. Indeed, if $c \le |V_0|$, we have $R_1(\gamma) < c \le |V_0|$. Hence, by Fact 1, we have K_γ or $\overline{K_\gamma}$ in $G[V_0]$, a contradiction to (P2).

Let $V(G) = V_0 \cup V_1 \cup \cdots \cup V_m$ be a standard partition. First, we show Claims 1, 2 and 3, which reduce the number of possible combinations of the types of subsets in the partition.

Claim 1. $n(C_+) = 0$ *or* $n(C_-) = 0$.

Proof. Suppose to a contradiction that $n(C_+) > 0$ and $n(C_-) > 0$. Without loss of generality, we may assume V_1 is of type C_+ and V_2 is of type C_-. Since $|V_1| = |V_2| = \gamma \ge R_2(\beta_1)$, by Fact 2, for $1 \le i \le 2$, we have $V_i' \subset V_i$ with $|V_i'| = \beta_1$ such that $V_1' \times V_2' \subset E(G)$ or $V_1' \times V_2' \subset E(\overline{G})$. Therefore, we have a subset $S \subset V_1' \cup V_2'$ of type B_+ or B_-, a contradiction to (P4). □

Claim 2. $n(B_+) = 0$ *or* $n(B_-) = 0$.

Proof. Suppose to a contradiction that $n(B_+) > 0$ and $n(B_-) > 0$. Without loss of generality, we may assume V_1 is of type B_+ and V_2 is of type B_-. For $1 \le i \le 2$, let $V_i = V_{i1} \cup V_{i2}$ such that $\binom{V_{11}}{2} \cup (V_{11} \times V_{12}) \subset E(G)$, $\binom{V_{12}}{2} \subset E(\overline{G})$, $\binom{V_{22}}{2} \subset E(G)$, $\binom{V_{21}}{2} \cup (V_{21} \times V_{22}) \subset E(\overline{G})$. By the definition of β_1 and β_2, we have $|V_{11}| = |V_{21}| = \beta_1 \ge \beta_2$ and $\beta_2 \ge R_2(\alpha_2)$. Hence, by Fact 2, for $1 \le i \le 2$, we have $V'_{i1} \subset V_{i1}$ with $|V'_{i1}| = \alpha_2$ such that $V'_{11} \times V'_{21} \subset E(G)$ or $V'_{11} \times V'_{21} \subset E(\overline{G})$. If $V'_{11} \times V'_{21} \subset E(G)$, then we have a subset $S \subset V'_{11} \cup V'_{21} \cup V_{22}$ of type A_+, and if $V'_{11} \times V'_{21} \subset E(\overline{G})$, then we have a subset $S \subset V'_{21} \cup V'_{11} \cup V_{12}$ of type A_-, a contradiction to (P3). □

Claim 3. *$n(A_+) = 0$ or $n(A_-) = 0$.*

Proof. Suppose to a contradiction that $n(A_+) > 0$ and $n(A_-) > 0$. Without loss of generality, we may assume V_1 is of type A_+ and V_2 is of type A_-. For $1 \le i \le 2$, let $V_i = V_{i1} \cup V_{i2} \cup V_{i3}$ such that $\binom{V_{11}}{2} \cup (V_{11} \times V_{12}) \cup \binom{V_{13}}{2} \subset E(G)$, $\binom{V_{12}}{2} \cup (V_{12} \times V_{13}) \subset E(\overline{G})$, $\binom{V_{22}}{2} \cup (V_{22} \times V_{23}) \subset E(G)$, $\binom{V_{21}}{2} \cup (V_{21} \times V_{22}) \cup \binom{V_{23}}{2} \subset E(\overline{G})$. Since $|V_{11}| = |V_{21}| = \alpha_1 \ge R_2(\varepsilon)$, by Fact 2, for $1 \le i \le 2$, we have $V'_{i1} \subset V_{i1}$ with $|V'_{i1}| = \varepsilon$ such that $V'_{11} \times V'_{21} \subset E(G)$ or $V'_{11} \times V'_{21} \subset E(\overline{G})$. If $V'_{11} \times V'_{21} \subset E(G)$, then we have a subset $S \subset V'_{11} \cup V'_{21} \cup V_{22} \cup V_{23}$ such that $G[S]$ is $\mathcal{B}_{k,\ell}$-good, and if $V'_{11} \times V'_{21} \subset E(\overline{G})$, then we have a subset $S \subset V'_{21} \cup V'_{11} \cup V_{12} \cup V_{13}$ such that $G[S]$ is $\mathcal{B}_{k,\ell}$-good, a contradiction. □

Next, we prepare Claims 4 and 5, which count the number of disjoint copies of induced subgraphs isomorphic to $H \in \mathcal{B}_{k,\ell}$ in a subset of type A_+ and type B_+.

Claim 4. *Let $S \subset V(G)$ be of type A_+. Let $a = |S|/(2k - 1 - \ell/k)$. Then we have*

(1) $N(G[S], K_k) \ge \left(1 + \dfrac{\ell}{k^2(k-1)}\right) a,$

(2) $N(G[S], \overline{K_k}) \ge a,$

(3) $N(G[S], K_\ell + \overline{K_{k-\ell}}) \ge a,$

(4) $N(G[S], \overline{K_\ell} \cup K_{k-\ell}) \ge \left(1 - \dfrac{\ell}{k(k-\ell)}\right) a.$

Proof. Indeed, let $S = S_1 \cup S_2 \cup S_3$ such that $|S_i| = \alpha_i$ for $1 \le i \le 3$ and $\binom{S_1}{2} \cup (S_1 \times S_2) \cup \binom{S_3}{2} \subset E(G)$, $\binom{S_2}{2} \cup (S_2 \times S_3) \subset E(\overline{G})$. By the definition of α_1, α_2 and α_3, we have $|S_1| = \ell a$, $|S_2| = (k-1)a$ and $|S_3| = (k - l - \ell/k)a$. Hence, we have

$$N(G[S], K_k) \ge \tfrac{|S_1|}{k-1} + \tfrac{|S_3|}{k} = \tfrac{\ell}{k-1}a + \tfrac{k-\ell-\ell/k}{k}a = \left(1 + \tfrac{\ell}{k^2(k-1)}\right) a,$$

$$N(G[S], \overline{K_k}) \ge \tfrac{|S_2|}{k-1} = a,$$

$$N(G[S], K_\ell + \overline{K_{k-\ell}}) \ge \tfrac{|S_1|}{\ell} = a,$$

and

$$N(G[S], \overline{K_\ell} \cup K_{k-\ell}) \ge \tfrac{|S_3|}{k-\ell} = \tfrac{k-\ell-\ell/k}{k-\ell}a = \left(1 - \tfrac{\ell}{k(k-\ell)}\right) a,$$

as required. □

Claim 5. *Let $S \subset V(G)$ be of type B_+. Let $S = S_1 \cup S_2$ such that $|S_i| = \beta_i$ for $i = 1, 2$ and $\binom{S_1}{2} \cup (S_1 \times S_2) \subset E(G)$, $\binom{S_2}{2} \subset E(\overline{G})$. Let $a = |S|/(2k - 1 - \ell/k)$. Then we have*

(1) $N(G[S], K_k) \geq a$,

(2) $N(G[S], K_\ell + \overline{K_{k-\ell}}) \geq \dfrac{|S_1|}{k-2}$.

Proof. By the definition of β_1 and β_2, if $\ell \geq k/2$ then $|S_1| = \ell|S|/k$ and $|S_2| = (k-\ell)|S|/k$, and if $\ell < k/2$ then $|S_1| = |S_2| = |S|/2$. Hence, in any case, we have $|S_1| \geq |S|/2$. Then we have

$$N(G[S], K_k) = \frac{|S_1|}{k-1} \geq \frac{|S|}{2(k-1)} \geq a.$$

Furthermore, for $\ell \geq k/2$, we have

$$N(G[S], K_\ell + \overline{K_{k-\ell}}) = \frac{|S_1|}{\ell},$$

and for $\ell < k/2$, we have

$$N(G[S], K_\ell + \overline{K_{k-\ell}}) = \frac{|S_2|}{k-\ell} = \frac{|S_1|}{k-\ell}.$$

Hence, in any case, we have $N(G[S], K_\ell + \overline{K_{k-\ell}}) \geq |S_1|/(k-2)$. □

By Claim 3, without loss of generality, we may assume $n(A_-) = 0$. Suppose that $m = p + q + r$ and $V_1, \ldots, V_p$ are of type A_+, $V_{p+1}, \ldots, V_{p+q}$ are of type B_+ or B_-, $V_{p+q+1}, \ldots, V_{p+q+r}$ are of type C_+ or C_-.

For $1 \leq i \leq p$, let $V_i = V_{i1} \cup V_{i2} \cup V_{i3}$ such that $|V_{ij}| = \alpha_j$ for $1 \leq j \leq 3$ and $\binom{V_{i1}}{2} \cup (V_{i1} \times V_{i2}) \cup \binom{V_{i3}}{2} \subset E(G)$, $\binom{V_{i2}}{2} \cup (V_{i2} \times V_{i3}) \subset E(\overline{G})$.

Case 1. $n(B_-) = 0$. For $1 \leq i \leq q$, let $V_{p+i} = V_{p+i,1} \cup V_{p+i,2}$ such that $|V_{p+i,j}| = \beta_j$, for $j = 1, 2$ and $\binom{V_{p+i,1}}{2} \cup (V_{p+i,1} \times V_{p+i,2}) \subset E(G)$, $\binom{V_{p+i,2}}{2} \subset E(\overline{G})$.

Case 1.1. $n(C_-) = 0$. In this case, V_i is one of types A_+, B_+, C_+ for $1 \leq i \leq m$. Hence, by Claims 4 and 5, we have

$$N(G[V_i], K_k) \geq |V_i|/(2k - 1 - \ell/k).$$

Therefore, since $|V_0| < c$, we have

$$N(G, K_k) \geq |V(G) \setminus V_0|/(2k - 1 - \ell/k) \geq n,$$

a contradiction.

Case 1.2. $n(C_+) = 0$. In this case, V_i is one of types A_+, B_+, C_- for $1 \le i \le m$. Let us define a partition $V(G) \setminus V_0 = S \cup T \cup U$ such that

$$S = \bigcup_{1\le i\le q} V_{p+i,1},$$
$$T = \bigcup_{1\le i\le q} V_{p+i,2} \cup \bigcup_{1\le i\le r} V_{p+q+i},$$
$$U = \bigcup_{1\le i\le p} V_i,$$

$s = |S|$, $t = |T|$, $u = |U|$. Let $n_1 = u/(2k-1-\ell/k)$, $n_2 = (s+t)/(2k-1-\ell/k)$. Since $|V_0| < c$, we have $n_1 + n_2 = |V(G) \setminus V_0|/(2k-1-\ell/k) \ge n$. Furthermore, by Claim 4, we have $N(G[U], K_\ell + \overline{K_{k-\ell}}) \ge n_1$ and $N(G[U], \overline{K_k}) \ge n_1$. On the other hand, since $s + t = (2k-1-\ell/k)n_2$, we have $s > (k-2)n_2$ or $t > kn_2$.

If $s > (k-2)n_2$, by Claim 5, we have

$$N(G[S \cup T], K_\ell + \overline{K_{k-\ell}}) \ge \frac{s}{k-2} > n_2.$$

If $t > kn_2$, we have

$$N(G[T], \overline{K_k}) \ge \frac{t}{k} > n_2.$$

In any case, we have

$$N(G, \{K_\ell + \overline{K_{k-\ell}}, \overline{K_k}\}) > n_1 + n_2 \ge n,$$

a contradiction.

Case 2. $n(B_+) = 0$. For $1 \le i \le q$, let $V_{p+i} = V_{p+i,1} \cup V_{p+i,2}$ such that $|V_{p+i,j}| = \beta_j$, for $j = 1, 2$ and $\binom{V_{p+i,2}}{2} \subset E(G)$, $\binom{V_{p+i,1}}{2} \cup (V_{p+i,1} \times V_{p+i,2}) \subset E(\overline{G})$.

Case 2.1. $n(C_+) = 0$. In this case, V_i is one of types A_+, B_-, C_- for $1 \le i \le m$. Hence, by Claim 4 and Claim 5, we have

$$N(G[V_i], \overline{K_k}) \ge |V_i|/(2k-1-\ell/k).$$

Therefore, since $|V_0| < c$, we have

$$N(G, \overline{K_k}) \ge |V(G) \setminus V_0|/(2k-1-\ell/k) \ge n,$$

a contradiction.

Case 2.2. $n(C_-) = 0$. In this case, V_i is one of types A_+, B_-, C_+ for $1 \le i \le m$. We may assume $q \ge 1$ and $r \ge 1$. Indeed, if $q = 0$ or $r = 0$, as in Case 1 or in Case 2.1, we have a contradiction.

In order to show the assertion of the theorem, we will modify the original partition. Let us prepare two claims. Let $X = \bigcup_{1\le i\le r} V_{p+q+i}$.

Claim 6. *Suppose that $\ell < k/2$. Let V_i be of type B_- such that $V_i = V_{i,1} \cup V_{i,2}$, $|V_{i,j}| = \beta_j$ with $j = 1,2$, $\binom{V_{i,2}}{2} \subset E(G)$, $\binom{V_{i,1}}{2} \cup (V_{i,1} \times V_{i,2}) \subset E(\overline{G})$. Let $s_0 = \beta_2 \ell/(k-\ell)$. Let $T \subset X \cup V_0$ such that $G[T] \cong K_\gamma$. Then we have partitions $V_{i,1} = S_0 \cup S_1 \cup \cdots \cup S_w$, $T = T_0 \cup T_1 \cup \cdots \cup T_w$ such that*

(1) $|S_0| = s_0$,

(2) $|S_j| = \alpha_2 \ell$, $|T_j| = \alpha_2(k-\ell)$ *for* $1 \le j \le w$,

(3) $S_j \times T_j \subset E(\overline{G})$ *for* $1 \le j \le w$.

Proof. Since $\ell < k/2$, we have $\beta_1 = \beta_2$. By the definition of β_2 and γ, we have $s_0 = \beta_2\ell/(k-\ell) \ge R_2(\alpha_2 k)$ and $\gamma \ge R_3(\alpha_2 k, \beta_1)$. Hence, by Lemma 8, we have partitions $V_{i,1} = S_0 \cup S_1 \cup \cdots \cup S_w$, $T = T_0 \cup T_1 \cup \cdots \cup T_w$ such that (1) $|S_0| = s_0$, (2) $|S_j| = \alpha_2\ell$, $|T_j| = \alpha_2(k-\ell)$ for $1 \le j \le w$, and (3')$S_j \times T_j \subset E(G)$ or $S_j \times T_j \subset E(\overline{G})$ for $1 \le j \le w$. If $S_j \times T_j \subset E(G)$ for some j, we have a set of type A_+ in $T_j \cup S_j \cup V_{i,2}$, which contradicts the property (P3) of the partition. Thus we have $S_j \times T_j \subset E(\overline{G})$ for $1 \le j \le w$, as claimed. □

Claim 7. *Let V_i be of type A_+ such that $V_i = V_{i,1} \cup V_{i,2} \cup V_{i,3}$, $|V_{i,j}| = \alpha_j$ with $1 \le j \le 3$, $\binom{V_{i,1}}{2} \cup (V_{i,1} \times V_{i,2}) \cup \binom{V_{i,3}}{2} \subset E(G)$, $\binom{V_{i,2}}{2} \cup (V_{i,2} \times V_{i,3}) \subset E(\overline{G})$. Let $T \subset X \cup V_0$ such that $G[T] \cong K_\gamma$. Let $s_0 = \alpha_3\ell/(k-\ell)$. Then we have partitions $V_{i,2} = S_0 \cup S_1 \cup \cdots \cup S_w$, $T = T_0 \cup T_1 \cup \cdots \cup T_w$ such that*

(1) $|S_0| = s_0$,

(2) $|S_j| = \varepsilon\ell$, $|T_j| = \varepsilon(k-\ell)$ for $1 \le j \le w$,

(3) $S_j \times T_j \subset E(\overline{G})$ for $1 \le j \le w$.

Proof. Let $\lambda = \max\{\alpha_2, R_3(\varepsilon k, \alpha_2)\}$. Let $V_{p+1} = V_{p+1,1} \cup V_{p+1,2}$ such that $|V_{p+1,j}| = \beta_j$, for $j = 1,2$ and $\binom{V_{p+1,2}}{2} \subset E(G)$, $\binom{V_{p+1,1}}{2} \cup (V_{p+1,1} \times V_{p+1,2}) \subset E(\overline{G})$. Since $|V_{p+1,1}| = \beta_1 \ge R_2(\lambda)$ and $|T| = \gamma \ge R_2(\lambda)$, by Fact 2, we have $V'_{p+1,1} \subset V_{p+1,1}$, $T' \subset T$ with $|V'_{p+1,1}| = |T'| = \lambda$ such that $T' \times V'_{p+1,1} \subset E(G)$ or $T' \times V'_{p+1,1} \subset E(\overline{G})$.

If $T' \times V'_{p+1,1} \subset E(G)$, we have a set of type A_+ in $T' \cup V'_{p+1,1} \cup V_{p+1,2}$, which contradicts the property (P3) of the partition. Hence, we have $T' \times V'_{p+1,1} \subset E(\overline{G})$.

Since $s_0 = \alpha_3\ell/(k-\ell) \ge R_2(\varepsilon k)$ and $|T'| = \lambda \ge R_3(\varepsilon k, \alpha_2)$, by Lemma 8, we have partitions $V_{i,2} = S_0 \cup S_1 \cup \cdots \cup S_w$, $T' = T_0 \cup T_1 \cup \cdots \cup T_w$ such that (1) $|S_0| = s_0$, (2) $|S_j| = \varepsilon\ell$, $|T_j| = \varepsilon(k-\ell)$ for $1 \le j \le w$, and (3') $S_j \times T_j \subset E(G)$ or $S_j \times T_j \subset E(\overline{G})$ for $1 \le j \le w$. If $S_j \times T_j \subset E(G)$ for some j, we have a $\mathcal{B}_{k,\ell}$-good subgraph in $G[V'_{p+1,1} \cup T_j \cup S_j \cup V_{i,3}]$, a contradiction. Hence, we have $S_j \times T_j \subset E(\overline{G})$ for $1 \le j \le w$, as claimed. □

Under the situation of Claim 6, we have $N(G[V_i \cup T], \overline{K_\ell} \cup K_{k-\ell}) \ge |V_{i,1}|/\ell$, and under the situation of Claim 7, we have $N(G[V_i \cup T], \overline{K_\ell} \cup K_{k-\ell}) \ge |V_{i,2}|/\ell$.

By using Claims 6 and 7, let us modify the partition $V = V_0 \cup V_1 \cup \cdots \cup V_m$ according to the following algorithm.

Algorithm 1.
Step 0. $X_0 = (X \setminus V_m) \cup V_0$. If $\ell < k/2$, then put $y = p+q$, otherwise put $y = p$. Set 1 to i.
Step 1. If $i > y$, then set 0 to x and stop.
Step 2. If there exists $W \subset X_0$ such that $G[W] \cong K_\gamma$, then take W, otherwise set 1 to x, set $i-1$ to y' and stop.
Step 3. For V_i and W, take $T_1, T_2, \ldots, T_w \subset W$ satisfying the condition of Claim 6 or 7. Let $Z_i = T_1 \cup \cdots \cup T_w$. Set $X_0 = X_0 \setminus Z_i$. Add 1 to i. Go to Step 1.

Note that in the procedure of the algorithm, $G[X_0]$ contains no $\overline{K_\gamma}$ as a subgraph. Indeed, if there exists $W \subset X_0$ with $G[W] \cong \overline{K_\gamma}$, since $\gamma \geq R_2(\beta_1)$, by Fact 2, we have $V'_m \subset V_m$ and $W' \subset W$ such that $V'_m \cup W'$ is of type B_+ or B_-, a contradiction to (P4) of the original partition.

After executing the algorithm, we consider the two subcases according to the return status x.

Case 2.2.1. $x = 0$. Let us take a partition $X_0 = W_0 \cup W_1 \cup \cdots \cup W_\mu$ such that $G[W_i] \cong K_\gamma$ for $1 \leq i \leq \mu$ and $|W_0| < R_1(\gamma)$.
Let us define

$$S = \bigcup_{1 \leq i \leq p} V_{i,2} \cup \bigcup_{1 \leq i \leq q} V_{p+i,1},$$
$$T = \bigcup_{1 \leq i \leq p} (V_{i,1} \cup V_{i,3}) \cup \bigcup_{1 \leq i \leq q} V_{p+i,2} \cup \bigcup_{1 \leq i \leq y} Z_i \cup \bigcup_{1 \leq i \leq \mu} W_i \cup V_m,$$

$s = |S|$, $t = |T|$. Then we have $s + t \geq (2k - 1 - \ell/k)n$.

On the other hand, we have $n > N(G[S \cup T], \overline{K_\ell} \cup K_{k-\ell}) \geq s/\ell$. Hence, we have $\ell n > s$. Furthermore, we have $n > N(G[T], K_k) \geq t/k$. Hence, we have $kn > t$. Therefore, we have $(k+\ell)n > s + t \geq (2k - 1 - \ell/k)n$, a contradiction.

Case 2.2.2. $x = 1$. Let us define

$$S = \bigcup_{1 \leq i \leq q} V_{p+i,1},$$
$$T = \bigcup_{1 \leq i \leq q} V_{p+i,2} \cup \bigcup_{1 \leq i \leq y'} Z_i,$$
$$U = \bigcup_{1 \leq i \leq p} V_i,$$

$s = |S|$, $t = |T|$, $u = |U|$. Let $n_1 = u/(2k - 1 - \ell/k)$, $n_2 = (s+t)/(2k - 1 - \ell/k)$. Then we have $n_1 + n_2 \geq n$, since $|V(G) \setminus (S \cup T \cup U)| = |V_m \cup X_0| < \gamma + R_1(\gamma) = c$.

By a similar argument to that in Case 1.2, if $s \geq (k-1)n_2$, we have $N(G[S \cup T], \overline{K_k}) = s/(k-1) \geq n_2$, a contradiction. Hence, we have $s < (k-1)n_2$. Therefore, we have

$$\left(k - \frac{\ell}{k}\right) n_2 < t. \tag{1}$$

In the same way, we have $N(G[T], K_k) + N(G[U], K_k) < n$. Hence, by Claim 4, we have

$$\frac{t}{k} + \left(1 + \frac{\ell}{k^2(k-1)}\right) n_1 < n. \tag{2}$$

By (1) and (2), we have

$$n < kn_2. \tag{3}$$

On the other hand, by using Claim 4, we have $n > N(G, \overline{K_\ell} \cup K_{k-\ell}) \geq t/(k-\ell) + (1 - \ell/k(k-\ell))n_1$. Hence, we have

$$kt - (k^2 - k\ell - \ell)n_2 < \ell n. \tag{4}$$

By (3) and (4), we have $t < (k - \ell/k)n_2$, a contradiction to (1). This completes the proof. ■

4. Concluding Remarks

Let $\mathcal{G}_k$ be the family of all graphs with k vertices, as defined in Section 1. For $k \geq 4$, $f(n, \mathcal{G}_k)$ is not known well. For $k = 4$, let $G = K_{2n-1} \cup (K_{n-1} + \overline{K_{3n-1}})$. Then we have $N(G, \mathcal{G}_4) < n$. It follows that $f(n, \mathcal{G}_4) \geq 6n - 2$. On the other hand, from Theorem 5 by substituting 4 for k and 2 for ℓ, we have $f(n, \mathcal{G}_4) \leq f(n, \mathcal{B}_{4,2}) = (13/2)n + O(1)$. It is conjectured that $f(n, \mathcal{G}_4) = 6n + O(1)$ ([5]).

References

[1] S.A. Burr, *On the Ramsey numbers $r(G, nH)$ and $r(nG, nH)$ when n is large*, Discrete Math. **65** (1987) 215–229.
doi:10.1016/0012-365X(87)90053-7

[2] S.A. Burr, *On Ramsey numbers for large disjoint unions of graphs*, Discrete Math. **70** (1988) 277–293.
doi:10.1016/0012-365X(88)90004-0

[3] S.A. Burr, P. Erdős and J.H. Spencer, *Ramsey theorems for multiple copies of graphs*, Trans. Amer. Math. Soc. **209** (1975) 87–99.
doi:10.1090/S0002-9947-1975-0409255-0

[4] R.L. Graham, B.L. Rothschild and J.H. Spencer, Ramsey Theory, 2nd Edition (Wiley, New York, 1990).

[5] T. Nakamigawa, *Vertex disjoint equivalent subgraphs of order* 3, J. Graph Theory **56** (2007) 159–166.
doi:10.1002/jgt.20263

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