# A RAMSEY-TYPE THEOREM FOR MULTIPLE DISJOINT COPIES OF INDUCED SUBGRAPHS 

Tomoki Nakamigawa ${ }^{1}$<br>Department of Information Science<br>Shonan Institute of Technology<br>1-1-25 Tsujido-Nishikaigan, Fujisawa<br>Kanagawa 251-8511, Japan<br>e-mail: nakami@info.shonan-it.ac.jp


#### Abstract

Let $k$ and $\ell$ be positive integers with $\ell \leq k-2$. It is proved that there exists a positive integer $c$ depending on $k$ and $\ell$ such that every graph of order $(2 k-1-\ell / k) n+c$ contains $n$ vertex disjoint induced subgraphs, where these subgraphs are isomorphic to each other and they are isomorphic to one of four graphs: (1) a clique of order $k,(2)$ an independent set of order $k$, (3) the join of a clique of order $\ell$ and an independent set of order $k-\ell$, or (4) the union of an independent set of order $\ell$ and a clique of order $k-\ell$.


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## 1. Introduction

Let $G$ and $H$ denote finite undirected graphs without multiple edges and loops. For a graph $G$, let $V(G)$ and $E(G)$ denote the set of vertices of $G$ and the set of edges of $G$. For a subset $S \subset V(G)$, the subgraph of $G$ induced by $S$ is denoted by $G[S]$.

For two graphs $G$ and $H$, let us define $N(G, H)$ as the maximum integer $n$ such that there exists a vertex partition $V(G)=V_{0} \cup V_{1} \cup \cdots \cup V_{n}$ satisfying $G\left[V_{i}\right] \cong$ $H$ for $1 \leq i \leq n$. For a family of graphs $\mathcal{H}$, let us define $N(G, \mathcal{H})$ as the maximum of $N(G, H)$ over $H \in \mathcal{H}$. Furthermore, for a positive integer $n$, we define an

[^0]integer valued function $f(n, \mathcal{H})$ as the minimum integer $s$ such that $N(G, \mathcal{H}) \geq n$ for every graph $G$ with $|V(G)| \geq s$. By the definition, $f\left(1,\left\{K_{k}, \overline{K_{\ell}}\right\}\right)$ is the classical Ramsey number of 2-edge colored graphs, where $\overline{K_{\ell}}$ is the complement of $K_{\ell}$.

We remark that if $\mathcal{H}$ does not contain $K_{k}$ or $\overline{K_{k}}$ for all $k \geq 1$, then $f(n, \mathcal{H})$ is not determined as a finite value, because we have $N\left(K_{s}, \mathcal{H}\right)=0$ or $N\left(\overline{K_{s}}, \mathcal{H}\right)=0$ for $s \geq 1$. Hence, in the following, we always assume that $\left\{K_{k}, \overline{K_{\ell}}\right\} \subset \mathcal{H}$ for some $k$ and $\ell$.

Our aim is to study $f(n, \mathcal{H})$ for some family of graphs $\mathcal{H}$ with $n$ sufficiently large. In order to explain related results, let us introduce a few more notations. For two graphs $G_{1}$ and $G_{2}$, the union $G_{1} \cup G_{2}$ is the graph such that $V\left(G_{1} \cup G_{2}\right)=$ $V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and $E\left(G_{1} \cup G_{2}\right)=E\left(G_{1}\right) \cup E\left(G_{2}\right)$. The join $G_{1}+G_{2}$ is the graph such that $V\left(G_{1}+G_{2}\right)=V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and $E\left(G_{1}+G_{2}\right)=\left(V\left(G_{1}\right) \times V\left(G_{2}\right)\right) \cup$ $E\left(G_{1}\right) \cup E\left(G_{2}\right)$.

Let $\mathcal{G}_{k}$ be the family of all graphs with $k$ vertices. It is not difficult to see that $f\left(n, \mathcal{G}_{2}\right)=3 n-1$ for $n \geq 1$. Indeed, the inequality $3 n-1 \leq f\left(n, \mathcal{G}_{2}\right)$ is followed by the fact $N\left(K_{2 n-1} \cup \overline{K_{n-1}}, \mathcal{G}_{2}\right) \leq n-1$. The following result is a classical one in the graph Ramsey theory.

Theorem 1 [3]. Let $n \geq 2$. Then $f\left(n,\left\{K_{3}, \overline{K_{3}}\right\}\right)=5 n$.
The above result is extended for complete graphs with any number of vertices.
Theorem 2 [1, 2]. Let $k, \ell \geq 2$. Then
$f\left(n,\left\{K_{k}, \overline{K_{\ell}}\right\}\right)=(k+l-1) n+f\left(1,\left\{K_{k-1}, \overline{K_{\ell-1}}\right\}\right)-2$ for $n$ sufficiently large.
Let $\mathcal{A}_{k}=\left\{K_{k}, \overline{K_{k}}, K_{1, k-1}, \overline{K_{1, k-1}}\right\}$ for $k \geq 3$. Recently, the author proved the following result.

Theorem 3 [5]. Let $k \geq 3$. Then $f\left(n, \mathcal{A}_{k}\right)=\left(2 k-1-\frac{1}{k}\right) n+O(1)$.
Since $\mathcal{G}_{3}=\mathcal{A}_{3}$, we have an immediate consequence of Theorem 3.
Corollary 4. $f\left(n, \mathcal{G}_{3}\right)=\frac{14}{3} n+O(1)$.
We will discuss shortly $f\left(n, \mathcal{G}_{4}\right)$ in Section 4 .

## 2. Main Results

For $1 \leq \ell \leq k-2$, let $\mathcal{B}_{k, \ell}=\left\{K_{k}, \overline{K_{k}}, K_{\ell}+\overline{K_{k-\ell}}, \overline{K_{\ell}} \cup K_{k-\ell}\right\}$. The main result of the paper is as follows.

Theorem 5. Let $k$ and $\ell$ be positive integers with $2 \leq \ell \leq k-2$. Then $f\left(n, \mathcal{B}_{k, \ell}\right)=$ $\left(2 k-1-\frac{\ell}{k}\right) n+O(1)$.

The proof of Theorem 5 will be given in Section 3. Since $\mathcal{B}_{k, 1}=\mathcal{A}_{k}$ for $k \geq 3$, by combining Theorem 3 and Theorem 5, we have $f\left(n, \mathcal{B}_{k, \ell}\right)=(2 k-1-\ell / k) n+O(1)$ for $1 \leq \ell \leq k-2$.

In this problem, $\mathcal{B}_{k, \ell}$ is in a special position.
Proposition 6. Let $k \geq 3$. Let $\mathcal{H}$ be a family of graphs having $k$ vertices such that $\mathcal{H} \cap \mathcal{B}_{k, \ell}=\left\{K_{k}, \overline{K_{k}}\right\}$ for $1 \leq \ell \leq k-2$. Then we have $f(n, \mathcal{H})=(2 k-1) n+O(1)$.

In particular, for a graph $H$ with $k$ vertices such that $H \notin \mathcal{B}_{k, \ell}$ for $1 \leq \ell \leq$ $k-2$, we have $f\left(n,\left\{K_{k}, \overline{K_{k}}, H, \bar{H}\right\}\right)=(2 k-1) n+O(1)$.

Proof. It suffices to prove the claim in the first half. For a lower bound, let $G=K_{(k-1) n-1}+\overline{K_{k n-1}}$. Then we have $N\left(G, K_{k}\right)=N\left(G, \overline{K_{k}}\right)=n-1$ and $N(G, H)=N(G, \bar{H})=0$ for $H \in \mathcal{H} \backslash\left\{K_{k}, \overline{K_{k}}\right\}$. Hence, we have $f(n, \mathcal{H})>$ $|V(G)|=(2 k-1) n-2$. For an upper bound, by Theorem 2, we have $f(n, \mathcal{H}) \leq$ $f\left(n,\left\{K_{k}, \overline{K_{k}}\right\}\right)=(2 k-1) n+f\left(1,\left\{K_{k-1}, \overline{K_{k-1}}\right\}\right)-2$ for $n$ sufficiently large.

## 3. Proof of Theorem 5

Proof. Lower bound. Let $G=K_{m}+\left(K_{(k-\ell) n-1} \cup \overline{K_{(k-1) n-1}}\right)$, where $m=$ $\lfloor(\ell-\ell / k) n\rfloor$.
Claim. $N\left(G, \mathcal{B}_{k, \ell}\right)<n$.
Proof. Let $V(G)=V_{1} \cup V_{2} \cup V_{3}$ such that $\left|V_{1}\right|=m,\left|V_{2}\right|=(k-\ell) n-1$, $\left|V_{3}\right|=(k-1) n-1$ and $E(G)=\binom{V_{1}}{2} \cup\binom{V_{2}}{2} \cup\left(V_{1} \times V_{2}\right) \cup\left(V_{1} \times V_{3}\right)$.

Firstly, we have $N\left(G, \overline{K_{n}}\right)<n$. Indeed, each $\overline{K_{k}}$ of $G$ contains at least $k-1$ vertices of $V_{3}$. Hence, we have $N\left(G, \overline{K_{k}}\right) \leq\left\lfloor\left|V_{3}\right| /(k-1)\right\rfloor=n-1$.

In the same manner, we have $N\left(G, \overline{K_{\ell}} \cup K_{k-\ell}\right)<n$. Indeed, each $\overline{K_{\ell}} \cup K_{k-\ell}$ of $G$ contains at least $k-\ell$ vertices of $V_{2}$. Hence, we have $N\left(G, \overline{K_{\ell}} \cup K_{k-\ell}\right) \leq$ $\left\lfloor\left|V_{2}\right| /(k-\ell)\right\rfloor=n-1$.
Next, we show that $N\left(G, K_{\ell}+\overline{K_{k-\ell}}\right)<n$. Indeed, each $K_{\ell}+\overline{K_{k-\ell}}$ of $G$ contains at least $\ell$ vertices of $V_{1}$. Hence, we have $N\left(G, K_{\ell}+\overline{K_{k-\ell}}\right) \leq\left\lfloor\left|V_{1}\right| / \ell\right\rfloor<n$.

Lastly, we show that $N\left(G, K_{k}\right)<n$. For $v \in V(G)$, let us assign a weight $w(v)$ such that $w(v)=1 /(k-1)$ for $v \in V_{1}, w(v)=1 / k$ for $v \in V_{2}$, and $w(v)=0$ for $v \in V_{3}$. Furthermore, for $S \subset V(G)$, let $w(S)=\sum_{v \in S} w(v)$. Then we have $w(S) \geq 1$ for any $S \subset V(G)$ such that $G[S] \cong K_{k}$. On the other hand, the total weight is calculated as

$$
\begin{aligned}
w(V(G)) & =\frac{\left|V_{1}\right|}{k-1}+\frac{\left|V_{2}\right|}{k} \\
& \leq \frac{1}{k-1}\left(\ell-\frac{\ell}{k}\right) n+\frac{1}{k}((k-\ell) n-1)=n-\frac{1}{k} .
\end{aligned}
$$

Hence, we have $N\left(G, K_{k}\right)<n$. Therefore, we have $N\left(G, \mathcal{B}_{k, \ell}\right)<n$.

By the claim, we have $f\left(n, \mathcal{B}_{k, \ell}\right)>|V(G)|>(2 k-1-\ell / k) n-3$.
Upper bound. Before we start the proof, let us show its outline. The main idea of the proof is a variant of a "bow tie argument", which is originated from the proof of Theorem $1([3]$, see also [4]). A bow tie is a graph with 5 vertices containing both $K_{3}$ and $\overline{K_{3}}$. Let us summarize how to prove $f\left(n,\left\{K_{3}, \overline{K_{3}}\right\}\right) \leq 5 n$ by a bow tie argument. Let $G$ be an underlying graph with $5 n$ vertices. What we want to show is that $N\left(G,\left\{K_{3}, \overline{K_{3}}\right\}\right) \geq n$. If $G$ contains no bow tie, it turns out that the structure of $G$ becomes very simple, and we can easily show that $N\left(G,\left\{K_{3}, \overline{K_{3}}\right\}\right) \geq n$. Otherwise, let $S$ be a bow tie of $G$. We partition $G$ into two graphs $G[S]$ and $G^{\prime}=G[V(G) \backslash S]$. Since $\left|V\left(G^{\prime}\right)\right|=5(n-1)$, by inductive hypothesis, we have $N\left(G^{\prime},\left\{K_{3}, \overline{K_{3}}\right\}\right) \geq n-1$. Then with an additional $K_{3}$ or $\overline{K_{3}}$ in $G[S]$, we have $N\left(G,\left\{K_{3}, \overline{K_{3}}\right\}\right) \geq n$, as required.

Now, we go back to the proof of Theorem 5 . We will show that for $n \geq 1$, there exists a positive constant $c=c(k, \ell)$ depending on $k$ and $\ell$, such that $f\left(n, \mathcal{B}_{k, \ell}\right) \leq(2 k-1-\ell / k) n+c$. We will define the value of $c$ just after Lemma 8. Suppose to a contradiction that $G$ is a counterexample with the smallest number of vertices. We assume $|V(G)| \geq(2 k-1-\ell / k) n+c$ and $N\left(G, \mathcal{B}_{k, \ell}\right)<n$.

Let us introduce a family of graphs, $\mathcal{B}_{k, \ell^{-}}$good graphs, which is considered as a variant of a bow tie. We call a graph $G_{0} \mathcal{B}_{k, \ell^{-}}$good if there exists a positive integer $n_{0}$ such that $(1)\left|V\left(G_{0}\right)\right|=(2 k-1-\ell / k) n_{0}$ and $(2) N\left(G_{0}, H\right) \geq n_{0}$ for all $H \in \mathcal{B}_{k, \ell}$. Then a crucial observation is that a smallest counterexample $G$
 $\mathcal{B}_{k, \ell^{-}} \operatorname{good}$ induced subgraph $G_{0}$ with $\left|V\left(G_{0}\right)\right|=(2 k-1-\ell / k) n_{0}$, let us partition $G$ into two graphs $G_{0}$ and $G_{1}=G\left[V(G) \backslash V\left(G_{0}\right)\right]$. Then we have $\left|V\left(G_{1}\right)\right| \geq$ $(2 k-1-\ell / k)\left(n-n_{0}\right)+c$. Furthermore, since $N\left(G_{0}, H\right) \geq n_{0}$ and $N(G, H)<n$ for all $H \in \mathcal{B}_{k, \ell}$, we have $N\left(G_{1}, \mathcal{B}_{k, \ell}\right)<n-n_{0}$. Hence, $G_{1}$ is also a counterexample, a contradiction to the minimality of $G$.

The following lemma is a key for the proof.
Lemma 7. Let $n_{0}=k^{2}(k-1) \ell(k-\ell)$. Let $G_{0}$ be a graph with $(2 k-1-\ell / k) n_{0}$ vertices. Suppose that there exists a vertex partition $V\left(G_{0}\right)=V_{1} \cup V_{2} \cup V_{3} \cup V_{4}$ such that $\left|V_{1}\right|=(\ell-1) n_{0},\left|V_{2}\right|=\ell n_{0},\left|V_{3}\right|=(k-\ell) n_{0},\left|V_{4}\right|=(k-\ell-\ell / k) n_{0}$, and $E\left(G_{0}\right) \supset\binom{V_{1}}{2} \cup\left(V_{1} \times V_{2}\right) \cup\binom{V_{3}}{2} \cup\left(V_{3} \times V_{4}\right), E\left(\overline{G_{0}}\right) \supset\binom{V_{2}}{2} \cup\left(V_{2} \times V_{3}\right) \cup\binom{V_{4}}{2}$. Then both $G_{0}$ and $\overline{G_{0}}$ are $\mathcal{B}_{k, \ell \text {-good. }}$

Proof. Since $\mathcal{B}_{k, \ell^{-} \text {goodness }}$ is symmetric for a graph and its complement, it suffices to show that $N\left(G_{0}, H\right) \geq n_{0}$ for all $H \in \mathcal{B}_{k, \ell}$.

For $H=K_{k}$, we have

$$
N\left(G_{0}, K_{k}\right) \geq \frac{\left|V_{1}\right|}{k-1}+\frac{\left|V_{3}\right|}{k-1}=\frac{\ell-1}{k-1} n_{0}+\frac{k-\ell}{k-1} n_{0}=n_{0}
$$

For $H=\overline{K_{k}}$, we have

$$
N\left(G_{0}, \overline{K_{k}}\right) \geq \frac{\left|V_{2}\right|}{k-1}+\frac{\left|V_{4}\right|}{k}=\frac{\ell}{k-1} n_{0}+\frac{k-\ell-\ell / k}{k} n_{0}>n_{0}
$$

For $H=K_{\ell}+\overline{K_{k-\ell}}$, we have

$$
\begin{aligned}
& N\left(G_{0}, K_{\ell}+\overline{K_{k-\ell}}\right) \\
& \quad \geq \min \left\{\frac{\left|V_{1}\right|}{\ell}, \frac{\left|V_{2}\right|}{k-\ell}\right\}+\min \left\{\frac{\left|V_{3}\right|}{\ell}, \frac{\left|V_{4}\right|}{k-\ell}\right\} \\
& \quad=\min \left\{\frac{\left|V_{1}\right|}{\ell}+\frac{\left|V_{3}\right|}{\ell}, \frac{\left|V_{1}\right|}{\ell}+\frac{\left|V_{4}\right|}{k-\ell}, \frac{\left|V_{2}\right|}{k-\ell}+\frac{\left|V_{3}\right|}{\ell}, \frac{\left|V_{2}\right|}{k-\ell}+\frac{\left|V_{4}\right|}{k-\ell}\right\} \\
& \quad \geq \min \left\{\frac{k-1}{\ell}, \frac{\ell-1}{\ell}+\frac{k-\ell-1}{k-\ell}, \frac{\ell}{k-\ell}+\frac{k-\ell}{\ell}, \frac{k-1}{k-\ell}\right\} n_{0} \geq n_{0}
\end{aligned}
$$

Finally, for $H=\overline{K_{\ell}} \cup K_{k-\ell}$, we have

$$
N\left(G_{0}, \overline{K_{\ell}} \cup K_{k-\ell}\right) \geq \frac{\left|V_{2}\right|}{\ell}=n_{0}
$$

Hence, we have $N\left(G_{0}, \mathcal{B}_{k, \ell}\right) \geq n_{0}$.
We also use the following basic facts on graph Ramsey theory. (For example, see [4].)

Fact 1. Let $k \geq 1$. There exists a positive integer $N_{1}$ depending on $k$ such that for any $n \geq N_{1}$, every graph with $n$ vertices contains $K_{k}$ or $\overline{K_{k}}$ as a subgraph.

Fact 2. Let $k \geq 1$. There exists a positive integer $N_{2}$ depending on $k$ such that for any $n \geq N_{2}$, every bipartite graph $G=G(A, B)$ with $|A|=|B|=n$, where $A$ and $B$ are bipartitions of $G$, contains two sets of vertices $A^{\prime} \subset A$ and $B^{\prime} \subset B$ with $\left|A^{\prime}\right|=\left|B^{\prime}\right|=k$ satisfying $A^{\prime} \times B^{\prime} \subset E(G)$ or $A^{\prime} \times B^{\prime} \subset E(\bar{G})$.

By $R_{1}(k)$ or $R_{2}(k)$ we denote, respectively, the minimum integers $N_{1}$ in Fact 1 and $N_{2}$ in Fact 2.

Lemma 8. Let $k, k_{1}, k_{2}, s, s_{0}, w$ be positive integers such that $\max \left\{k_{1}, k_{2}\right\} \leq k$, $R_{2}(k)-k_{1} \leq s_{0}$ and $s-s_{0}=k_{1} w$. Let $G$ be a graph. Let $S \subset V(G)$ with $|S|=s$. Then there exists a positive integer $N_{3}$ depending on $k$ and $s$ such that for any $t \geq N_{3}$ and for any $T \subset V(G)$ with $S \cap T=\emptyset,|T|=t$, we have partitions $S=S_{0} \cup S_{1} \cup \cdots \cup S_{w}$ and $T=T_{0} \cup T_{1} \cup \cdots \cup T_{w}$ satisfying that
(1) $\left|S_{0}\right|=s_{0}$,
(2) $\left|S_{i}\right|=k_{1},\left|T_{i}\right|=k_{2}$ for $1 \leq i \leq w$, and
(3) $S_{i} \times T_{i} \subset E(G)$ or $S_{i} \times T_{i} \subset E(\bar{G})$ for $1 \leq i \leq w$.

Proof. Let $N_{3}$ be a positive integer such that $N_{3} \geq k(s+k)$. Then, for $1 \leq i \leq w$, we have

$$
|S|-(i-1) k_{1} \geq s-(w-1) k_{1} \geq R_{2}(k)
$$

and

$$
\begin{aligned}
|T|-(i-1) k_{2} & \geq t-(w-1) k_{2}=t-\left(\frac{s-s_{0}}{k_{1}}-1\right) k_{2} \\
& \geq t-\left(s-s_{0}\right) k \geq\left(s_{0}+k\right) k \geq R_{2}(k)
\end{aligned}
$$

Hence, by using Fact $2 w$ times, we can take subsets $S_{i} \subset S$ and $T_{i} \subset T$ one by one such that $\left|S_{i}\right|=k_{1},\left|T_{i}\right|=k_{2}$, and $S_{i} \times T_{i} \subset E(G)$ or $S_{i} \times T_{i} \subset E(\bar{G})$ for $1 \leq i \leq w$.

By $R_{3}(k, s)$ we denote the minimum integer $N_{3}$ in Lemma 8. In the proof of Theorem 5, we use the existence of $R_{1}(k), R_{2}(k)$ and $R_{3}(k, s)$, but we will not need their exact values.

Let $n_{0}=k^{2}(k-1) \ell(k-\ell)$, which is appeared in Lemma 7. Let us define positive integers $\varepsilon, \alpha_{1}, \alpha_{2}, \alpha_{3}, \beta_{1}, \beta_{2}$, and $\gamma$ satisfying the following conditions:

- $\varepsilon=k n_{0}$,
- $\alpha_{1}, \alpha_{2}, \alpha_{3}, \beta_{1}, \beta_{2}, \gamma$ are multiples of $\varepsilon k(2 k-1-\ell / k)$,
- $\frac{\alpha_{1}}{\ell}=\frac{\alpha_{2}}{k-1}=\frac{\alpha_{3}}{k-\ell-\ell / k}$,
- $\alpha_{i} \geq R_{2}(\varepsilon)$ for $1 \leq i \leq 3$,
- $\frac{\ell}{k-\ell} \alpha_{3} \geq R_{2}(\varepsilon k)$,
- $\alpha_{2}-\frac{\alpha_{3} \ell}{k-\ell}$ is a multiple of $\varepsilon \ell$,
- $\frac{\beta_{1}}{\max \{k / 2, \ell\}}=\frac{\beta_{2}}{\min \{k / 2, k-\ell\}}$,
- $\frac{\ell}{k-\ell} \beta_{2} \geq R_{2}\left(\alpha_{2} k\right)$,
- $\beta_{2} \geq R_{2}\left(\max \left\{\alpha_{2}, R_{3}\left(\varepsilon k, \alpha_{2}\right)\right\}\right)$,
- $\beta_{1}-\frac{\beta_{2} \ell}{k-\ell}$ is a multiple of $\alpha_{2} \ell$,
- $\gamma \geq R_{2}\left(\beta_{1}\right)$,
- $\gamma \geq R_{3}\left(\alpha_{2} k, \beta_{1}\right)$.

Finally, we define a positive integer $c$ as $c=R_{1}(\gamma)+\gamma$. Next, we define a family of subsets of the vertices.

A subset $S \subset V(G)$ is called of type $A_{+}$if there exists a partition $S=$ $S_{1} \cup S_{2} \cup S_{3}$ such that $\left|S_{i}\right|=\alpha_{i}$ for $1 \leq i \leq 3$ and $\binom{S_{1}}{2} \cup\left(S_{1} \times S_{2}\right) \cup\binom{S_{3}}{2} \subset E(G)$, $\binom{S_{2}}{2} \cup\left(S_{2} \times S_{3}\right) \subset E(\bar{G})$.

A subset $S \subset V(G)$ is called of type $B_{+}$if there exists a partition $S=S_{1} \cup S_{2}$ such that $\left|S_{i}\right|=\beta_{i}$ for $1 \leq i \leq 2$ and $\binom{S_{1}}{2} \cup\left(S_{1} \times S_{2}\right) \subset E(G),\binom{S_{2}}{2} \subset E(\bar{G})$.

A subset $S \subset V(G)$ is called of type $C_{+}$if $G[S] \cong K_{\gamma}$.
Furthermore, a subset $S \subset V(G)$ is called of type $A_{-}, B_{-}, C_{-}$, if in the complement $\bar{G}, S$ is of type $A_{+}, B_{+}, C_{+}$, respectively.

Let us consider a vertex partition $V(G)=V_{0} \cup V_{1} \cup \cdots \cup V_{m}$ such that
(P1) for $1 \leq i \leq m, V_{i}$ is one of the six types $A_{+}, A_{-}, B_{+}, B_{-}, C_{+}, C_{-}$,
(P2) $V_{0}$ contains no subset $S$ of these six types,
(P3) $n\left(A_{+}\right)+n\left(A_{-}\right)$is maximum with respect to (P1) and (P2),
(P4) $n\left(B_{+}\right)+n\left(B_{-}\right)$is maximum with respect to (P1), (P2) and (P3),
where for a type $X, n(X)$ denotes the number of indices $i$ with $1 \leq i \leq m$ such that $V_{i}$ is of type $X$.

We call a partition $V(G)=V_{0} \cup V_{1} \cup \cdots \cup V_{m}$ satisfying the above properties from (P1) to (P4) a standard partition.

Firstly, we remark that if there exists a partition $V(G)=V_{0}^{\prime} \cup V_{1} \cup \cdots \cup V_{s}$ such that $V_{i}$ is one of the six types for $1 \leq i \leq s$, then we can extend the partition to $V(G)=V_{0} \cup V_{1} \cup \cdots \cup V_{s} \cup V_{s+1} \cup \cdots \cup V_{m}$ satisfying (P1) and (P2), by taking suitable subsets greedily from $V_{0}^{\prime}$. In particular, starting from $s=0$, any graph admits at least one standard partition.

We also remark that for a standard partition $V(G)=V_{0} \cup V_{1} \cup \cdots \cup V_{m}$, we have $\left|V_{0}\right|<c$. Indeed, if $c \leq\left|V_{0}\right|$, we have $R_{1}(\gamma)<c \leq\left|V_{0}\right|$. Hence, by Fact 1 , we have $K_{\gamma}$ or $\overline{K_{\gamma}}$ in $G\left[V_{0}\right]$, a contradiction to (P2).

Let $V(G)=V_{0} \cup V_{1} \cup \cdots \cup V_{m}$ be a standard partition. First, we show Claims 1,2 and 3 , which reduce the number of possible combinations of the types of subsets in the partition.

Claim 1. $n\left(C_{+}\right)=0$ or $n\left(C_{-}\right)=0$.
Proof. Suppose to a contradiction that $n\left(C_{+}\right)>0$ and $n\left(C_{-}\right)>0$. Without loss of generality, we may assume $V_{1}$ is of type $C_{+}$and $V_{2}$ is of type $C_{-}$. Since $\left|V_{1}\right|=\left|V_{2}\right|=\gamma \geq R_{2}\left(\beta_{1}\right)$, by Fact 2, for $1 \leq i \leq 2$, we have $V_{i}^{\prime} \subset V_{i}$ with $\left|V_{i}^{\prime}\right|=\beta_{1}$ such that $V_{1}^{\prime} \times V_{2}^{\prime} \subset E(G)$ or $V_{1}^{\prime} \times V_{2}^{\prime} \subset E(\bar{G})$. Therefore, we have a subset $S \subset V_{1}^{\prime} \cup V_{2}^{\prime}$ of type $B_{+}$or $B_{-}$, a contradiction to (P4).

Claim 2. $n\left(B_{+}\right)=0$ or $n\left(B_{-}\right)=0$.
Proof. Suppose to a contradiction that $n\left(B_{+}\right)>0$ and $n\left(B_{-}\right)>0$. Without loss of generality, we may assume $V_{1}$ is of type $B_{+}$and $V_{2}$ is of type $B_{-}$. For $1 \leq i \leq 2$, let $V_{i}=V_{i 1} \cup V_{i 2}$ such that $\binom{V_{11}}{2} \cup\left(V_{11} \times V_{12}\right) \subset E(G),\binom{V_{12}}{2} \subset E(\bar{G})$, $\binom{V_{22}}{2} \subset E(G),\binom{V_{21}}{2} \cup\left(V_{21} \times V_{22}\right) \subset E(\bar{G})$. By the definition of $\beta_{1}$ and $\beta_{2}$, we have $\left|V_{11}\right|=\left|V_{21}\right|=\beta_{1} \geq \beta_{2}$ and $\beta_{2} \geq R_{2}\left(\alpha_{2}\right)$. Hence, by Fact 2 , for $1 \leq i \leq 2$, we
have $V_{i 1}^{\prime} \subset V_{i 1}$ with $\left|V_{i 1}^{\prime}\right|=\alpha_{2}$ such that $V_{11}^{\prime} \times V_{21}^{\prime} \subset E(G)$ or $V_{11}^{\prime} \times V_{21}^{\prime} \subset E(\bar{G})$. If $V_{11}^{\prime} \times V_{21}^{\prime} \subset E(G)$, then we have a subset $S \subset V_{11}^{\prime} \cup V_{21}^{\prime} \cup V_{22}$ of type $A_{+}$, and if $V_{11}^{\prime} \times V_{21}^{\prime} \subset E(\bar{G})$, then we have a subset $S \subset V_{21}^{\prime} \cup V_{11}^{\prime} \cup V_{12}$ of type $A_{-}$, a contradiction to (P3).

Claim 3. $n\left(A_{+}\right)=0$ or $n\left(A_{-}\right)=0$.
Proof. Suppose to a contradiction that $n\left(A_{+}\right)>0$ and $n\left(A_{-}\right)>0$. Without loss of generality, we may assume $V_{1}$ is of type $A_{+}$and $V_{2}$ is of type $A_{-}$. For $1 \leq i \leq 2$, let $V_{i}=V_{i 1} \cup V_{i 2} \cup V_{i 3}$ such that $\binom{V_{11}}{2} \cup\left(V_{11} \times V_{12}\right) \cup\binom{V_{13}}{2} \subset E(G)$, $\binom{V_{12}}{2} \cup\left(V_{12} \times V_{13}\right) \subset E(\bar{G}),\binom{V_{22}}{2} \cup\left(V_{22} \times V_{23}\right) \subset E(G),\binom{V_{21}}{2} \cup\left(V_{21} \times V_{22}\right) \cup\binom{V_{23}}{2} \subset$ $E(\bar{G})$. Since $\left|V_{11}\right|=\left|V_{21}\right|=\alpha_{1} \geq R_{2}(\varepsilon)$, by Fact 2 , for $1 \leq i \leq 2$, we have $V_{i 1}^{\prime} \subset V_{i 1}$ with $\left|V_{i 1}^{\prime}\right|=\varepsilon$ such that $V_{11}^{\prime} \times V_{21}^{\prime} \subset E(G)$ or $V_{11}^{\prime} \times V_{21}^{\prime} \subset E(\bar{G})$. If $V_{11}^{\prime} \times V_{21}^{\prime} \subset E(G)$, then we have a subset $S \subset V_{11}^{\prime} \cup V_{21}^{\prime} \cup V_{22} \cup V_{23}$ such that $G[S]$ is $\mathcal{B}_{k, \ell^{-} \text {-good, and if } V_{11}^{\prime} \times V_{21}^{\prime} \subset E(\bar{G}) \text {, then we have a subset } S \subset V_{21}^{\prime} \cup V_{11}^{\prime} \cup V_{12} \cup V_{13}, ~}^{\text {and }}$ such that $G[S]$ is $\mathcal{B}_{k, \ell^{-} \text {good, a contradiction. }}^{\text {- }}$

Next, we prepare Claims 4 and 5 , which count the number of disjoint copies of induced subgraphs isomorphic to $H \in \mathcal{B}_{k, \ell}$ in a subset of type $A_{+}$and type $B_{+}$.

Claim 4. Let $S \subset V(G)$ be of type $A_{+}$. Let $a=|S| /(2 k-1-\ell / k)$. Then we have
(1) $N\left(G[S], K_{k}\right) \geq\left(1+\frac{\ell}{k^{2}(k-1)}\right) a$,
(2) $N\left(G[S], \overline{K_{k}}\right) \geq a$,
(3) $N\left(G[S], K_{\ell}+\overline{K_{k-\ell}}\right) \geq a$,
(4) $N\left(G[S], \overline{K_{\ell}} \cup K_{k-\ell}\right) \geq\left(1-\frac{\ell}{k(k-\ell)}\right) a$.

Proof. Indeed, let $S=S_{1} \cup S_{2} \cup S_{3}$ such that $\left|S_{i}\right|=\alpha_{i}$ for $1 \leq i \leq 3$ and $\binom{S_{1}}{2} \cup\left(S_{1} \times S_{2}\right) \cup\binom{S_{3}}{2} \subset E(G),\binom{S_{2}}{2} \cup\left(S_{2} \times S_{3}\right) \subset E(\bar{G})$. By the definition of $\alpha_{1}$, $\alpha_{2}$ and $\alpha_{3}$, we have $\left|S_{1}\right|=\ell a,\left|S_{2}\right|=(k-1) a$ and $\left|S_{3}\right|=(k-l-\ell / k) a$. Hence, we have

$$
\begin{gathered}
N\left(G[S], K_{k}\right) \geq \frac{\left|S_{1}\right|}{k-1}+\frac{\left|S_{3}\right|}{k}=\frac{\ell}{k-1} a+\frac{k-\ell-\ell / k}{k} a=\left(1+\frac{\ell}{k^{2}(k-1)}\right) a \\
N\left(G[S], \overline{K_{k}}\right) \geq \frac{\left|S_{2}\right|}{k-1}=a \\
N\left(G[S], K_{\ell}+\overline{K_{k-\ell}}\right) \geq \frac{\left|S_{1}\right|}{\ell}=a
\end{gathered}
$$

and

$$
N\left(G[S], \overline{K_{\ell}} \cup K_{k-\ell}\right) \geq \frac{\left|S_{3}\right|}{k-\ell}=\frac{k-\ell-\ell / k}{k-\ell} a=\left(1-\frac{\ell}{k(k-\ell)}\right) a
$$

as required.

Claim 5. Let $S \subset V(G)$ be of type $B_{+}$. Let $S=S_{1} \cup S_{2}$ such that $\left|S_{i}\right|=\beta_{i}$ for $i=1,2$ and $\binom{S_{1}}{2} \cup\left(S_{1} \times S_{2}\right) \subset E(G),\binom{S_{2}}{2} \subset E(\bar{G})$. Let $a=|S| /(2 k-1-\ell / k)$. Then we have
(1) $N\left(G[S], K_{k}\right) \geq a$,
(2) $N\left(G[S], K_{\ell}+\overline{K_{k-\ell}}\right) \geq \frac{\left|S_{1}\right|}{k-2}$.

Proof. By the definition of $\beta_{1}$ and $\beta_{2}$, if $\ell \geq k / 2$ then $\left|S_{1}\right|=\ell|S| / k$ and $\left|S_{2}\right|=$ $(k-\ell)|S| / k$, and if $\ell<k / 2$ then $\left|S_{1}\right|=\left|S_{2}\right|=|S| / 2$. Hence, in any case, we have $\left|S_{1}\right| \geq|S| / 2$. Then we have

$$
N\left(G[S], K_{k}\right)=\frac{\left|S_{1}\right|}{k-1} \geq \frac{|S|}{2(k-1)} \geq a
$$

Furthermore, for $\ell \geq k / 2$, we have

$$
N\left(G[S], K_{\ell}+\overline{K_{k-\ell}}\right)=\frac{\left|S_{1}\right|}{\ell}
$$

and for $\ell<k / 2$, we have

$$
N\left(G[S], K_{\ell}+\overline{K_{k-\ell}}\right)=\frac{\left|S_{2}\right|}{k-\ell}=\frac{\left|S_{1}\right|}{k-\ell}
$$

Hence, in any case, we have $N\left(G[S], K_{\ell}+\overline{K_{k-\ell}}\right) \geq\left|S_{1}\right| /(k-2)$.
By Claim 3, without loss of generality, we may assume $n\left(A_{-}\right)=0$. Suppose that $m=p+q+r$ and $V_{1}, \ldots, V_{p}$ are of type $A_{+}, V_{p+1}, \ldots, V_{p+q}$ are of type $B_{+}$or $B_{-}, V_{p+q+1}, \ldots, V_{p+q+r}$ are of type $C_{+}$or $C_{-}$.

For $1 \leq i \leq p$, let $V_{i}=V_{i 1} \cup V_{i 2} \cup V_{i 3}$ such that $\left|V_{i j}\right|=\alpha_{j}$ for $1 \leq j \leq 3$ and $\binom{V_{i 1}}{2} \cup\left(V_{i 1} \times V_{i 2}\right) \cup\binom{V_{i 3}}{2} \subset E(G),\binom{V_{i 2}}{2} \cup\left(V_{i 2} \times V_{i 3}\right) \subset E(\bar{G})$.

Case 1. $n\left(B_{-}\right)=0$. For $1 \leq i \leq q$, let $V_{p+i}=V_{p+i, 1} \cup V_{p+i, 2}$ such that $\left|V_{p+i, j}\right|=\beta_{j}$, for $j=1,2$ and $\binom{V_{p+i, 1}}{2} \cup\left(V_{p+i, 1} \times V_{p+i, 2}\right) \subset E(G),\binom{V_{p+i, 2}}{2} \subset E(\bar{G})$.

Case 1.1. $n\left(C_{-}\right)=0$. In this case, $V_{i}$ is one of types $A_{+}, B_{+}, C_{+}$for $1 \leq i \leq m$. Hence, by Claims 4 and 5 , we have

$$
N\left(G\left[V_{i}\right], K_{k}\right) \geq\left|V_{i}\right| /(2 k-1-\ell / k)
$$

Therefore, since $\left|V_{0}\right|<c$, we have

$$
N\left(G, K_{k}\right) \geq\left|V(G) \backslash V_{0}\right| /(2 k-1-\ell / k) \geq n
$$

a contradiction.

Case 1.2. $n\left(C_{+}\right)=0$. In this case, $V_{i}$ is one of types $A_{+}, B_{+}, C_{-}$for $1 \leq i \leq m$. Let us define a partition $V(G) \backslash V_{0}=S \cup T \cup U$ such that

$$
\begin{aligned}
& S=\bigcup_{1 \leq i \leq q} V_{p+i, 1} \\
& T=\bigcup_{1 \leq i \leq q} V_{p+i, 2} \cup \bigcup_{1 \leq i \leq r} V_{p+q+i} \\
& U=\bigcup_{1 \leq i \leq p} V_{i}
\end{aligned}
$$

$s=|S|, t=|T|, u=|U|$. Let $n_{1}=u /(2 k-1-\ell / k), n_{2}=(s+t) /(2 k-1-\ell / k)$. Since $\left|V_{0}\right|<c$, we have $n_{1}+n_{2}=\left|V(G) \backslash V_{0}\right| /(2 k-1-\ell / k) \geq n$. Furthermore, by Claim 4, we have $N\left(G[U], K_{\ell}+\overline{K_{k-\ell}}\right) \geq n_{1}$ and $N\left(G[U], \overline{K_{k}}\right) \geq n_{1}$. On the other hand, since $s+t=(2 k-1-\ell / k) n_{2}$, we have $s>(k-2) n_{2}$ or $t>k n_{2}$.

If $s>(k-2) n_{2}$, by Claim 5 , we have

$$
N\left(G[S \cup T], K_{\ell}+\overline{K_{k-\ell}}\right) \geq \frac{s}{k-2}>n_{2}
$$

If $t>k n_{2}$, we have

$$
N\left(G[T], \overline{K_{k}}\right) \geq \frac{t}{k}>n_{2}
$$

In any case, we have

$$
N\left(G,\left\{K_{\ell}+\overline{K_{k-\ell}}, \overline{K_{k}}\right\}\right)>n_{1}+n_{2} \geq n
$$

a contradiction.
Case 2. $n\left(B_{+}\right)=0$. For $1 \leq i \leq q$, let $V_{p+i}=V_{p+i, 1} \cup V_{p+i, 2}$ such that $\left|V_{p+i, j}\right|=\beta_{j}$, for $j=1,2$ and $\binom{V_{p+i, 2}}{2} \subset E(G),\binom{V_{p+i, 1}}{2} \cup\left(V_{p+i, 1} \times V_{p+i, 2}\right) \subset E(\bar{G})$.

Case 2.1. $n\left(C_{+}\right)=0$. In this case, $V_{i}$ is one of types $A_{+}, B_{-}, C_{-}$for $1 \leq i \leq m$. Hence, by Claim 4 and Claim 5, we have

$$
N\left(G\left[V_{i}\right], \overline{K_{k}}\right) \geq\left|V_{i}\right| /(2 k-1-\ell / k)
$$

Therefore, since $\left|V_{0}\right|<c$, we have

$$
N\left(G, \overline{K_{k}}\right) \geq\left|V(G) \backslash V_{0}\right| /(2 k-1-\ell / k) \geq n
$$

a contradiction.
Case 2.2. $n\left(C_{-}\right)=0$. In this case, $V_{i}$ is one of types $A_{+}, B_{-}, C_{+}$for $1 \leq i \leq m$. We may assume $q \geq 1$ and $r \geq 1$. Indeed, if $q=0$ or $r=0$, as in Case 1 or in Case 2.1, we have a contradiction.

In order to show the assertion of the theorem, we will modify the original partition. Let us prepare two claims. Let $X=\bigcup_{1 \leq i \leq r} V_{p+q+i}$.

Claim 6. Suppose that $\ell<k / 2$. Let $V_{i}$ be of type $B_{-}$such that $V_{i}=V_{i, 1} \cup V_{i, 2}$, $\left|V_{i, j}\right|=\beta_{j}$ with $j=1,2,\binom{V_{i, 2}}{2} \subset E(G),\binom{V_{i, 1}}{2} \cup\left(V_{i, 1} \times V_{i, 2}\right) \subset E(\bar{G})$. Let $s_{0}=\beta_{2} \ell /(k-\ell)$. Let $T \subset X \cup V_{0}$ such that $G[T] \cong K_{\gamma}$. Then we have partitions $V_{i, 1}=S_{0} \cup S_{1} \cup \cdots \cup S_{w}, T=T_{0} \cup T_{1} \cup \cdots \cup T_{w}$ such that
(1) $\left|S_{0}\right|=s_{0}$,
(2) $\left|S_{j}\right|=\alpha_{2} \ell,\left|T_{j}\right|=\alpha_{2}(k-\ell)$ for $1 \leq j \leq w$,
(3) $S_{j} \times T_{j} \subset E(\bar{G})$ for $1 \leq j \leq w$.

Proof. Since $\ell<k / 2$, we have $\beta_{1}=\beta_{2}$. By the definition of $\beta_{2}$ and $\gamma$, we have $s_{0}=\beta_{2} \ell /(k-\ell) \geq R_{2}\left(\alpha_{2} k\right)$ and $\gamma \geq R_{3}\left(\alpha_{2} k, \beta_{1}\right)$. Hence, by Lemma 8, we have partitions $V_{i, 1}=S_{0} \cup S_{1} \cup \cdots \cup S_{w}, T=T_{0} \cup T_{1} \cup \cdots \cup T_{w}$ such that (1) $\left|S_{0}\right|=s_{0}$, (2) $\left|S_{j}\right|=\alpha_{2} \ell,\left|T_{j}\right|=\alpha_{2}(k-\ell)$ for $1 \leq j \leq w$, and $\left(3^{\prime}\right) S_{j} \times T_{j} \subset E(G)$ or $S_{j} \times T_{j} \subset E(\bar{G})$ for $1 \leq j \leq w$. If $S_{j} \times T_{j} \subset E(G)$ for some $j$, we have a set of type $A_{+}$in $T_{j} \cup S_{j} \cup V_{i, 2}$, which contradicts the property ( P 3 ) of the partition. Thus we have $S_{j} \times T_{j} \subset E(\bar{G})$ for $1 \leq j \leq w$, as claimed.

Claim 7. Let $V_{i}$ be of type $A_{+}$such that $V_{i}=V_{i, 1} \cup V_{i, 2} \cup V_{i, 3},\left|V_{i, j}\right|=\alpha_{j}$ with $1 \leq j \leq 3,\binom{V_{i, 1}}{2} \cup\left(V_{i, 1} \times V_{i, 2}\right) \cup\binom{V_{i, 3}}{2} \subset E(G),\binom{V_{i, 2}}{2} \cup\left(V_{i, 2} \times V_{i, 3}\right) \subset E(\bar{G})$. Let $T \subset X \cup V_{0}$ such that $G[T] \cong K_{\gamma}$. Let $s_{0}=\alpha_{3} \ell /(k-\ell)$. Then we have partitions $V_{i, 2}=S_{0} \cup S_{1} \cup \cdots \cup S_{w}, T=T_{0} \cup T_{1} \cup \cdots \cup T_{w}$ such that
(1) $\left|S_{0}\right|=s_{0}$,
(2) $\left|S_{j}\right|=\varepsilon \ell,\left|T_{j}\right|=\varepsilon(k-\ell)$ for $1 \leq j \leq w$,
(3) $S_{j} \times T_{j} \subset E(\bar{G})$ for $1 \leq j \leq w$.

Proof. Let $\lambda=\max \left\{\alpha_{2}, R_{3}\left(\varepsilon k, \alpha_{2}\right)\right\}$. Let $V_{p+1}=V_{p+1,1} \cup V_{p+1,2}$ such that $\left|V_{p+1, j}\right|=\beta_{j}$, for $j=1,2$ and $\binom{V_{p+1,2}}{2} \subset E(G),\binom{V_{p+1,1}}{2} \cup\left(V_{p+1,1} \times V_{p+1,2}\right) \subset$ $E(\bar{G})$. Since $\left|V_{p+1,1}\right|=\beta_{1} \geq R_{2}(\lambda)$ and $|T|=\gamma \geq R_{2}(\lambda)$, by Fact 2 , we have $V_{p+1,1}^{\prime} \subset V_{p+1,1}, T^{\prime} \subset T$ with $\left|V_{p+1,1}^{\prime}\right|=\left|T^{\prime}\right|=\lambda$ such that $T^{\prime} \times V_{p+1,1}^{\prime} \subset E(G)$ or $T^{\prime} \times V_{p+1,1}^{\prime} \subset E(\bar{G})$.

If $T^{\prime} \times V_{p+1,1}^{\prime} \subset E(G)$, we have a set of type $A_{+}$in $T^{\prime} \cup V_{p+1,1}^{\prime} \cup V_{p+1,2}$, which contradicts the property ( P 3 ) of the partition. Hence, we have $T^{\prime} \times V_{p+1,1}^{\prime} \subset E(\bar{G})$.

Since $s_{0}=\alpha_{3} \ell /(k-\ell) \geq R_{2}(\varepsilon k)$ and $\left|T^{\prime}\right|=\lambda \geq R_{3}\left(\varepsilon k, \alpha_{2}\right)$, by Lemma 8 , we have partitions $V_{i, 2}=S_{0} \cup S_{1} \cup \cdots \cup S_{w}, T^{\prime}=T_{0} \cup T_{1} \cup \cdots \cup T_{w}$ such that (1) $\left|S_{0}\right|=s_{0},(2)\left|S_{j}\right|=\varepsilon \ell,\left|T_{j}\right|=\varepsilon(k-\ell)$ for $1 \leq j \leq w$, and (3') $S_{j} \times T_{j} \subset E(G)$ or $S_{j} \times T_{j} \subset E(\bar{G})$ for $1 \leq j \leq w$. If $S_{j} \times T_{j} \subset E(G)$ for some $j$, we have a $\mathcal{B}_{k, \ell}$-good subgraph in $G\left[V_{p+1,1}^{\prime} \cup T_{j} \cup S_{j} \cup V_{i, 3}\right]$, a contradiction. Hence, we have $S_{j} \times T_{j} \subset E(\bar{G})$ for $1 \leq j \leq w$, as claimed.

Under the situation of Claim 6, we have $N\left(G\left[V_{i} \cup T\right], \overline{K_{\ell}} \cup K_{k-\ell}\right) \geq\left|V_{i, 1}\right| / \ell$, and under the situation of Claim 7, we have $N\left(G\left[V_{i} \cup T\right], \overline{K_{\ell}} \cup K_{k-\ell}\right) \geq\left|V_{i, 2}\right| / \ell$.

By using Claims 6 and 7 , let us modify the partition $V=V_{0} \cup V_{1} \cup \cdots \cup V_{m}$ according to the following algorithm.

## Algorithm 1.

Step 0. $X_{0}=\left(X \backslash V_{m}\right) \cup V_{0}$. If $\ell<k / 2$, then put $y=p+q$, otherwise put $y=p$.
Set 1 to $i$.
Step 1. If $i>y$, then set 0 to $x$ and stop.
Step 2. If there exists $W \subset X_{0}$ such that $G[W] \cong K_{\gamma}$, then take $W$, otherwise set 1 to $x$, set $i-1$ to $y^{\prime}$ and stop.
Step 3. For $V_{i}$ and $W$, take $T_{1}, T_{2}, \ldots, T_{w} \subset W$ satisfying the condition of Claim 6 or 7 . Let $Z_{i}=T_{1} \cup \cdots \cup T_{w}$. Set $X_{0}=X_{0} \backslash Z_{i}$. Add 1 to $i$. Go to Step 1.
Note that in the procedure of the algorithm, $G\left[X_{0}\right]$ contains no $\overline{K_{\gamma}}$ as a subgraph. Indeed, if there exists $W \subset X_{0}$ with $G[W] \cong \overline{K_{\gamma}}$, since $\gamma \geq R_{2}\left(\beta_{1}\right)$, by Fact 2 , we have $V_{m}^{\prime} \subset V_{m}$ and $W^{\prime} \subset W$ such that $V_{m}^{\prime} \cup W^{\prime}$ is of type $B_{+}$or $B_{-}$, a contradiction to ( P 4 ) of the original partition.

After executing the algorithm, we consider the two subcases according to the return status $x$.

Case 2.2.1. $x=0$. Let us take a partition $X_{0}=W_{0} \cup W_{1} \cup \cdots \cup W_{\mu}$ such that $G\left[W_{i}\right] \cong K_{\gamma}$ for $1 \leq i \leq \mu$ and $\left|W_{0}\right|<R_{1}(\gamma)$.
Let us define
$S=\bigcup_{1 \leq i \leq p} V_{i, 2} \cup \bigcup_{1 \leq i \leq q} V_{p+i, 1}$,
$T=\bigcup_{1 \leq i \leq p}\left(V_{i, 1} \cup V_{i, 3}\right) \cup \bigcup_{1 \leq i \leq q} V_{p+i, 2} \cup \bigcup_{1 \leq i \leq y} Z_{i} \cup \bigcup_{1 \leq i \leq \mu} W_{i} \cup V_{m}$,
$s=|S|, t=|T|$. Then we have $s+t \geq(2 k-1-\ell / k) n$.
On the other hand, we have $n>N\left(G[S \cup T] \overline{K_{\ell}} \cup K_{k-\ell}\right) \geq s / \ell$. Hence, we have $\ell n>s$. Furthermore, we have $n>N\left(G[T], K_{k}\right) \geq t / k$. Hence, we have $k n>t$. Therefore, we have $(k+\ell) n>s+t \geq(2 k-1-\ell / k) n$, a contradiction.

Case 2.2.2. $x=1$. Let us define

$$
\begin{aligned}
& S=\bigcup_{1 \leq i \leq q} V_{p+i, 1} \\
& T=\bigcup_{1 \leq i \leq q} V_{p+i, 2} \cup \bigcup_{1 \leq i \leq y^{\prime}} Z_{i} \\
& U=\bigcup_{1 \leq i \leq p} V_{i}
\end{aligned}
$$

$s=|S|, t=|T|, u=|U|$. Let $n_{1}=u /(2 k-1-\ell / k), n_{2}=(s+t) /(2 k-1-\ell / k)$.
Then we have $n_{1}+n_{2} \geq n$, since $|V(G) \backslash(S \cup T \cup U)|=\left|V_{m} \cup X_{0}\right|<\gamma+R_{1}(\gamma)=c$.
By a similar argument to that in Case 1.2 , if $s \geq(k-1) n_{2}$, we have $N(G[S \cup$ $\left.T], \overline{K_{k}}\right)=s /(k-1) \geq n_{2}$, a contradiction. Hence, we have $s<(k-1) n_{2}$. Therefore, we have

$$
\begin{equation*}
\left(k-\frac{\ell}{k}\right) n_{2}<t \tag{1}
\end{equation*}
$$

In the same way, we have $N\left(G[T], K_{k}\right)+N\left(G[U], K_{k}\right)<n$. Hence, by Claim 4, we have

$$
\begin{equation*}
\frac{t}{k}+\left(1+\frac{\ell}{k^{2}(k-1)}\right) n_{1}<n \tag{2}
\end{equation*}
$$

By (1) and (2), we have

$$
\begin{equation*}
n<k n_{2} . \tag{3}
\end{equation*}
$$

On the other hand, by using Claim 4, we have $n>N\left(G, \overline{K_{\ell}} \cup K_{k-\ell}\right) \geq t /(k-$ $\ell)+(1-\ell / k(k-\ell)) n_{1}$. Hence, we have

$$
\begin{equation*}
k t-\left(k^{2}-k \ell-\ell\right) n_{2}<\ell n \tag{4}
\end{equation*}
$$

By (3) and (4), we have $t<(k-\ell / k) n_{2}$, a contradiction to (1). This completes the proof.

## 4. Concluding Remarks

Let $\mathcal{G}_{k}$ be the family of all graphs with $k$ vertices, as defined in Section 1. For $k \geq 4, f\left(n, \mathcal{G}_{k}\right)$ is not known well. For $k=4$, let $G=K_{2 n-1} \cup\left(K_{n-1}+\overline{K_{3 n-1}}\right)$. Then we have $N\left(G, \mathcal{G}_{4}\right)<n$. It follows that $f\left(n, \mathcal{G}_{4}\right) \geq 6 n-2$. On the other hand, from Theorem 5 by substituting 4 for $k$ and 2 for $\ell$, we have $f\left(n, \mathcal{G}_{4}\right) \leq$ $f\left(n, \mathcal{B}_{4,2}\right)=(13 / 2) n+O(1)$. It is conjectured that $f\left(n, \mathcal{G}_{4}\right)=6 n+O(1)([5])$.

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