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# ON THE UNIQUENESS OF *D*-VERTEX MAGIC CONSTANT

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**Abstract**

Let $G = (V, E)$ be a graph of order $n$ and let $D \subseteq \{0, 1, 2, 3, \ldots\}$. For $v \in V$, let $N_D(v) = \{u \in V : d(u, v) \in D\}$. The graph $G$ is said to be $D$-vertex magic if there exists a bijection $f : V(G) \to \{1, 2, \ldots, n\}$ such that for all $v \in V$, $\sum_{u \in N_D(v)} f(u)$ is a constant, called $D$-vertex magic constant. O'Neal and Slater have proved the uniqueness of the $D$-vertex magic constant by showing that it can be determined by the $D$-neighborhood fractional domination number of the graph. In this paper we give a simple and elegant proof of this result. Using this result, we investigate the existence of distance magic labelings of complete $r$-partite graphs where $r \geq 4$.

**Keywords:** distance magic graph, $D$-vertex magic graph, magic constant, dominating function, fractional domination number.

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## 1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. We further assume that $G$ has no isolated vertices. The order $|V|$ and the size $|E|$ are denoted by $n$ and $m$, respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [3].

The concept of distance magic labeling has been motivated by the construction of magic squares. A magic square of order $n$ is an $n \times n$ array whose entries are an arrangement of the integers $1, 2, 3, \ldots, n^2$ in which all elements in any row, any column, the main diagonal or the main back diagonal add to the same sum $r$. Now if we label the vertices of a complete $n$-partite graph with parts $|V_i| = n$, $1 \leq i \leq n$ in such a way that the vertices of $V_i$ are labeled with the integers in the $i^{th}$ row of the magic square, then the sum of the labels of all the vertices in the open neighborhood of each vertex is the same and is equal to $r(n-1)$. Motivated by this observation Vilfred [10] in his doctoral thesis introduced the following concept of $\Sigma$-labeling.

**Definition 1.1.** A $\Sigma$-labeling of a graph $G = (V, E)$ of order $n$ is a bijection $f : V \to \{1, 2, \ldots, n\}$ such that $\sum_{u \in N(v)} f(u) = k$ for all $v \in V$, where $N(v)$ is the open neighborhood of $v$. The constant $k$ is called the magic constant of the labeling $f$. A graph which admits a $\Sigma$-labeling is called a $\Sigma$-graph.

The same concept was introduced and studied by different authors with different terminology. Miller *et al.* [6] used the term 1-vertex magic labeling and Sugeng *et al.* [9] used the term distance magic labeling for the same concept. Beena [2] introduced the concept of $\Sigma'$-labeling in which the closed neighborhood sums are all equal. O'Neal and Slater [7] introduced the following concept of $D$-vertex magic labeling, which includes the notion of distance magic labeling as well as $\Sigma'$-labeling as special cases.

**Definition 1.2.** Let $G = (V, E)$ be a graph of order $n$ and let $D \subseteq \{0, 1, 2, 3, \ldots\}$. For $v \in V$, the set $N_D(v) = \{u \in V : d(u, v) \in D\}$ is called the $D$-neighborhood of $v$. A bijection $f : V(G) \to \{1, 2, \ldots, n\}$ is called a $D$-vertex magic labeling of $G$ if $\sum_{u \in N_D(v)} f(u) = k$ for all $v \in V$. The constant $k$ is called the $D$-vertex magic constant and a graph which admits a $D$-vertex magic labeling is called $D$-vertex magic graph.

If $D = \{1\}$, then $N_D(v)$ is the open neighborhood $N(v)$ and the $D$-vertex magic labeling is the usual distance magic labeling. If $D = \{0, 1\}$, then $N_D(v)$ is the closed neighborhood $N[v]$ and the corresponding $D$-vertex magic labeling is the $\Sigma'$-labeling.

The first author posed the following problem in his talk in IWOGL 2010 at the University of Minnesota, Duluth, U.S.A.

**Problem 1.3** [1]**.** *Does there exist a distance magic graph with two different distance magic labelings having different magic constants?*

In the revision note given in [1], we have given a simple and short proof, using algebraic concepts, that for any distance magic graph, the distance magic constant is unique, thus answering Problem 1.3. The same proof technique gives the uniqueness of the $D$-vertex magic constant for any $D$-vertex magic graph.

O'Neal and Slater obtained a formula for the $D$-vertex magic constant in terms of a fractional domination parameter of the graph, which implies the uniqueness of the magic constant.

**Definition 1.4.** Let $G = (V, E)$ be a graph. A function $f : V(G) \to [0, 1]$ is said to be a $D$-neighborhood fractional dominating function if for every vertex $v \in V(G)$, $\sum_{u \in N_D(v)} f(u) \geq 1$. The $D$-neighborhood fractional domination number of a graph is denoted by $\gamma_f(G; D)$ and is defined as $\gamma_f(G; D)=$ min $\{|f| : f$ is a $D$-neighborhood fractional dominating function$\}$, where $|f|=\sum_{v \in V(G)} f(v)$. If $D$-neighborhood fractional dominating function for $G$ does not exist, then we define $\gamma_f(G; D) = \infty$.

Also it is clear from the definition that when $D = \{0, 1\}$, $\gamma_f(G; D) = \gamma_f(G)$ is the fractional domination number and when $D = \{1\}$, $\gamma_f(G; D) = \gamma_{ft}(G)$ is the fractional total domination number. The concepts of fractional domination and fractional total domination have been investigated by several authors and for a survey on these topics we refer to Chapter 3 of Haynes *et al.* [5].

O'Neal and Slater have proved the following:

**Theorem 1.5** [8]**.** *If a graph $G$ is $D$-vertex magic, then its $D$-vertex magic constant* $k = \frac{n(n+1)}{2\gamma_f(G;D)}$.

In this paper we give a simple and elegant proof of the above theorem. We use this result to prove the existence and non-existence of distance magic labelings for complete $r$-partite graphs with $r \geq 4$.

## 2. Main Results

**Definition 2.1.** Let $G = (V, E)$ be a graph of order $n$ with $V = \{v_1, v_2, \ldots, v_n\}$. Let $D \subseteq \{0, 1, 2, \ldots\}$. The $n \times n$ matrix $A_D = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } d(v_i, v_j) \in D, \\ 0 & \text{otherwise,} \end{cases}$$

is called the $D$-distance matrix of $G$.

**Theorem 2.2.** *If a graph $G$ admits a distance magic labeling $f$ with magic constant $k$, then $k = \frac{n(n+1)}{2\gamma_{ft}}$.*

***Proof.*** Let $G = (V, E)$ be a graph of order $n$. Let $A$ be its adjacency matrix. Let $h : V \to \{1, 2, \dots, n\}$ be a distance magic labeling of $G$ with magic constant $k$. Then $AX = k\overline{u}$, where $X = (h(v_1), h(v_2), \dots, h(v_n))^T$ and $\overline{u} = (1, 1, \dots, 1)^T$ is a $n \times 1$ matrix. Now let $g$ be a fractional total dominating function of $G$ with $|g| = \sum_{i=1}^{n} g(v_i) = \gamma_{ft}(G)$. Then $AY \geq 1$ where $Y = (g(v_1), g(v_2), \dots, g(v_n))^T$. Let $AY = M = (l_1, l_2, \dots, l_n)^T$, so that each $l_i \geq 1$. Now, $X^T AY$ is a $1 \times 1$ matrix and hence $X^T AY = (X^T AY)^T = Y^T AX$. Thus

$$\begin{aligned} X^T AY &= Y^T AX = Y^T (AX) = Y^T k\overline{u} = kY^T \overline{u} \\ &= k(g(v_1) + g(v_2) + \cdots + g(v_n)) = k\gamma_{ft}. \end{aligned}$$

Also

$$\begin{aligned} X^T AY &= X^T M = (h(v_1), h(v_2), \dots, h(v_n))(l_1, l_2, \dots, l_n)^T \\ &= \sum\nolimits_{i=1}^{n} h(v_i) l_i \geq \sum\nolimits_{i=1}^{n} h(v_i) = \frac{n(n+1)}{2}. \end{aligned}$$

Thus, $X^T AY \geq \frac{n(n+1)}{2}$. Hence it follows that $k\gamma_{ft} \geq \frac{n(n+1)}{2}$. Thus $k \geq \frac{n(n+1)}{2\gamma_{ft}}$. We now prove the reverse inequality. Define $\theta : V \to [0, 1]$ by $\theta(v) = \frac{h(v)}{k}$. Since $h$ is a distance magic labeling of $G$, it follows that $0 < \frac{h(v)}{k} \leq 1$. Also for any $v \in V$, $\sum_{u \in N(v)} \theta(u) = \frac{1}{k} \sum_{u \in N(v)} h(u) = 1$. Thus $\theta$ is a fractional total dominating function of $G$. Hence $\gamma_{ft}(G) \leq |\theta| = \sum_{v \in V(G)} \theta(v) = \sum_{v \in V(G)} \frac{h(v)}{k} = \frac{n(n+1)}{2k}$. Thus $k \leq \frac{n(n+1)}{2\gamma_{ft}}$. ∎

**Theorem 2.3.** *If a graph $G$ is $D$-vertex magic, then its $D$-vertex magic constant $k = \frac{n(n+1)}{2\gamma_f(G;D)}$.*

***Proof.*** Replace $A$ by $A_D$ in the proof of Theorem 2.2. ∎

Theorem 2.3 serves as a powerful tool in proving the existence and nonexistence of $\Sigma'$-labelings as well as distance magic labelings for graphs, which we illustrate in the following theorems.

**Theorem 2.4.** *The hypercube $Q_n$ is not a $\Sigma'$-graph when $n$ is even and $n > 0$.*

***Proof.*** Since $Q_n$ is an $n$-regular graph of order $2^n$, we have $\gamma_f(Q_n) = \frac{2^n}{n+1}$ ([4]). If $Q_n$ admits a $\Sigma'$-labeling, then by Theorem 2.3, the magic constant $k = \frac{2^n(2^n+1)}{2\gamma_f(Q_n)} = \frac{(n+1)(2^n+1)}{2}$, which is not an integer when $n$ is even. Hence $Q_n$ is not a $\Sigma'$-graph when $n$ is even. ∎

**Theorem 2.5.** *Let $G$ be a graph of order $2n$ consisting of two edge disjoint cycles $C_1 = (v_1, v_2, v_3, \ldots, v_{2n-1}, v_{2n}, v_1)$ and $C_2 = (v_2, v_4, v_6, \ldots, v_{2n-2}, v_{2n}, v_2)$. Then the graph $G$ is not distance magic.*

***Proof.*** The function $f : V(G) \to [0,1]$ defined by

$$f(v_i) = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ \frac{1}{2} & \text{if } i \text{ is even,} \end{cases}$$

is a fractional total dominating function of $G$ and hence $\gamma_{ft}(G) \leq |f| = \frac{n}{2}$. Now let $g$ be any fractional total dominating function of $G$ with $g(v_i) = b_i$, $1 \leq i \leq 2n$. Since $N(v_{2i+1}) = \{v_{2i}, v_{2i+2}\}$ for each $i = 1, 2, \ldots, n$ it follows that $b_{2i}+b_{2i+2} \geq 1$, for all $i = 1, 2, \ldots, n$, where the addition in the suffix is taken modulo $2n$. Adding these $n$ inequalities we obtain $2\sum_{i=1}^{n} b_{2i} \geq n$. Thus $|g| \geq \sum_{i=1}^{n} b_{2i} \geq \frac{n}{2}$ and hence $\gamma_{ft}(G) = \frac{n}{2}$. Now, if $G$ is distance magic, then by Theorem 2.3, the magic constant $k = 2(2n+1)$. However the maximum possible weight for any vertex of degree 2 is $4n-1$. Hence $G$ is not distance magic. ∎

Beena [2] characterized complete bipartite graphs which are distance magic. Also Miller *et al.* [6] characterized complete tripartite graphs which are distance magic. The problem of characterizing complete $k$-partite graphs which are distance magic remains open for $k \geq 4$. We prove some results in this direction.

**Theorem 2.6.** *Let $G = K_{a_1,a_2,a_3,\ldots,a_r}$, $2 \leq a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_r$ be a complete $r$-partite graph with $r \geq 4$. If $G$ is distance magic, then $2r|n(n+1)$.*

***Proof.*** First we prove that $\gamma_{ft}(G) = \frac{r}{r-1}$. Let $V_1, V_2, V_3, \ldots, V_r$ be the partite sets with $|V_i| = a_i$. Define $f : V(G) \to [0,1]$ by $f(v_i) = \frac{1}{a_i(r-1)}$ if $v_i \in V_i$. Clearly $f$ is a fractional total dominating function. Hence $\gamma_{ft}(G) \leq |f| = \sum_{v\in V(G)} f(v) = \frac{r}{r-1}$. Now, let $f$ be any fractional total dominating function of $G$. Then $|f| - \sum_{v\in V_i} f(v) \geq 1$ for $i = 1, 2, \ldots, r$. Adding these $r$ inequalities we get $r|f|-|f| \geq r$. Therefore $|f| \geq \frac{r}{r-1}$. Thus $\gamma_{ft}(G) \geq \frac{r}{r-1}$. Hence $\gamma_{ft}(G) = \frac{r}{r-1}$. Now, if $G$ is distance magic, then by Theorem 2.3, $k = \frac{n(n+1)(r-1)}{2r}$. Hence $2r|n(n+1)$. ∎

**Corollary 2.7.** *If $G$ is a complete 4-partite distance magic graph of order $n$, then $n \equiv 0$ or $7 (\bmod 8)$.*

**Corollary 2.8.** *If $G$ is a complete 5-partite distance magic graph of order $n$, then $n \equiv 0$ or $4 (\bmod 5)$.*

**Corollary 2.9.** *The complete $r$-partite graph $G = K_{a,a,\ldots,a,a+1}$, $a \geq 2$ and $r \geq 4$, is not distance magic.*

***Proof.*** If $G$ is distance magic, then the magic constant $k = \frac{(ra+1)(ra+2)(r-1)}{2r}$. If $r$ is a power of 2, then trivially $k$ is not an integer. If $r$ is not a power of 2, let $p$ be an odd prime factor of $r$. Then $p$ does not divide $(ra+1)(ra+2)(r-1)$ and hence $k$ is not an integer. Thus $G$ is not distance magic. ■

We now proceed to characterize complete $r$-partite distance magic graphs of small order for $r = 4$ and $r = 5$. Let $G$ be a complete 4-partite distance magic graph of order $n$ with partite sets $V_i, 1 \leq i \leq 4$. By Corollary 2.7, $n \equiv 0$ or $7 (\bmod 8)$. If $n = 7$, then $k = 21$ and $G = K_{1,2,2,2}$. Also $V_1 = \{7\}, V_2 = \{1,6\}, V_3 = \{2,5\}$ and $V_4 = \{3,4\}$ gives a distance magic labeling of $G$. If $n = 8$, then $k = 27$, so that the sum of the labels of the vertices in each partite set is 9. In this case $G = K_{2,2,2,2}$ and $V_1 = \{1,8\}, V_2 = \{2,7\}, V_3 = \{3,6\}$ and $V_4 = \{4,5\}$ gives a distance magic labeling of $G$.

**Theorem 2.10.** *A complete* 4*-partite graph of order* 15 *is distance magic if and only if it is isomorphic to one of the graphs* $K_{4,4,4,3}$, $K_{5,4,3,3}$ *and* $K_{6,3,3,3}$.

***Proof.*** Let $G$ be a complete 4-partite distance magic graph of order 15. By Theorem 2.6, we have $k = 90$. Thus sum of the labels in each partite set is 30. Thus $|V_i| \geq 3$ and $G$ is isomorphic to one of the graphs $K_{4,4,4,3}$, $K_{5,4,3,3}$ and $K_{6,3,3,3}$.

To prove the converse, we take $V_1 = \{6,7,8,9\}, V_2 = \{2,3,12,13\}, V_3 = \{4,5,10,11\}$ and $V_4 = \{1,14,15\}$ if $G = K_{4,4,4,3}$; $V_1 = \{2,3,4,10,11\}, V_2 = \{6,7,8,9\}, V_3 = \{5,12,13\}$ and $V_4 = \{1,14,15\}$ if $G = K_{5,4,3,3}$; $V_1 = \{2,3,4,6,7,8\}, V_2 = \{5,12,13\}, V_3 = \{1,14,15\}$ and $V_4 = \{9,10,11\}$ if $G = K_{6,3,3,3}$. This gives a distance magic labeling of $G$. ■

**Theorem 2.11.** *A complete* 4*-partite graph of order* 16 *is distance magic if and only if it is isomorphic to one of the graphs* $K_{7,3,3,3}$, $K_{6,4,3,3}$, $K_{5,5,3,3}$, $K_{5,4,4,3}$ *and* $K_{4,4,4,4}$.

***Proof.*** Let $G$ be a complete 4-partite graph of order 16. If $G$ is distance magic, then by Theorem 2.6 we have $k = 102$. Thus sum of the labels in each partite set is 34. Thus $|V_i| \geq 3$ and $G$ is isomorphic to one of the graphs $K_{7,3,3,3}$, $K_{6,4,3,3}$, $K_{5,5,3,3}$, $K_{5,4,4,3}$ and $K_{4,4,4,4}$.

To prove the converse, we take $V_1 = \{1,2,4,5,6,7,9\}, V_2 = \{3,15,16\}, V_3 = \{10,11,13\}$ and $V_4 = \{8,12,14\}$ if $G = K_{7,3,3,3}$; $V_1 = \{1,4,5,7,8,9\}, V_2 = \{2,6,12,14\}, V_3 = \{3,15,16\}$ and $V_4 = \{10,11,13\}$ if $G = K_{6,4,3,3}$; $V_1 = \{1,4,7,8,14\}, V_2 = \{2,5,6,9,12\}, V_3 = \{3,15,16\}$ and $V_4 = \{10,11,13\}$ if $G = K_{5,5,3,3}$; $V_1 = \{2,5,6,9,12\}, V_2 = \{4,7,8,15\}, V_3 = \{1,3,14,16\}$ and $V_4 = \{10,11,13\}$ if $G = K_{5,4,4,3}$; $V_1 = \{5,6,10,13\}, V_2 = \{1,3,14,16\}, V_3 = \{4,7,8,15\}$ and $V_4 = \{2,9,11,12\}$ if $G = K_{4,4,4,4}$. This gives a distance magic labeling of $G$. ■

The proofs of the following theorems are similar to that of Theorem 2.11.

**Theorem 2.12.** *A complete* 4*-partite graph of order* 24 *is distance magic if and only if it is isomorphic to one of the graphs* $K_{11,5,4,4}$, $K_{10,6,4,4}$, $K_{10,5,5,4}$, $K_{9,7,4,4}$, $K_{9,5,5,5}$, $K_{9,6,5,4}$, $K_{8,8,4,4}$, $K_{8,7,5,4}$, $K_{8,6,6,4}$, $K_{8,6,5,5}$, $K_{7,7,5,5}$, $K_{7,7,6,4}$, *and* $K_{6,6,6,6}$.

Let $G$ be a complete 5-partite distance magic graphs of order $n$ with partite sets $V_i, 1 \leq i \leq 5$. By Corollary 2.8, $n \equiv 0$ or $4(\bmod 5)$. If $n = 9$, then $k = 36$ and $G = K_{1,2,2,2,2}$. If $n = 10$, then $k = 44$ and $G = K_{2,2,2,2,2}$.

**Theorem 2.13.** *A complete* 5*-partite graph of order* 14 *is distance magic if and only if it is isomorphic to one of the graphs* $K_{6,2,2,2,2}$, $K_{5,3,2,2,2}$, $K_{4,4,2,2,2}$, $K_{4,3,3,2,2}$ *and* $K_{3,3,3,3,2}$.

***Proof.*** Let $G$ be a complete 5-partite graph of order 14. It follows from Theorem 2.6, that $k = 84$. Thus sum of the labels in each partite set is 21. Thus $|V_i| \geq 2$ and $G$ is isomorphic to one of the graphs $K_{6,2,2,2,2}$, $K_{5,3,2,2,2}$, $K_{4,4,2,2,2}$, $K_{4,3,3,2,2}$ and $K_{3,3,3,3,2}$.

To prove the converse, we take $V_1 = \{1,2,3,4,5,6\}, V_2 = \{10,11\}, V_3 = \{9,12\}, V_4 = \{8,13\}$ and $V_5 = \{7,14\}$ if $G = K_{6,2,2,2,2}$; $V_1 = \{1,2,3,5,10\}, V_2 = \{4,6,11\}, V_3 = \{9,12\}, V_4 = \{8,13\}$ and $V_5 = \{7,14\}$ if $G = K_{5,3,2,2,2}$; $V_1 = \{2,3,6,10\}, V_2 = \{1,4,5,11\}, V_3 = \{9,12\}, V_4 = \{8,13\}$ and $V_5 = \{7,14\}$ if $G = K_{4,4,2,2,2}$; $V_1 = \{2,3,6,10\}, V_2 = \{1,9,11\}, V_3 = \{4,5,12\}, V_4 = \{8,13\}$ and $V_5 = \{7,14\}$ if $G = K_{4,3,3,2,2}$; $V_1 = \{3,8,10\}, V_2 = \{1,9,11\}, V_3 = \{4,5,12\}, V_4 = \{2,6,13\}$ and $V_5 = \{7,14\}$ if $G = K_{3,3,3,3,2}$. This gives a distance magic labeling of $G$. ∎

The proofs of the following theorems are similar to that of Theorem 2.11.

**Theorem 2.14.** *A complete* 5*-partite graph of order* 15 *is distance magic if and only if it is isomorphic to one of the graphs* $K_{6,3,2,2,2}$, $K_{5,4,2,2,2}$, $K_{5,3,3,2,2}$, $K_{4,4,3,2,2}$, $K_{4,3,3,3,2}$ *and* $K_{3,3,3,3,3}$.

**Theorem 2.15.** *A complete* 5*-partite graph of order* 19 *is distance magic if and only if it is isomorphic to one of the graphs* $K_{7,3,3,3,3}$, $K_{6,4,3,3,3}$, $K_{5,5,3,3,3}$, $K_{5,4,4,3,3}$ *and* $K_{4,4,4,4,3}$.

**Theorem 2.16.** *A complete* 5*-partite graph of order* 20 *is distance magic if and only if it is isomorphic to one of the graphs* $K_{8,3,3,3,3}$, $K_{7,4,3,3,3}$, $K_{6,5,3,3,3}$, $K_{5,5,4,3,3}$, $K_{6,4,4,3,3}$, $K_{5,4,4,4,3}$ *and* $K_{4,4,4,4,4}$.

## 3. CONCLUSION AND SCOPE

In this paper we have given a simple and elegant proof of a formula for the distance magic constant $k$ of a distance magic graph in terms of the fractional total domination number. We have also illustrated the use of this result in determining whether a given graph is distance magic or not. One can further explore the application of this technique for getting new results on $\Sigma'$-graphs and distance magic graphs.

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