# ON MONOCHROMATIC SUBGRAPHS OF EDGE-COLORED COMPLETE GRAPHS 

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#### Abstract

In a red-blue coloring of a nonempty graph, every edge is colored red or blue. If the resulting edge-colored graph contains a nonempty subgraph $G$ without isolated vertices every edge of which is colored the same, then $G$ is said to be monochromatic.

For two nonempty graphs $G$ and $H$ without isolated vertices, the monochromatic Ramsey number $\operatorname{mr}(G, H)$ of $G$ and $H$ is the minimum integer $n$ such that every red-blue coloring of $K_{n}$ results in a monochromatic $G$ or a monochromatic $H$. Thus, the standard Ramsey number of $G$ and $H$ is bounded below by $\operatorname{mr}(G, H)$. The monochromatic Ramsey numbers of graphs belonging to some common classes of graphs are studied.

We also investigate another concept closely related to the standard Ramsey numbers and monochromatic Ramsey numbers of graphs. For a fixed integer $n \geq 3$, consider a nonempty subgraph $G$ of order at most $n$ containing no isolated vertices. Then $G$ is a common monochromatic subgraph of $K_{n}$ if every red-blue coloring of $K_{n}$ results in a monochromatic copy of


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$G$. Furthermore, $G$ is a maximal common monochromatic subgraph of $K_{n}$ if $G$ is a common monochromatic subgraph of $K_{n}$ that is not a proper subgraph of any common monochromatic subgraph of $K_{n}$. Let $\mathcal{S}(n)$ and $\mathcal{S}^{*}(n)$ be the sets of common monochromatic subgraphs and maximal common monochromatic subgraphs of $K_{n}$, respectively. Thus, $G \in \mathcal{S}(n)$ if and only if $R(G, G)=\operatorname{mr}(G, G) \leq n$. We determine the sets $\mathcal{S}(n)$ and $\mathcal{S}^{*}(n)$ for $3 \leq n \leq 8$. Keywords: Ramsey number, monochromatic Ramsey number, common monochromatic subgraph, maximal common monochromatic subgraph.


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## 1. Introduction

Graph coloring is said to be one of the most popular areas of graph theory. When restricting our attention to coloring edges, proper edge colorings of graphs have been studied extensively. In a proper edge coloring of a nonempty graph, no two adjacent edges are assigned the same color. The chromatic index of a graph is then the smallest number of colors required in a proper edge coloring of that graph. Of course, there are other interesting edge colorings that are not necessarily proper. In an edge-colored graph, where adjacent edges may be assigned the same color, a subgraph $G$ all of whose edges are assigned the same color is referred to as a monochromatic $G$ while a subgraph $H$ no two of whose edges are assigned the same color is referred to as a rainbow $H$. Therefore, a proper edge coloring of a graph is an edge coloring in which every subgraph that is isomorphic to a star is rainbow, whereas an edge coloring of a graph that is not proper is an edge coloring that results in a monochromatic $P_{3}$ as a subgraph.

Ramsey theory is one of the most studied areas of research in extremal graph theory, which often deals with determining the values of or bounds for Ramsey numbers of two graphs. In a red-blue coloring of a graph, every edge is colored red or blue. If the resulting edge-colored graph contains a subgraph $G$ every edge of which is colored red (blue), then this subgraph is called a red $G$ (a blue $G$ ). For two graphs $G$ and $H$, the Ramsey number $R(G, H)$ is the minimum positive integer $n$ for which every red-blue coloring of $K_{n}$ contains either a red $G$ or a blue $H$. In fact, for every $k$ graphs $G_{1}, G_{2}, \ldots, G_{k}$, it is known that there exist positive integers $n$ such that every edge coloring of $K_{n}$ with $k$ colors $1,2, \ldots, k$ produces a subgraph $G_{i}$ all of whose edges are colored $i$ for some $i \in\{1,2, \ldots, k\}$. Then the smallest positive integer $n$ with this property is the Ramsey number $R\left(G_{1}, G_{2}, \ldots, G_{k}\right)$ of the graphs $G_{1}, G_{2}, \ldots, G_{k}$. Ramsey numbers $R(G, H)$ and $R\left(G_{1}, G_{2}, \ldots, G_{k}\right)$ are defined for any nonempty graphs $G, H$ and $G_{1}, G_{2}, \ldots, G_{k}$ and have been investigated for many years.

In this paper, we study two concepts that arise when considering red-blue colorings of complete graphs. We refer to the book [1] for graph-theoretical notation and terminology not described in this paper. (In particular, we use $G+H$ and $G \vee H$ to denote the union and join of two vertex-disjoint graphs $G$ and $H$, respectively.) Also, see [2] for well-known Ramsey numbers, some of which are summarized below.

- For $1 \leq m \leq n, R\left(K_{1, m}, K_{1, n}\right)=m+n-\epsilon$, where $\epsilon=1$ if both $m$ and $n$ are even and $\epsilon=0$ otherwise.
- For $2 \leq m \leq n, R\left(P_{m}, P_{n}\right)=\lfloor m / 2\rfloor+n-1$.
- For $m, n \geq 3$,

$$
\begin{aligned}
& R\left(C_{3}, C_{n}\right)=\left\{\begin{array}{cl}
6 & \text { if } n=3, \\
2 n-1 & \text { if } n \geq 4 .
\end{array}\right. \\
& R\left(C_{4}, C_{n}\right)= \begin{cases}6 & \text { if } n=4, \\
7 & \text { if } n=3,5, \\
n+1 & \text { if } n \geq 6 .\end{cases} \\
& R\left(C_{m}, C_{n}\right)= \begin{cases}2 n-1 & \text { if } 5 \leq m \leq n \text { and } m \text { is odd }, \\
m / 2+n-1 & \text { if } 6 \leq m \leq n \text { and } \\
\max \{2 m-1, m / 2+n-1\} & \text { if } 6 \leq m<n \text { and } n \text { are even }, \\
m \text { is even while } n \text { is odd. }\end{cases}
\end{aligned}
$$

## 2. Monochromatic Ramsey Numbers: Definitions and Observations

The monochromatic Ramsey number $\operatorname{mr}(G, H)$ of two nonempty graphs $G$ and $H$ without isolated vertices is defined to be the minimum integer $n$ such that every red-blue edge coloring of $K_{n}$ results in either a monochromatic $G$ or a monochromatic $H$. Therefore, $\operatorname{mr}(G, G)=R(G, G)$. In fact, observe that

$$
\begin{equation*}
\min \{|V(G)|,|V(H)|\} \leq \operatorname{mr}(G, H) \leq \min \{R(G, G), R(H, H), R(G, H)\} \tag{1}
\end{equation*}
$$

and so $\operatorname{mr}(G, H)$ is defined for every pair $G, H$ of nonempty graphs.
Figure 1 shows all red-blue colorings of $K_{3}$ and $K_{4}$. Let us make some elementary observations.

Observation 1. (a) $\operatorname{mr}(G, H)=2$ if and only if $P_{2} \in\{G, H\}$.
(b) $\operatorname{mr}\left(G, P_{3}\right)=3$ for every nonempty graph $G$ unless $G=P_{2}$.
(c) Every red-blue coloring of $K_{4}$ contains either a monochromatic $C_{3}$ or a monochromatic $P_{4}$.





Figure 1. The red-blue colorings of $K_{3}$ and $K_{4}$.

Let $H_{1}$ be the graph obtained from the 4 -path $\left(v_{1}, v_{2}, v_{3}, v_{4}\right)$ by joining $v_{1}$ and $v_{3}$. As an example, we consider the three graphs $C_{3}, P_{4}$ and $H_{1}$. Since there is no 3 regular graph of order 7 , every red-blue coloring of $K_{7}$ contains a monochromatic $K_{1,4}$. Then there is a monochromatic $H_{1}$ of one color or a monochromatic $K_{4}$ of the other color. Since $C_{3}, P_{4} \subseteq H_{1} \subseteq K_{4}$, it follows that each of $R\left(C_{3}, P_{4}\right)$, $R\left(H_{1}, P_{4}\right)$ and $R\left(C_{3}, H_{1}\right)$ is at most 7 . Thus, the red-blue coloring of $K_{6}$ resulting in a red $K_{3,3}$ and a blue $2 C_{3}$ shows that

$$
R\left(C_{3}, P_{4}\right)=R\left(H_{1}, P_{4}\right)=R\left(C_{3}, H_{1}\right)=7 .
$$

Let us now determine the corresponding monochromatic Ramsey numbers. Note first that $\operatorname{mr}\left(C_{3}, H_{1}\right) \leq R\left(C_{3}, C_{3}\right)=6$ while $\operatorname{mr}\left(C_{3}, P_{4}\right), \operatorname{mr}\left(H_{1}, P_{4}\right) \leq R\left(P_{4}, P_{4}\right)$ $=5$ by (1). In fact, $\operatorname{mr}\left(C_{3}, H_{1}\right)=6$ as the red-blue coloring of $K_{5}$ in Figure 2 shows since neither $C_{3}$ nor $H_{1}$ is a subgraph of $C_{5}$. Similarly, the red-blue coloring


Figure 2. A red-blue coloring of $K_{5}$.
of $K_{4}$ inducing a red $C_{3}+K_{1}$ (the union of $C_{3}$ and $K_{1}$ ) and a blue $K_{1,3}$ shows that $\operatorname{mr}\left(H_{1}, P_{4}\right)=5$. Finally, Observation 1(c) implies that $\operatorname{mr}\left(C_{3}, P_{4}\right) \leq 4$ while the red-blue colorings of $K_{3}$ using both colors show that $\operatorname{mr}\left(C_{3}, P_{4}\right) \geq 4$. Hence,

$$
\operatorname{mr}\left(C_{3}, P_{4}\right)=4, \operatorname{mr}\left(H_{1}, P_{4}\right)=5, \operatorname{mr}\left(C_{3}, H_{1}\right)=6 .
$$

On the other hand, it can be similarly verified that

$$
\operatorname{mr}\left(K_{1,3}, C_{4}\right)=R\left(K_{1,3}, K_{1,3}\right)=R\left(C_{4}, C_{4}\right)=R\left(K_{1,3}, C_{4}\right)=6 .
$$

Thus, equality and strict inequality are both possible in (1).
We present a few more useful observations. First, if $G$ and $H$ are nonempty graphs and $G \subseteq H$, then there is a red-blue coloring of $K_{R(G, G)-1}$ that does not
contain a monochromatic $G$, which certainly does not contain a monochromatic $H$ either, while every red-blue coloring of $K_{R(G, G)}$ contains a monochromatic $G$. Therefore, $\operatorname{mr}(G, H)=\operatorname{mr}(G, G)=R(G, G)$ in this case. Also, if $F$ is a nonempty subgraph of $G$, then every red-blue coloring of $K_{\operatorname{mr}(G, H)}$ contains either a monochromatic $G$, which certainly contains a monochromatic $F$, or a monochromatic $H$. Thus, $\operatorname{mr}(F, H) \leq \operatorname{mr}(G, H)$. We summarize these facts as follows.

Observation 2. Let $G, H$ be a pair of nonempty graphs.
(a) If $G \subseteq H$, then $\operatorname{mr}(G, H)=\operatorname{mr}(G, G)=R(G, G)$.
(b) If $G^{\prime}$ and $H^{\prime}$ are nonempty subgraphs of $G$ and $H$, respectively, then $\operatorname{mr}\left(G^{\prime}, H^{\prime}\right) \leq \operatorname{mr}(G, H)$.

## 3. The Monochromatic Ramsey Numbers Involving Paths and Cycles

It is well-known that $R\left(P_{m}, P_{n}\right)=\lfloor m / 2\rfloor+n-1$ for integers $m$ and $n$ with $2 \leq$ $m \leq n$. Hence, the following is an immediate consequence of Observation 2(a).
Theorem 3. For integers $m$ and $n$ with $2 \leq m \leq n, \operatorname{mr}\left(P_{m}, P_{n}\right)=\lfloor 3 m / 2\rfloor-1$. Also, determining the exact value of the monochromatic Ramsey number of two nonempty graphs is possible when one of the two graphs is either $C_{3}$ or $C_{4}$.
Theorem 4. For every nonempty graph $G$ without isolated vertices,

$$
\operatorname{mr}\left(G, C_{3}\right)=\left\{\begin{array}{cl}
6 & \text { if } G \nsubseteq C_{5} \\
|V(G)| & \text { otherwise }
\end{array}\right.
$$

Proof. By Observation 1(a)(b), we assume that $G \notin\left\{P_{2}, P_{3}\right\}$. If $G$ is not a subgraph of $C_{5}$, then the red-blue coloring of $K_{5}$ shown in Figure 2 shows that $\operatorname{mr}\left(G, C_{3}\right) \geq 6$. Thus the result is immediate by (1) since $R\left(C_{3}, C_{3}\right)=6$.

Now assume that $G \subseteq C_{5}$. Since the coloring shown in Figure 2 is the only red-blue coloring of $K_{5}$ inducing no monochromatic triangles, it follows that $\operatorname{mr}\left(G, C_{3}\right) \leq 5$. On the other hand, since there are red-blue colorings of $K_{3}$ and $K_{4}$ resulting in no monochromatic triangles (see Figure 1), it follows that $\operatorname{mr}\left(G, C_{3}\right) \geq|V(G)|$ when $|V(G)|=4,5$. Finally, since the red-blue coloring of $K_{4}$ avoids monochromatic triangles if and only if it induces either (i) a red (blue) $C_{4}$ and a blue (red) $2 P_{2}$ or (ii) a red (blue) $P_{4}$ and a blue (red) $P_{4}$, it follows that $\operatorname{mr}\left(2 P_{2}, C_{3}\right)=\operatorname{mr}\left(P_{4}, C_{3}\right)=4$.

Theorem 5. For every nonempty graph $G$ without isolated vertices,

$$
\operatorname{mr}\left(G, C_{4}\right)= \begin{cases}2 & \text { if } G=P_{2} \\ 3 & \text { if } G=P_{3}, \\ 5 & \text { if } G \in\left\{P_{4}, P_{5}, 2 P_{2}, P_{2}+P_{3}\right\} \\ 6 & \text { otherwise }\end{cases}
$$

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Proof. We may assume that $G \notin\left\{P_{2}, P_{3}\right\}$. First, $\operatorname{mr}\left(G, C_{4}\right) \leq R\left(C_{4}, C_{4}\right)=6$ by (1). Note also that a red-blue coloring $c$ of $K_{5}$ induces no monochromatic 4 -cycles if and only if $c$ is either the coloring in Figure 2 or the coloring in Figure 3(a). If

(a)

(b)

Figure 3. A red-blue coloring of $K_{5}$ and the induced monochromatic subgraph $H_{2}$.
$c$ is the coloring in Figure 3(a), then $c$ results in a red $H_{2}$ and a blue $H_{2}$, where $H_{2}$ is the graph shown in Figure 3(b). Therefore, $\operatorname{mr}\left(G, C_{4}\right)=6$ if $G \nsubseteq C_{5}$ and $G \nsubseteq H_{2}$.

Suppose next that $G \subseteq C_{5}$. Since the coloring in Figure 3(a) shows that $\operatorname{mr}\left(C_{5}, C_{4}\right)=6$, suppose further that $G \subseteq P_{5}$. Then both colorings in Figures 2 and 3(a) contain a monochromatic $P_{5}$ and so $\operatorname{mr}\left(G, C_{4}\right) \leq 5$. Thus, $\operatorname{mr}\left(G, C_{4}\right)=5$ if $|V(G)|=5$. Furthermore, there is a red-blue coloring of $K_{4}$ that contains neither a monochromatic $P_{4}$ nor a monochromatic $2 P_{2}$ (see Figure 1) and so $\operatorname{mr}\left(P_{4}, C_{4}\right)=\operatorname{mr}\left(2 P_{2}, C_{4}\right)=5$ as well.

Finally, suppose that $G \subseteq H_{2}$ and $G \nsubseteq C_{5}$. Then $G \nsubseteq P_{5}$ and so either $C_{3} \subseteq G$ or $K_{1,3} \subseteq G$. Then the red-blue coloring of $K_{5}$ in Figure 2 shows that $\operatorname{mr}\left(G, C_{4}\right)=6$.

Corollary 6. $\operatorname{mr}\left(C_{3}, C_{5}\right)+1=\operatorname{mr}\left(C_{4}, C_{5}\right)=6$ while $\operatorname{mr}\left(C_{3}, C_{n}\right)=\operatorname{mr}\left(C_{4}, C_{n}\right)=$ 6 for every integer $n \geq 3$ and $n \neq 5$.

Now let us suppose that $m, n \geq 5$. What is $\operatorname{mr}\left(C_{m}, C_{n}\right)$ then?
Theorem 7. For integers $m$ and $n$ with $5 \leq m \leq n$,

$$
\operatorname{mr}\left(C_{m}, C_{n}\right)=\left\{\begin{array}{cl}
3 m / 2-1 & \text { if } m \text { is even, } \\
2 m-1 & \text { if } m \text { is odd and either } n \text { is odd or } n \geq 2 m .
\end{array}\right.
$$

If $m$ is odd and $n$ is even with $m+1 \leq n \leq 2 m-2$, then

$$
\begin{equation*}
m+n / 2-1 \leq \operatorname{mr}\left(C_{m}, C_{n}\right) \leq \min \{2 m-1,3 n / 2-1\} . \tag{2}
\end{equation*}
$$

Proof. We consider the following two cases, according to the parity of $m$.
Case 1. $m$ is even. Since the largest monochromatic cycle in the red-blue coloring of $K_{3 m / 2-2}$ inducing a red $K_{m / 2-1}+K_{m-1}$ and a blue $K_{m / 2-1, m-1}$ is $C_{m-1}$, it follows that $\operatorname{mr}\left(C_{m}, C_{n}\right)>3 m / 2-2$. Then the result is immediate by (1) since $m \geq 6$ and $R\left(C_{m}, C_{m}\right)=3 m / 2-1$.

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Case 2. $m$ is odd. Then $\operatorname{mr}\left(C_{m}, C_{n}\right) \leq R\left(C_{m}, C_{m}\right)=2 m-1$. If $n$ is odd or $n \geq 2 m$, then the red-blue coloring of $K_{2 m-2}$ inducing a red $2 K_{m-1}$ and a blue $K_{m-1, m-1}$ shows that $\operatorname{mr}\left(C_{m}, C_{n}\right) \geq 2 m-1$. If $n$ is even and $m+1 \leq n \leq$ $2 m-2$, on the other hand, then the red-blue coloring of $K_{m+n / 2-2}$ inducing a red $K_{m-1}+K_{n / 2-1}$ and a blue $K_{m-1, n / 2-1}$ shows that $\operatorname{mr}\left(C_{m}, C_{n}\right) \geq m+n / 2-1$, while $\operatorname{mr}\left(C_{m}, C_{n}\right) \leq R\left(C_{n}, C_{n}\right)=3 n / 2-1$.

Let us also consider $\operatorname{mr}\left(C_{m}, P_{n}\right)$. Note that $\operatorname{mr}\left(C_{m}, P_{n}\right) \leq \operatorname{mr}\left(C_{m}, C_{n}\right)$.
Theorem 8. For integers $m \geq 5$ and $n \geq 4$,

$$
\operatorname{mr}\left(C_{m}, P_{n}\right)=\left\{\begin{array}{cl}
\lfloor 3 n / 2\rfloor-1 & \text { if } n \leq m \\
3 m / 2-1 & \text { if } n \geq m+1 \text { and } m \text { is even } \\
2 m-1 & \text { if } n \geq 2 m-1 \text { and } m \text { is odd }
\end{array}\right.
$$

If $m$ is odd and $m+1 \leq n \leq 2 m-2$, then

$$
\begin{equation*}
m+\lfloor n / 2\rfloor-1 \leq \operatorname{mr}\left(C_{m}, P_{n}\right) \leq \min \{2 m-1,\lfloor 3 n / 2\rfloor-1\} \tag{3}
\end{equation*}
$$

Proof. If $n \leq m$, then $\operatorname{mr}\left(C_{m}, P_{n}\right)=R\left(P_{n}, P_{n}\right)=\lfloor 3 n / 2\rfloor-1$. Hence, let us suppose that $5 \leq m<n$. If $m$ is even, then the red-blue coloring of $K_{3 m / 2-2}$ inducing a red $K_{m / 2-1}+K_{m-1}$ and a blue $K_{m / 2-1, m-1}$ shows that $\operatorname{mr}\left(C_{m}, P_{n}\right)>$ $3 m / 2-2$. Then the result follows since $\operatorname{mr}\left(C_{m}, P_{n}\right) \leq \operatorname{mr}\left(C_{m}, C_{n}\right)=3 m / 2-1$. If $m$ is odd, then $\operatorname{mr}\left(C_{m}, P_{n}\right) \leq R\left(C_{m}, C_{m}\right)=2 m-1$. In the red-blue coloring of $K_{2 m-2}$ inducing a red $2 K_{m-1}$ and a blue $K_{m-1, m-1}$, a longest monochromatic path is of order $2 m-2$ and a longest monochromatic odd cycle is of order $m-2$. Therefore, if $m$ is odd and $n \geq 2 m-1$, then $\operatorname{mr}\left(C_{m}, P_{n}\right)=2 m-1$.

It remains to consider the case where $m$ is odd and $m+1 \leq n \leq 2 m-2$. In the red-blue coloring of $K_{m+\lfloor n / 2\rfloor-2}$ resulting in a red $K_{m-1}+K_{\lfloor n / 2\rfloor-1}$ and a blue $K_{m-1,\lfloor n / 2\rfloor-1}$, a longest monochromatic path is of order $n-1$ and a longest monochromatic odd cycle is of order $m-2$. Hence, $m+\lfloor n / 2\rfloor-1 \leq \operatorname{mr}\left(C_{m}, P_{n}\right) \leq$ $\min \{2 m-1,\lfloor 3 n / 2\rfloor-1\}$ by (1).

It turns out that the values $\operatorname{mr}\left(C_{m}, C_{n}\right)$ and $\operatorname{mr}\left(C_{m}, P_{n}\right)$ equal the lower bounds in (2) and (3), respectively, for $m=5$.

Theorem 9. $\operatorname{mr}\left(C_{5}, P_{n}\right)=\lfloor n / 2\rfloor+4$ for $6 \leq n \leq 8$.
Proof. First we prove that $\operatorname{mr}\left(C_{5}, P_{7}\right) \leq 7$ by verifying that every red-blue coloring of $K_{7}$ induces a monochromatic $C_{5}$ or a monochromatic $P_{7}$. Assume, to the contrary, that there is a red-blue coloring of $K_{7}$ with $V\left(K_{7}\right)=\left\{v_{1}, v_{2}, \ldots, v_{7}\right\}$ with neither a monochromatic $C_{5}$ nor a monochromatic $P_{7}$. We may assume that $\left(v_{1}, v_{2}, v_{3}, v_{4}, v_{1}\right)$ is a red 4-cycle since $R\left(C_{4}, C_{4}\right)=6$. Assume further that $v_{1} v_{5}$ is a red edge since there is no blue $P_{7}$. Then $v_{2} v_{5}$ and $v_{4} v_{5}$ are blue edges to avoid red 5-cycles.

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If $v_{6} v_{7}$ is red, then $v_{2} v_{6}, v_{2} v_{7}$ and $v_{4} v_{7}$ must be all blue edges. However then, a red $P_{7}$ or a blue $C_{5}$ results regardless of the color of the edge $v_{5} v_{6}$, which is a contradiction. Hence, $v_{6} v_{7}$ must be a blue edge. Then at least one of $v_{2} v_{6}$ and $v_{4} v_{7}$, say the former, must be red since a blue $C_{5}$ results otherwise. This in turn implies that the edges $v_{1} v_{6}, v_{3} v_{6}$ and $v_{5} v_{7}$ are all blue. Since a red $C_{5}$ or a blue $P_{7}$ is produced if $v_{3} v_{7}$ and $v_{4} v_{7}$ are assigned the same color, let us assume, by the symmetry, that $v_{3} v_{7}$ is red and $v_{4} v_{7}$ is blue. Then both $v_{4} v_{6}$ and $v_{5} v_{6}$ are blue to avoid red Hamiltonian paths. However then, there is a monochromatic $C_{5}$ regardless of the color of the edge $v_{2} v_{7}$. Thus, there is no such coloring and the result follows by Observation 2(b) and (3).

To verify that $\operatorname{mr}\left(C_{5}, P_{8}\right) \leq 8$, let there be given a red-blue coloring of $K_{8}$ with $V\left(K_{8}\right)=\left\{v_{1}, v_{2}, \ldots, v_{8}\right\}$ avoiding monochromatic 5 -cycles. From the previous result, we may assume that $\left(v_{1}, v_{2}, \ldots, v_{7}\right)$ is a red $P_{7}$. If this cannot be extended to a red $P_{8}$, then $v_{1} v_{8}$ and $v_{7} v_{8}$ are blue. Also, the edges $v_{1} v_{5}$, $v_{2} v_{6}$ and $v_{3} v_{7}$ are blue, which in turn implies that $v_{3} v_{5}$ is red and $v_{1} v_{6}$ is blue. However then, this still produces a monochromatic $C_{5}$ regardless of the color of $v_{2} v_{7}$. This implies that there must be a red $P_{8}$, that is, $\operatorname{mr}\left(C_{5}, P_{8}\right) \leq 8$ and so $\operatorname{mr}\left(C_{5}, P_{8}\right)=8$ by (3).

Theorem 10. $\operatorname{mr}\left(C_{5}, C_{n}\right)=n / 2+4$ for $n=6,8$.
Proof. Let us first show that $\operatorname{mr}\left(C_{5}, C_{6}\right) \leq 7$. Consider a red-blue coloring of $K_{7}$ with $V\left(K_{7}\right)=\left\{v_{1}, v_{2}, \ldots, v_{7}\right\}$ containing no monochromatic 5 -cycles. Then we may assume that $\left(v_{1}, v_{2}, \ldots, v_{7}\right)$ is a red $P_{7}$ by Theorem 9 . If there is no red $C_{6}$, then $v_{1} v_{6}$ and $v_{2} v_{7}$ are blue edges. Also, $v_{1} v_{5}, v_{2} v_{6}$ and $v_{3} v_{7}$ are blue since there is no red $C_{5}$. Then there is a blue $C_{6}$ or the edge $v_{3} v_{5}$ is red. If the latter occurs, however, then a monochromatic $C_{5}$ results no matter how the edges $v_{1} v_{4}$ and $v_{4} v_{7}$ are colored. This verifies the claim and so $\operatorname{mr}\left(C_{5}, C_{6}\right)=7$ by (2).

To show that $\operatorname{mr}\left(C_{5}, C_{8}\right) \leq 8$, consider a red-blue coloring of $K_{8}$ with $V\left(K_{8}\right)=\left\{v_{1}, v_{2}, \ldots, v_{8}\right\}$ containing no monochromatic 5 -cycles. Assume further that $\left(v_{1}, v_{2}, \ldots, v_{8}\right)$ is a red $P_{8}$ and the edges $v_{1} v_{5}, v_{1} v_{8}, v_{2} v_{6}, v_{3} v_{7}$ and $v_{4} v_{8}$ are all blue. We consider the following two cases.

Case 1. $v_{1} v_{3}$ or $v_{6} v_{8}$, say the former, is a red edge. Then $v_{2} v_{5}$ is a blue edge, which implies that $v_{6} v_{8}$ is red and $v_{4} v_{7}$ is blue. Thus, $\left(v_{3}, v_{7}, v_{4}, v_{8}, v_{1}, v_{5}, v_{2}, v_{6}\right)$ is a blue $P_{8}$. Let us therefore assume that $v_{3} v_{6}$ is red. Then either both $v_{2} v_{4}$ and $v_{5} v_{7}$ are red, creating a red $C_{8}$, or there is a blue $C_{5}$.

Case 2. Both $v_{1} v_{3}$ and $v_{6} v_{8}$ are blue edges. Then in order to avoid a monochromatic $C_{5}$, observe that $v_{2} v_{5}$ is red and $v_{1} v_{7}$ is blue, forcing $v_{3} v_{6}$ to be red. This then implies that $v_{2} v_{4}$ and $v_{5} v_{7}$ are blue. Thus, both $v_{1} v_{4}$ and $v_{5} v_{8}$ must be red, creating a red $C_{8}$.

The result is now immediate again by (2).

## 4. Common Monochromatic Subgraphs: Definitions and Observations

Let $G$ be a nonempty graph without isolated vertices. The Ramsey number $R(G, G)$ is the smallest integer $n$ such that every red-blue coloring of $K_{n}$ results in a monochromatic subgraph that is isomorphic to $G$ (and we have seen that $R(G, G)=\operatorname{mr}(G, G))$.

There is another approach to study red-blue colorings of $K_{n}$ and resulting monochromatic subgraphs. For a fixed integer $n \geq 3$, consider an arbitrary redblue coloring of $K_{n}$. Then for each nonempty subgraph $G$ of $K_{n}$ without isolated vertices, either there is a monochromatic $G$ or there is no monochromatic $G$. A natural question to ask is, then, which nonempty subgraph $G \subseteq K_{n}$ without isolated vertices has the property that every red-blue coloring of $K_{n}$ results in a monochromatic $G$ ?

Let $G$ be a nonempty graph without isolated vertices. For a fixed integer $n \geq 3$, the graph $G$ is a common monochromatic subgraph of $K_{n}$ if every red-blue coloring of $K_{n}$ results in a monochromatic copy of $G$. The following is therefore an immediate observation.

Observation 11. A nonempty graph $G$ without isolated vertices is a common monochromatic subgraph of $K_{n}$ if and only if $R(G, G) \leq n$.

Suppose that $G$ and $G^{\prime}$ are nonempty graphs without isolated vertices with $G^{\prime} \subseteq$ $G$. Then certainly $R\left(G^{\prime}, G^{\prime}\right) \leq R(G, G)$. Hence, Observation 11 implies the following.

Observation 12. Suppose that $G$ and $G^{\prime}$ are nonempty graphs without isolated vertices with $G^{\prime} \subseteq G$. If $G$ is a common monochromatic subgraph of $K_{n}$, then so is $G^{\prime}$.

Observation 12 suggests an additional important concept. For a fixed integer $n \geq 3$, a nonempty graph $G$ without isolated vertices is a maximal common monochromatic subgraph of $K_{n}$ if $G$ is a common monochromatic subgraph of $K_{n}$ that is not a proper subgraph of any common monochromatic subgraph of $K_{n}$.

For each integer $n \geq 3$, let us denote the sets of common monochromatic subgraphs and maximal common monochromatic subgraphs of $K_{n}$ by $\mathcal{S}(n)$ and $\mathcal{S}^{*}(n)$, respectively. Thus, $\mathcal{S}^{*}(n) \subseteq \mathcal{S}(n)$. Also, $\mathcal{S}(n) \neq \emptyset$ since $P_{2}$ is a common monochromatic subgraph of $K_{3}$ and

$$
\begin{equation*}
\mathcal{S}(3) \subseteq \mathcal{S}(4) \subseteq \mathcal{S}(5) \subseteq \cdots \subseteq \mathcal{S}(n) \subseteq \mathcal{S}(n+1) \subseteq \cdots \tag{4}
\end{equation*}
$$

(We will soon see that the sets $\mathcal{S}^{*}(n)$ for $n \geq 3$ do not have a similar nesting property.) Note also that a nonempty graph $G$ without isolated vertices belongs to the set $\mathcal{S}(n)$ if and only if $G$ is a subgraph of a graph belonging to $\mathcal{S}^{*}(n)$. Therefore,

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$\mathcal{S}^{*}(n) \neq \emptyset$. Moreover, once one of the sets $\mathcal{S}(n)$ and $\mathcal{S}^{*}(n)$ is determined, the other can be obtained as well.

For each $n \geq 3$, there are red-blue colorings of $K_{n}$ with $\lceil n(n-1) / 4\rceil$ red edges and $\lfloor n(n-1) / 4\rfloor$ blue edges. Therefore, if $G$ is a common monochromatic subgraph of $K_{n}$, then $|E(G)| \leq\lceil n(n-1) / 4\rceil$. From this, $\mathcal{S}(3) \subseteq\left\{P_{2}, P_{3}\right\}$ and it is straightforward to verify that each of $P_{2}$ and $P_{3}$ is a common monochromatic subgraph of $K_{3}$. Thus $\mathcal{S}(3)=\left\{P_{2}, P_{3}\right\}$ and $\mathcal{S}^{*}(3)=\left\{P_{3}\right\}$. Furthermore, by examining the red-blue colorings of $K_{4}$ in Figure 1, it follows that $\mathcal{S}(3)=\mathcal{S}(4)=$ $\left\{P_{2}, P_{3}\right\}$ and $\mathcal{S}^{*}(3)=\mathcal{S}^{*}(4)=\left\{P_{3}\right\}$.

## 5. Common Monochromatic Subgraphs of Complete Graphs of Small Order

In order to discuss the sets $\mathcal{S}(n)$ and $\mathcal{S}^{*}(n)$ for $n \geq 5$, let us state a few more results on common monochromatic subgraphs of $K_{n}$. First, recall for every integer $m \geq 2$ that

$$
R\left(K_{1, m}, K_{1, m}\right)=\left\{\begin{array}{cl}
2 m-1 & \text { if } m \text { is even, } \\
2 m & \text { if } m \text { is odd. }
\end{array}\right.
$$

For $K_{n}$, therefore,
(i) $K_{1, n / 2}$ is a common monochromatic subgraph while $K_{1, n / 2+1}$ is not when $n$ is even,
(ii) $K_{1,(n-1) / 2}$ is a common monochromatic subgraph while $K_{1,(n+1) / 2}$ is not when $n \equiv 1(\bmod 4)$ and
(iii) $K_{1,(n+1) / 2}$ is a common monochromatic subgraph while $K_{1,(n+3) / 2}$ is not when $n \equiv 3(\bmod 4)$. Also recall for every integer $m \geq 3$ that

$$
\begin{aligned}
& R\left(P_{m}, P_{m}\right)=\lfloor 3 m / 2\rfloor-1, \\
& R\left(C_{m}, C_{m}\right)=\left\{\begin{array}{cl}
6 & \text { if } m=3,4, \\
2 m-1 & \text { if } m \text { is odd and } m \geq 5 \\
3 m / 2-1 & \text { if } m \text { is even and } m \geq 6
\end{array}\right.
\end{aligned}
$$

Thus, the following is a consequence of Observation 12, providing us with some necessary conditions for a graph to be a common monochromatic subgraph of $K_{n}$.

Proposition 13. Suppose that $n \geq 3$ and $G \in \mathcal{S}(n)$.
(a) The maximum degree $\Delta(G)$ of $G$ satisfies the following.

$$
\Delta(G) \leq \begin{cases}\lceil n / 2\rceil & \text { if } n \equiv 3(\bmod 4), \\ \lfloor n / 2\rfloor & \text { otherwise. }\end{cases}
$$

(b) Every path in $G$ is of order at most $2 n / 3+1$. Also, every subgraph of $P_{\lfloor 2 n / 3+1\rfloor}$ without isolated vertices belongs to $\mathcal{S}(n)$.
(c) $G$ is acyclic for $3 \leq n \leq 5$. For $n \geq 6$, every odd cycle in $G$ is of order at most $(n+1) / 2$ while every even cycle in $G$ is of order at most $(2 n+2) / 3$. Also, every odd cycle of order at most $(n+1) / 2$ and every even cycle of order at most $(2 n+2) / 3$ belong to $\mathcal{S}(n)$.

Let $n \geq 4$. Consider the red-blue coloring of $K_{n}$ resulting in a red $K_{1}+K_{n-1}$ and a blue $K_{1, n-1}$. Also, for each integer $i$ with $1 \leq i \leq\lfloor n / 2\rfloor$, consider the red-blue coloring of $K_{n}$ resulting in a red $K_{i}+K_{n-i}$ and a blue $K_{i, n-i}$. These colorings of $K_{n}$ give us the following.

Proposition 14. Suppose that $n \geq 4$ and $G \in \mathcal{S}(n)$.
(a) $G$ is not a spanning subgraph of $K_{n}$.
(b) If $G$ is not bipartite, then $G \subseteq K_{i}+K_{n-i}$ for $1 \leq i \leq\lfloor n / 2\rfloor$.

We have seen that $\mathcal{S}(3)=\mathcal{S}(4)=\left\{P_{2}, P_{3}\right\}$ and $\mathcal{S}^{*}(3)=\mathcal{S}^{*}(4)=\left\{P_{3}\right\}$. We next determine the sets $\mathcal{S}(5)$ and $\mathcal{S}^{*}(5)$. Note first that $P_{4} \in \mathcal{S}(5)$. Also, if $G \in \mathcal{S}(5)$, then $G$ is acyclic and $\Delta(G) \leq 2$ by Proposition 13. It then follows by Observation 12 that

$$
\left\{P_{2}, P_{3}, P_{4}, 2 P_{2}\right\} \subseteq \mathcal{S}(5) \subseteq\left\{P_{2}, P_{3}, P_{4}, P_{5}, 2 P_{2}, P_{2}+P_{3}\right\}
$$

Therefore, $\mathcal{S}(5)=\left\{P_{2}, P_{3}, P_{4}, 2 P_{2}\right\}$ by Proposition 14(a). Furthermore, every graph in $\mathcal{S}(5)$ is a subgraph of $P_{4}$ and so $\mathcal{S}^{*}(5)=\left\{P_{4}\right\}$.

Thus far we have found that $\mathcal{S}^{*}(3)=\mathcal{S}^{*}(4)=\left\{P_{3}\right\}$ and $\mathcal{S}^{*}(5)=\left\{P_{4}\right\}$ (and so $\left.\mathcal{S}^{*}(4) \nsubseteq \mathcal{S}^{*}(5)\right)$. Let us next consider $\mathcal{S}^{*}(6)$ and $\mathcal{S}^{*}(7)$. In particular, we show that $\mathcal{S}^{*}(6)=\left\{C_{3}, G_{1}\right\}$ and $\mathcal{S}^{*}(7)=\left\{C_{3}+P_{2}, G_{2}, G_{3}\right\}$, where the graphs $G_{1}, G_{2}$ and $G_{3}$ are shown in Figure 4. We first show that both $C_{3}$ and $G_{1}$ are common

$G_{1}$

$G_{2}$

$G_{3}$

Figure 4. The graphs $G_{1}, G_{2}$ and $G_{3}$.
monochromatic subgraphs of $K_{6}$.
Lemma 15. $\left\{C_{3}, G_{1}\right\} \subseteq \mathcal{S}(6)$.
Proof. That $C_{3} \in \mathcal{S}(6)$ is immediate since $R\left(C_{3}, C_{3}\right)=6$. To see that $G_{1} \in \mathcal{S}(6)$, consider an arbitrary red-blue coloring of $K_{6}$ with $V\left(K_{6}\right)=\left\{v_{1}, v_{2}, \ldots, v_{6}\right\}$. Since
$R\left(C_{4}, C_{4}\right)=6$, we may assume, without loss of generality, that $\left(v_{1}, v_{2}, v_{3}, v_{4}, v_{1}\right)$ is a red 4 -cycle, say. Then either there is a red $G_{1}$ or the edges $v_{i} v_{5}$ and $v_{i} v_{6}$ for $1 \leq i \leq 4$ are all blue, creating a blue $G_{1}$.

Theorem 16. $\mathcal{S}^{*}(6)=\left\{C_{3}, G_{1}\right\}$.
Proof. First we show that $C_{3} \in \mathcal{S}^{*}(6)$. If this is not the case, then there is a common monochromatic subgraph $G$ of $K_{6}$ containing $C_{3}$ as a proper subgraph. Then $G$ is certainly not bipartite and so $G \subseteq 2 K_{3}$ by Proposition 14(b). Furthermore, $G$ is not a spanning subgraph of $K_{6}$ by Proposition 14(a). Thus, $G=C_{3}+P_{2}$. However, this is impossible as the red-blue coloring of $K_{6}$ inducing a red $2 K_{1}+K_{4}$ and a blue $K_{1,1,4}$ contains no monochromatic copy of $C_{3}+P_{2}$. Thus, $C_{3}$ is a maximal common monochromatic subgraph of $K_{6}$. In fact, $C_{3}$ is the only common monochromatic subgraph of $K_{6}$ that is not bipartite.

Next we show that $G_{1} \in \mathcal{S}^{*}(6)$ by verifying that $G_{1}$ is the only common monochromatic subgraph of $K_{6}$ containing $C_{4}$ as a proper subgraph. Since $C_{4}+P_{2}$ is a spanning subgraph of $K_{6}$ while $K_{1,1,2}$ is not bipartite, if $G \in \mathcal{S}(6)$ and $G$ contains $C_{4}$ as a proper subgraph, then $G$ must contain $G_{1}$ as a spanning subgraph and $G$ must be bipartite. Therefore, if $G \neq G_{1}$, then $G=K_{2,3}$. However, the red-blue coloring of $K_{6}$ inducing a red $P_{2} \square C_{3}$ (the cartesian product of $P_{2}$ and $C_{3}$ ) and a blue $C_{6}$ shows that this is impossible. Thus, $C_{4}$ and $G_{1}$ are the only common monochromatic subgraphs of $K_{6}$ that are bipartite and not acyclic. Furthermore, $G_{1} \in \mathcal{S}^{*}(6)$.

Finally, every forest of order at most 5 without isolated vertices is either $K_{1,4}$, which is not a common monochromatic subgraph of $K_{6}$ by Proposition 13(a), or a proper subgraph of $G_{1}$. Hence, no forest can be a maximal common monochromatic subgraph of $K_{6}$. This completes the proof.

By Theorem 16, we see that $G \in \mathcal{S}(6)$ if and only if $G$ is a nonempty graph without isolated vertices and either $G \subseteq C_{3}$ or $G \subseteq G_{1}$. Thus,

$$
\mathcal{S}(6)=\mathcal{S}^{*}(6) \cup\left\{P_{2}, P_{3}, P_{4}, P_{5}, 2 P_{2}, P_{2}+P_{3}, K_{1,3}, S_{2,3}, C_{4}\right\},
$$

where $S_{2,3}$ is the double star of order 5. (A double star is a tree whose diameter equals 3.)

As before, in order to show that $\mathcal{S}^{*}(7)=\left\{C_{3}+P_{2}, G_{2}, G_{3}\right\}$, we first show that the three graphs are common monochromatic subgraphs of $K_{7}$.

Lemma 17. $\left\{C_{3}+P_{2}, G_{2}, G_{3}\right\} \subseteq \mathcal{S}(7)$.
Proof. Let there be given a red-blue coloring of $K_{7}$ with $V\left(K_{7}\right)=\left\{v_{1}, v_{2}, \ldots, v_{7}\right\}$. Since $R\left(C_{3}, C_{3}\right)=6$, assume that ( $v_{1}, v_{2}, v_{3}, v_{1}$ ) is a red 3 -cycle.

If there is no red $C_{3}+P_{2}$, then the set $\left\{v_{4}, v_{5}, v_{6}, v_{7}\right\}$ induces a blue $K_{4}$. If one of $v_{1} v_{4}$ and $v_{2} v_{4}$ is blue, then a blue $C_{3}+P_{2}$ results. Otherwise, both $v_{1} v_{4}$
and $v_{2} v_{4}$ are red and a monochromatic $C_{3}+P_{2}$ is produced regardless of the color of the edge $v_{3} v_{5}$.

Next, if there is no red $G_{2}$, then at least one of the edges joining two vertices in the set $\left\{v_{4}, v_{5}, v_{6}, v_{7}\right\}$, say $v_{4} v_{5}$, is a blue edge. If one of $v_{1} v_{4}$ and $v_{1} v_{5}$ is red, then there is a red $G_{2}$. Otherwise, both $v_{1} v_{4}$ and $v_{1} v_{5}$ are blue and a monochromatic $G_{2}$ is produced regardless of the color of $v_{2} v_{4}$.

Finally, we show that $G_{3}$ is a common monochromatic subgraph of $K_{7}$. Let there be given a red-blue coloring of $K_{7}$. Suppose first that there is a monochromatic $K_{1,5}$, say $v_{i} v_{6}$ is red for $1 \leq i \leq 5$. If the set $\left\{v_{1}, v_{2}, \ldots, v_{5}\right\}$ induces a blue $K_{5}$, then either both $v_{1} v_{7}$ and $v_{2} v_{7}$ are red, producing a red $G_{3}$, or there is a blue $G_{3}$. Otherwise, we may assume that $v_{1} v_{2}$ is red. If there is no red $G_{3}$, then (i) at least one of $v_{1} v_{7}$ and $v_{2} v_{7}$ is blue and (ii) $v_{1} v_{i}$ and $v_{2} v_{i}$ are blue for $3 \leq i \leq 5$. This produces a blue $G_{3}$.

Since there is no 3 -regular graph of order 7 , we may now assume that $v_{i} v_{7}$ is red for $1 \leq i \leq 4$ while $v_{5} v_{7}$ and $v_{6} v_{7}$ are blue. If every edge joining a vertex in $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ and a vertex in $\left\{v_{5}, v_{6}\right\}$ is blue, then a blue $G_{3}$ is produced. Hence, assume that $v_{1} v_{5}$ is red. Then $v_{i} v_{5}$ is blue for $2 \leq i \leq 4$ if there is no red $G_{3}$. This in turn implies that either one of $v_{2} v_{6}$ and $v_{3} v_{6}$ is blue, producing a blue $G_{3}$, or there is a red $G_{3}$.

Theorem 18. $\mathcal{S}^{*}(7)=\left\{C_{3}+P_{2}, G_{2}, G_{3}\right\}$.
Proof. First note that $3 P_{2} \notin \mathcal{S}(7)$ as the red-blue coloring of $K_{7}$ inducing a red $2 K_{1}+K_{5}$ and a blue $K_{1,1,5}$ shows. Therefore, there is no common monochromatic subgraph of $K_{7}$ containing $3 P_{2}$ as a subgraph by Observation 12. Also, this coloring shows that $C_{3}+P_{3} \notin \mathcal{S}(7)$.

Let us first assume that $G$ is a common monochromatic subgraph of $K_{7}$ that is not bipartite. Therefore, $C_{3} \subseteq G \subseteq K_{3}+K_{4}, G \neq C_{3}+P_{3}$ and $3 P_{2} \nsubseteq G$. Also, $G$ is not a spanning subgraph of $K_{7}$ and so either $G \in\left\{C_{3}, C_{3}+P_{2}, G_{2}\right\}$ or $K_{1,1,2} \subseteq G$. However, the latter case is impossible as the red-blue coloring of $K_{7}$ in Figure 5 shows. Hence, Lemma 17 implies that $G$ is a common monochromatic


Figure 5. A red-blue coloring of $K_{7}$.
subgraph of $K_{7}$ that is not bipartite if and only if $G \in\left\{C_{3}, C_{3}+P_{2}, G_{2}\right\}$, that is,

$$
\left\{C_{3}+P_{2}, G_{2}\right\} \subseteq \mathcal{S}^{*}(7)
$$

Next we assume that $G$ is a common monochromatic subgraph of $K_{7}$ that is bipartite. If $G$ is not acyclic, then $C_{4}$ is a subgraph of $G$. Since $G$ is a bipartite graph of order at most 6 and $3 P_{2} \nsubseteq G$, it follows that either (i) $G \in\left\{C_{4}, G_{1}, G_{3}\right\}$ or (ii) $K_{2,3} \subseteq G$ or (iii) $G=G_{4}$, which is the graph shown in Figure 6. However,


Figure 6. The graph $G_{4}$.
the red-blue coloring of $K_{7}$ in Figure 5 shows that (ii) does not occur. Similarly, (iii) does not occur as the red-blue coloring of $K_{7}$ inducing a red $K_{1} \vee 2 K_{3}$ (the join of $K_{1}$ and $2 K_{3}$ ) and a blue $K_{1}+K_{3,3}$ shows. Thus, $G$ is a common monochromatic subgraph of $K_{7}$ containing $C_{4}$ as a subgraph if and only if $G \in\left\{C_{4}, G_{1}, G_{3}\right\}$, which implies that $G_{3} \in \mathcal{S}^{*}(7)$.

Finally, let $G$ be a forest of order at most 6 without isolated vertices. Therefore, if $G \in \mathcal{S}(7)$, then $\Delta(G) \leq 4$ and $3 P_{2} \nsubseteq G$ and so either $G \subseteq G_{3}$ or $G=S_{3,3}$ (the double star of order 6 whose maximum degree equals 3 ). However, the red-blue coloring of $K_{7}$ inducing a red $K_{2}+K_{5}$ and a blue $K_{2,5}$ contains no monochromatic copy of $S_{3,3}$, which implies that $G \subseteq G_{3}$. Therefore, there is no acyclic maximal common monochromatic subgraph of $K_{7}$. This completes the proof.

We see from Proposition 13 that $P_{6}, C_{6} \in \mathcal{S}(8)$ while $P_{7}, C_{7} \notin \mathcal{S}(8)$. For $K_{8}$, in fact, while there is no common monochromatic subgraph of order 8 by Proposition 14(a), there is no connected common monochromatic subgraph of order 7. The following two results show why.

Proposition 19. For each integer $n \geq 8$, no tree of order $n-1$ is a common monochromatic subgraph of $K_{n}$.

Proof. If $T$ is a tree of order $n-1 \geq 7$, then there is a unique integer $i$ with $1 \leq i \leq\lfloor(n-1) / 2\rfloor$ such that $T$ is a spanning subgraph of $K_{i, n-1-i}$. If $1 \leq i \leq$ $\lfloor n / 2\rfloor-2$, then the red-blue coloring of $K_{n}$ with a red $K_{\lfloor n / 2\rfloor}+K_{\lceil n / 2\rceil}$ and a blue $K_{\lfloor n / 2\rfloor,\lceil n / 2\rceil}$ shows that $T \notin \mathcal{S}(n)$. Similarly, if $\lfloor n / 2\rfloor-1 \leq i \leq\lfloor(n-1) / 2\rfloor$, then $T \notin \mathcal{S}(n)$ as the red-blue coloring of $K_{n}$ with a red $K_{2}+K_{n-2}$ and a blue $K_{2, n-2}$ shows.

We therefore obtain the following by Observation 12 and Propositions 14(a) and 19 for $n \geq 8$.

Corollary 20. Suppose that $n \geq 8$ and $G \in \mathcal{S}(n)$. If $G$ is connected, then $|V(G)| \leq n-2$.

On Monochromatic Subgraphs of Edge-colored ...

We are prepared to determine the set $\mathcal{S}^{*}(8)$. We first present two lemmas.
Lemma 21. $\left\{C_{3}+P_{3}, K_{1,3}+P_{3}, G_{2}+P_{2}, G_{3}\right\} \subseteq \mathcal{S}(8)$.
Proof. Clearly $G_{3} \in \mathcal{S}(7) \subseteq \mathcal{S}(8)$. Let there be given an arbitrary red-blue coloring of $K_{8}$ with $V\left(K_{8}\right)=\left\{v_{1}, v_{2}, \ldots, v_{8}\right\}$. Since $R\left(C_{6}, C_{6}\right)=8$, we may assume that $C=\left(v_{1}, v_{2}, \ldots, v_{6}, v_{1}\right)$ is a red 6 -cycle.

If there is a red edge joining a vertex belonging to $C$ and one of $v_{7}$ and $v_{8}$, then there is a red $K_{1,3}+P_{3}$. Otherwise, there is a blue $K_{1,3}+P_{3}$. Also, if there is neither a red $C_{3}+P_{3}$ nor a red $G_{2}+P_{2}$, then $\left(v_{1}, v_{3}, v_{5}, v_{1}\right)$ and $\left(v_{2}, v_{4}, v_{6}, v_{2}\right)$ are blue triangles. Furthermore, we may assume that $v_{1} v_{7}$ is blue. Therefore, blue copies of $C_{3}+P_{3}$ and $G_{2}+P_{2}$ result.

Lemma 22. $\left\{P_{2} \square P_{3}, G_{4}\right\} \subseteq \mathcal{S}(8)$.
Proof. Let there be given a red-blue coloring of $K_{8}$ with $V\left(K_{8}\right)=\left\{v_{1}, v_{2}, \ldots, v_{8}\right\}$, where $C=\left(v_{1}, v_{2}, \ldots, v_{6}, v_{1}\right)$ is a red 6 -cycle.

Assume, to the contrary, that there is no monochromatic $P_{2} \square P_{3}$. Hence, $v_{1} v_{4}, v_{2} v_{5}$ and $v_{3} v_{6}$ are blue. Furthermore, we may assume that $v_{1} v_{3}$ is red. We consider the following three cases.

Case 1. $v_{2} v_{4}$ or $v_{2} v_{6}$, say the former, is red. Then $v_{2} v_{6}$ and $v_{3} v_{5}$ are blue.
Subcase 1.1. $v_{1} v_{5}$ is red and $v_{4} v_{6}$ is blue. If both $v_{4} v_{7}$ and $v_{4} v_{8}$ are red, then a red $P_{2} \square P_{3}$ is produced or the edges $v_{2} v_{7}, v_{2} v_{8}, v_{3} v_{7}, v_{6} v_{7}$ and $v_{6} v_{8}$ are blue, producing a blue $P_{2} \square P_{3}$. Hence, assume that $v_{4} v_{7}$ is blue. Then there is a blue $P_{2} \square P_{3}$ or both $v_{2} v_{7}$ and $v_{3} v_{7}$ are red, producing a red $P_{2} \square P_{3}$. This is a contradiction.

Subcase 1.2. $v_{1} v_{5}$ and $v_{4} v_{6}$ are red. For $i=7,8$, note that at most one of $v_{2} v_{i}, v_{3} v_{i}, v_{5} v_{i}$ and $v_{6} v_{i}$ is red. We may therefore assume that $v_{2} v_{7}, v_{3} v_{7}, v_{5} v_{7}$ and $v_{2} v_{8}$ are blue. Then there exists a blue $P_{2} \square P_{3}$ if $v_{5} v_{8}$ is blue and so assume further that $v_{3} v_{8}$ and $v_{6} v_{8}$ are blue and $v_{5} v_{8}$ and $v_{6} v_{7}$ are red. Then either both $v_{1} v_{7}$ and $v_{1} v_{8}$ are blue, creating a blue $P_{2} \square P_{3}$, or a red $P_{2} \square P_{3}$ results. This cannot occur.

Subcase 1.3. $v_{1} v_{5}$ and $v_{4} v_{6}$ are blue. Let $i=7,8$. If $v_{2} v_{i}$ or $v_{3} v_{i}$ is red, then $v_{5} v_{i}$ and $v_{6} v_{i}$ are blue. Also, if $v_{2} v_{i}$ or $v_{3} v_{i}$ is blue, then $v_{1} v_{i}$ and $v_{4} v_{i}$ are red.

If $v_{2} v_{7}, v_{2} v_{8}, v_{3} v_{7}$ and $v_{3} v_{8}$ are all blue, therefore, then $v_{1} v_{7}, v_{1} v_{8}, v_{4} v_{7}$ and $v_{4} v_{8}$ are red. This creates a monochromatic copy of $P_{2} \square P_{3}$ regardless of the colors of $v_{5} v_{7}$ and $v_{5} v_{8}$.

Thus, let us assume that $v_{2} v_{7}$ is red and so $v_{5} v_{7}$ and $v_{6} v_{7}$ are blue. If $v_{2} v_{8}$ is red, then $v_{5} v_{8}$ and $v_{6} v_{8}$ are blue. Then either $v_{1} v_{7}$ or $v_{4} v_{8}$ is blue and there is a blue $P_{2} \square P_{3}$ or there is a red $P_{2} \square P_{3}$. On the other hand, if $v_{2} v_{8}$ is blue, then $v_{1} v_{8}$ is red and there is a monochromatic copy of $P_{2} \square P_{3}$ regardless of the color of $v_{7} v_{8}$.

Case 2. $v_{1} v_{5}$ or $v_{3} v_{5}$, say the former, is red. By Case 1, assume that $\left(v_{2}, v_{4}, v_{6}, v_{2}\right)$ is a blue triangle.

Subcase 2.1. $v_{3} v_{5}$ is red. If $v_{1} v_{7}, v_{3} v_{7}$ and $v_{5} v_{7}$ are blue, then $v_{2} v_{7}, v_{4} v_{7}$ and $v_{6} v_{7}$ must be red. However, this creates a red $P_{2} \square P_{3}$. Therefore, assume that $v_{1} v_{7}$ is red. Then $v_{2} v_{7}, v_{4} v_{7}$ and $v_{6} v_{7}$ are blue. Similarly, $v_{2} v_{8}, v_{4} v_{8}$ and $v_{6} v_{8}$ are also blue. Then either $v_{3} v_{7}$ or $v_{5} v_{7}$ is blue and there is a blue $P_{2} \square P_{3}$ or a red $P_{2} \square P_{3}$ results.

Subcase 2.2. $v_{3} v_{5}$ is blue. If $v_{1} v_{7}$ or $v_{1} v_{8}$, say the former, is red, then $v_{2} v_{7}$ and $v_{6} v_{7}$ are blue. Then both $v_{3} v_{7}$ and $v_{5} v_{7}$ are red, producing a red $P_{2} \square P_{3}$, or there is a blue $P_{2} \square P_{3}$. Hence, assume next that $v_{1} v_{7}$ and $v_{1} v_{8}$ are blue. Thus, at least one of $v_{2} v_{7}$ and $v_{6} v_{8}$, say $v_{2} v_{7}$, is red. Then either $v_{3} v_{7}$ or $v_{4} v_{7}$ is red and there is a red $P_{2} \square P_{3}$ or a blue $P_{2} \square P_{3}$ results.

Case 3. Neither Case 1 nor Case 2 occurs. Then one can verify that a blue $P_{2} \square P_{3}$ is produced.

Therefore, none of the three cases is possible and so $P_{2} \square P_{3}$ must be a common monochromatic subgraph of $K_{8}$.

To show that $G_{4} \in \mathcal{S}(8)$, assume that $v_{1} v_{4}$ is red so that the cycle $C$ and $v_{1} v_{4}$ form a red $P_{2} \square P_{3}$. Then either there is a red $G_{4}$ or each edge joining a vertex in $\left\{v_{2}, v_{3}, v_{5}, v_{6}\right\}$ and a vertex in $\left\{v_{7}, v_{8}\right\}$ is blue, producing a blue $G_{4}$.

Theorem 23. $\mathcal{S}^{*}(8)=\left\{P_{2} \square P_{3}, C_{3}+P_{3}, K_{1,3}+P_{3}, G_{2}+P_{2}, G_{3}, G_{4}\right\}$.
Proof. Observe that $P_{3}+2 P_{2} \notin \mathcal{S}(8)$ as the red-blue coloring of $K_{8}$ with a red $2 K_{1}+K_{6}$ and a blue $K_{1,1,6}$ shows. Also, neither $K_{1,1,2}$ nor $K_{2,3}$ belongs to $\mathcal{S}(8)$ as the red-blue coloring of $K_{8}$ in Figure 7(a) shows.

If $G \in \mathcal{S}(8)$ and $G$ is not bipartite, then $C_{3} \subseteq G \subseteq 2 K_{4}$. Since neither $P_{3}+2 P_{2}$ nor $K_{1,1,2}$ is a subgraph of $G$, it follows that either (i) $G$ is a subgraph of $C_{3}+P_{3}$ or $G_{2}+P_{2}$ or (ii) $G \in\left\{C_{3}+K_{1,3}, 2 C_{3}\right\}$. However, $2 C_{3} \notin \mathcal{S}(8)$ as the red-blue coloring of $K_{8}$ resulting in a red $P_{3}+K_{5}$ and a blue $\overline{P_{3}+K_{5}}$ shows.

Similarly, $C_{3}+K_{1,3} \notin \mathcal{S}(8)$ as the red-blue coloring of $K_{8}$ resulting in a red $2 K_{1}+K_{6}$ and a blue $K_{1,1,6}$ shows. Therefore, (ii) does not occur and so $C_{3}+P_{3}$ and $G_{2}+P_{2}$ are the maximal common monochromatic subgraphs of $K_{8}$ that are not bipartite.

Next assume that $G \in \mathcal{S}(8)$ and $G$ is bipartite. If $G$ contains a 6 -cycle as a subgraph, then $|V(G)|=6$ by Corollary 20. Furthermore, $K_{2,3} \nsubseteq G$ and so $G \in\left\{C_{6}, P_{2} \square P_{3}\right\}$.

If $G$ is bipartite and $C_{6} \nsubseteq G$, then either $C_{4} \subseteq G$ or $G$ is a forest. First suppose that $C_{4} \subseteq G$. Recall that neither $P_{3}+2 P_{2}$ nor $K_{2,3}$ can be a subgraph of $G$. Thus, either $G=C_{4}+P_{2} \subseteq P_{2} \square P_{3}$ or $G$ is connected. In particular, if $G$ is connected, then $|V(G)| \leq 6$ and $G$ must be a subgraph of one of $P_{2} \square P_{3}$,
$G_{3}$ and $G_{4}$. Hence, $P_{2} \square P_{3}, G_{3}$ and $G_{4}$ are the maximal common monochromatic subgraphs of $K_{8}$ containing even cycles.

Finally, suppose that $G$ is an acyclic common monochromatic subgraph of $K_{8}$. The red-blue coloring of $K_{8}$ with a red $2 K_{4}$ and a blue $K_{4,4}$ shows that $G \neq K_{1,4}+P_{2}$. Since $\Delta(G) \leq 4$ and $P_{3}+2 P_{2} \nsubseteq G$, it follows that either $G=K_{1,3}+P_{3}$ or $G$ is a subgraph of one of $P_{2} \square P_{3}$ and $G_{3}$. Furthermore, we have already seen that neither $C_{3}+K_{1,3}$ nor $G_{2}+P_{3}$ belongs to $\mathcal{S}(8)$. Therefore, $K_{1,3}+P_{3}$ is the only acyclic maximal common monochromatic subgraph of $K_{8}$. This completes the proof.


Figure 7. A red-blue coloring of $K_{8}$ and the monochromatic subgraph.
(The resulting red subgraph and blue subgraph are both isomorphic to the graph in Figure 7(b).)

The Figures 8-12 summarize the sets $\mathcal{S}^{*}(n)$ for $3 \leq n \leq 8$.

$$
0-0-0
$$

Figure 8. The member of $\mathcal{S}^{*}(3)=\mathcal{S}^{*}(4)$.

$$
0-0-0
$$

Figure 9. The member of $\mathcal{S}^{*}(5)$.


Figure 10. The members of $\mathcal{S}^{*}(6)$.






Figure 11. The members of $\mathcal{S}^{*}(7)$.







Figure 12. The members of $\mathcal{S}^{*}(8)$.

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