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ON MONOCHROMATIC SUBGRAPHS OF EDGE-COLORED COMPLETE GRAPHS

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Abstract

In a red-blue coloring of a nonempty graph, every edge is colored red or blue. If the resulting edge-colored graph contains a nonempty subgraph G without isolated vertices every edge of which is colored the same, then G is said to be monochromatic.

For two nonempty graphs G and H without isolated vertices, the monochromatic Ramsey number $\mathrm{mr}(G, H)$ of G and H is the minimum integer n such that every red-blue coloring of K_n results in a monochromatic G or a monochromatic H. Thus, the standard Ramsey number of G and H is bounded below by $\mathrm{mr}(G, H)$. The monochromatic Ramsey numbers of graphs belonging to some common classes of graphs are studied.

We also investigate another concept closely related to the standard Ramsey numbers and monochromatic Ramsey numbers of graphs. For a fixed integer $n \geq 3$, consider a nonempty subgraph G of order at most n containing no isolated vertices. Then G is a common monochromatic subgraph of K_n if every red-blue coloring of K_n results in a monochromatic copy of

G. Furthermore, G is a maximal common monochromatic subgraph of K_n if G is a common monochromatic subgraph of K_n that is not a proper subgraph of any common monochromatic subgraph of K_n. Let $\mathcal{S}(n)$ and $\mathcal{S}^*(n)$ be the sets of common monochromatic subgraphs and maximal common monochromatic subgraphs of K_n, respectively. Thus, $G \in \mathcal{S}(n)$ if and only if $R(G,G) = \mathrm{mr}(G,G) \le n$. We determine the sets $\mathcal{S}(n)$ and $\mathcal{S}^*(n)$ for $3 \le n \le 8$.

Keywords: Ramsey number, monochromatic Ramsey number, common monochromatic subgraph, maximal common monochromatic subgraph.

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1. Introduction

Graph coloring is said to be one of the most popular areas of graph theory. When restricting our attention to coloring edges, proper edge colorings of graphs have been studied extensively. In a proper edge coloring of a nonempty graph, no two adjacent edges are assigned the same color. The chromatic index of a graph is then the smallest number of colors required in a proper edge coloring of that graph. Of course, there are other interesting edge colorings that are not necessarily proper. In an edge-colored graph, where adjacent edges may be assigned the same color, a subgraph G all of whose edges are assigned the same color is referred to as a *monochromatic* G while a subgraph H no two of whose edges are assigned the same color is referred to as a *rainbow* H. Therefore, a proper edge coloring of a graph is an edge coloring in which every subgraph that is isomorphic to a star is rainbow, whereas an edge coloring of a graph that is not proper is an edge coloring that results in a monochromatic P_3 as a subgraph.

Ramsey theory is one of the most studied areas of research in extremal graph theory, which often deals with determining the values of or bounds for Ramsey numbers of two graphs. In a *red-blue coloring* of a graph, every edge is colored red or blue. If the resulting edge-colored graph contains a subgraph G every edge of which is colored red (blue), then this subgraph is called a *red* G (a *blue* G). For two graphs G and H, the *Ramsey number* $R(G,H)$ is the minimum positive integer n for which every red-blue coloring of K_n contains either a red G or a blue H. In fact, for every k graphs $G_1, G_2, \ldots, G_k$, it is known that there exist positive integers n such that every edge coloring of K_n with k colors $1, 2, \ldots, k$ produces a subgraph G_i all of whose edges are colored i for some $i \in \{1, 2, \ldots, k\}$. Then the smallest positive integer n with this property is the *Ramsey number* $R(G_1, G_2, \ldots, G_k)$ of the graphs $G_1, G_2, \ldots, G_k$. Ramsey numbers $R(G,H)$ and $R(G_1, G_2, \ldots, G_k)$ are defined for any nonempty graphs G, H and $G_1, G_2, \ldots, G_k$ and have been investigated for many years.

In this paper, we study two concepts that arise when considering red-blue colorings of complete graphs. We refer to the book [1] for graph-theoretical notation and terminology not described in this paper. (In particular, we use $G + H$ and $G \vee H$ to denote the union and join of two vertex-disjoint graphs G and H, respectively.) Also, see [2] for well-known Ramsey numbers, some of which are summarized below.

- For $1 \le m \le n$, $R(K_{1,m}, K_{1,n}) = m + n - \epsilon$, where $\epsilon = 1$ if both m and n are even and $\epsilon = 0$ otherwise.
- For $2 \le m \le n$, $R(P_m, P_n) = \lfloor m/2 \rfloor + n - 1$.
- For $m, n \ge 3$,

$$R(C_3, C_n) = \begin{cases} 6 & \text{if } n = 3, \\ 2n-1 & \text{if } n \ge 4. \end{cases}$$

$$R(C_4, C_n) = \begin{cases} 6 & \text{if } n = 4, \\ 7 & \text{if } n = 3, 5, \\ n+1 & \text{if } n \ge 6. \end{cases}$$

$$R(C_m, C_n) = \begin{cases} 2n-1 & \text{if } 5 \le m \le n \text{ and } m \text{ is odd}, \\ m/2 + n - 1 & \text{if } 6 \le m \le n \text{ and} \\ & \text{both } m \text{ and } n \text{ are even}, \\ \max\{2m-1, m/2+n-1\} & \text{if } 6 \le m < n \text{ and} \\ & m \text{ is even while } n \text{ is odd}. \end{cases}$$

2. Monochromatic Ramsey Numbers: Definitions and Observations

The *monochromatic Ramsey number* $\mathrm{mr}(G, H)$ of two nonempty graphs G and H without isolated vertices is defined to be the minimum integer n such that every red-blue edge coloring of K_n results in either a monochromatic G or a monochromatic H. Therefore, $\mathrm{mr}(G, G) = R(G, G)$. In fact, observe that

$$\min\{|V(G)|, |V(H)|\} \le \mathrm{mr}(G, H) \le \min\{R(G, G), R(H, H), R(G, H)\} \tag{1}$$

and so $\mathrm{mr}(G, H)$ is defined for every pair G, H of nonempty graphs.

Figure 1 shows all red-blue colorings of K_3 and K_4. Let us make some elementary observations.

Observation 1. (a) $\mathrm{mr}(G, H) = 2$ *if and only if* $P_2 \in \{G, H\}$.

(b) $\mathrm{mr}(G, P_3) = 3$ *for every nonempty graph* G *unless* $G = P_2$.

(c) *Every red-blue coloring of* K_4 *contains either a monochromatic* C_3 *or a monochromatic* P_4.

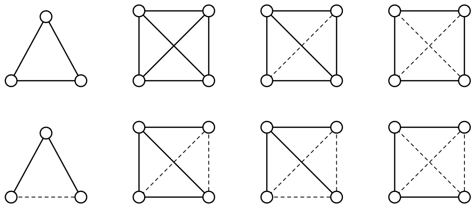


Figure 1. The red-blue colorings of K_3 and K_4.

Let H_1 be the graph obtained from the 4-path (v_1, v_2, v_3, v_4) by joining v_1 and v_3. As an example, we consider the three graphs C_3, P_4 and H_1. Since there is no 3-regular graph of order 7, every red-blue coloring of K_7 contains a monochromatic $K_{1,4}$. Then there is a monochromatic H_1 of one color or a monochromatic K_4 of the other color. Since $C_3, P_4 \subseteq H_1 \subseteq K_4$, it follows that each of $R(C_3, P_4)$, $R(H_1, P_4)$ and $R(C_3, H_1)$ is at most 7. Thus, the red-blue coloring of K_6 resulting in a red $K_{3,3}$ and a blue $2C_3$ shows that

$$R(C_3, P_4) = R(H_1, P_4) = R(C_3, H_1) = 7.$$

Let us now determine the corresponding monochromatic Ramsey numbers. Note first that $\mathrm{mr}(C_3, H_1) \leq R(C_3, C_3) = 6$ while $\mathrm{mr}(C_3, P_4), \mathrm{mr}(H_1, P_4) \leq R(P_4, P_4) = 5$ by (1). In fact, $\mathrm{mr}(C_3, H_1) = 6$ as the red-blue coloring of K_5 in Figure 2 shows since neither C_3 nor H_1 is a subgraph of C_5. Similarly, the red-blue coloring

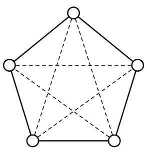


Figure 2. A red-blue coloring of K_5.

of K_4 inducing a red $C_3 + K_1$ (the union of C_3 and K_1) and a blue $K_{1,3}$ shows that $\mathrm{mr}(H_1, P_4) = 5$. Finally, Observation 1(c) implies that $\mathrm{mr}(C_3, P_4) \leq 4$ while the red-blue colorings of K_3 using both colors show that $\mathrm{mr}(C_3, P_4) \geq 4$. Hence,

$$\mathrm{mr}(C_3, P_4) = 4,\ \mathrm{mr}(H_1, P_4) = 5,\ \mathrm{mr}(C_3, H_1) = 6.$$

On the other hand, it can be similarly verified that

$$\mathrm{mr}(K_{1,3}, C_4) = R(K_{1,3}, K_{1,3}) = R(C_4, C_4) = R(K_{1,3}, C_4) = 6.$$

Thus, equality and strict inequality are both possible in (1).

We present a few more useful observations. First, if G and H are nonempty graphs and $G \subseteq H$, then there is a red-blue coloring of $K_{R(G,G)-1}$ that does not contain a monochromatic G, which certainly does not contain a monochromatic H either, while every red-blue coloring of $K_{R(G,G)}$ contains a monochromatic G. Therefore, $\mathrm{mr}(G,H) = \mathrm{mr}(G,G) = R(G,G)$ in this case. Also, if F is a nonempty subgraph of G, then every red-blue coloring of $K_{\mathrm{mr}(G,H)}$ contains either a monochromatic G, which certainly contains a monochromatic F, or a monochromatic H. Thus, $\mathrm{mr}(F,H) \le \mathrm{mr}(G,H)$. We summarize these facts as follows.

Observation 2. *Let G, H be a pair of nonempty graphs.*
(a) *If $G \subseteq H$, then* $\mathrm{mr}(G,H) = \mathrm{mr}(G,G) = R(G,G)$.
(b) *If G' and H' are nonempty subgraphs of G and H, respectively, then* $\mathrm{mr}(G',H') \le \mathrm{mr}(G,H)$.

3. The Monochromatic Ramsey Numbers Involving Paths and Cycles

It is well-known that $R(P_m,P_n) = \lfloor m/2 \rfloor + n - 1$ for integers m and n with $2 \le m \le n$. Hence, the following is an immediate consequence of Observation 2(a).

Theorem 3. *For integers m and n with $2 \le m \le n$,* $\mathrm{mr}(P_m,P_n) = \lfloor 3m/2 \rfloor - 1$.

Also, determining the exact value of the monochromatic Ramsey number of two nonempty graphs is possible when one of the two graphs is either C_3 or C_4.

Theorem 4. *For every nonempty graph G without isolated vertices,*

$$\mathrm{mr}(G,C_3) = \begin{cases} 6 & \textit{if } G \not\subseteq C_5, \\ |V(G)| & \textit{otherwise.} \end{cases}$$

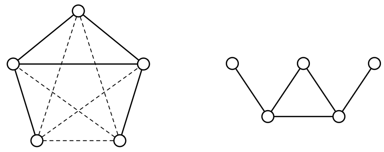
Proof. By Observation 1(a)(b), we assume that $G \notin \{P_2,P_3\}$. If G is not a subgraph of C_5, then the red-blue coloring of K_5 shown in Figure 2 shows that $\mathrm{mr}(G,C_3) \ge 6$. Thus the result is immediate by (1) since $R(C_3,C_3) = 6$.

Now assume that $G \subseteq C_5$. Since the coloring shown in Figure 2 is the only red-blue coloring of K_5 inducing no monochromatic triangles, it follows that $\mathrm{mr}(G,C_3) \le 5$. On the other hand, since there are red-blue colorings of K_3 and K_4 resulting in no monochromatic triangles (see Figure 1), it follows that $\mathrm{mr}(G,C_3) \ge |V(G)|$ when $|V(G)| = 4,5$. Finally, since the red-blue coloring of K_4 avoids monochromatic triangles if and only if it induces either (i) a red (blue) C_4 and a blue (red) $2P_2$ or (ii) a red (blue) P_4 and a blue (red) P_4, it follows that $\mathrm{mr}(2P_2,C_3) = \mathrm{mr}(P_4,C_3) = 4$. ■

Theorem 5. *For every nonempty graph G without isolated vertices,*

$$\mathrm{mr}(G,C_4) = \begin{cases} 2 & \textit{if } G = P_2, \\ 3 & \textit{if } G = P_3, \\ 5 & \textit{if } G \in \{P_4,P_5,2P_2,P_2+P_3\}, \\ 6 & \textit{otherwise.} \end{cases}$$

Proof. We may assume that $G \notin \{P_2, P_3\}$. First, $\mathrm{mr}(G, C_4) \le R(C_4, C_4) = 6$ by (1). Note also that a red-blue coloring c of K_5 induces no monochromatic 4-cycles if and only if c is either the coloring in Figure 2 or the coloring in Figure 3(a). If



(a) (b)

Figure 3. A red-blue coloring of K_5 and the induced monochromatic subgraph H_2.

c is the coloring in Figure 3(a), then c results in a red H_2 and a blue H_2, where H_2 is the graph shown in Figure 3(b). Therefore, $\mathrm{mr}(G, C_4) = 6$ if $G \not\subseteq C_5$ and $G \not\subseteq H_2$.

Suppose next that $G \subseteq C_5$. Since the coloring in Figure 3(a) shows that $\mathrm{mr}(C_5, C_4) = 6$, suppose further that $G \subseteq P_5$. Then both colorings in Figures 2 and 3(a) contain a monochromatic P_5 and so $\mathrm{mr}(G, C_4) \le 5$. Thus, $\mathrm{mr}(G, C_4) = 5$ if $|V(G)| = 5$. Furthermore, there is a red-blue coloring of K_4 that contains neither a monochromatic P_4 nor a monochromatic $2P_2$ (see Figure 1) and so $\mathrm{mr}(P_4, C_4) = \mathrm{mr}(2P_2, C_4) = 5$ as well.

Finally, suppose that $G \subseteq H_2$ and $G \not\subseteq C_5$. Then $G \not\subseteq P_5$ and so either $C_3 \subseteq G$ or $K_{1,3} \subseteq G$. Then the red-blue coloring of K_5 in Figure 2 shows that $\mathrm{mr}(G, C_4) = 6$. ■

Corollary 6. $\mathrm{mr}(C_3, C_5)+1 = \mathrm{mr}(C_4, C_5) = 6$ *while* $\mathrm{mr}(C_3, C_n) = \mathrm{mr}(C_4, C_n) = 6$ *for every integer* $n \ge 3$ *and* $n \ne 5$.

Now let us suppose that $m, n \ge 5$. What is $\mathrm{mr}(C_m, C_n)$ then?

Theorem 7. *For integers* m *and* n *with* $5 \le m \le n$,

$$\mathrm{mr}(C_m, C_n) = \begin{cases} 3m/2 - 1 & \text{if } m \text{ is even,} \\ 2m - 1 & \text{if } m \text{ is odd and either } n \text{ is odd or } n \ge 2m. \end{cases}$$

If m *is odd and* n *is even with* $m + 1 \le n \le 2m - 2$, *then*

$$m + n/2 - 1 \le \mathrm{mr}(C_m, C_n) \le \min\{2m - 1, 3n/2 - 1\}. \tag{2}$$

Proof. We consider the following two cases, according to the parity of m.

Case 1. m is even. Since the largest monochromatic cycle in the red-blue coloring of $K_{3m/2-2}$ inducing a red $K_{m/2-1} + K_{m-1}$ and a blue $K_{m/2-1,m-1}$ is C_{m-1}, it follows that $\mathrm{mr}(C_m, C_n) > 3m/2 - 2$. Then the result is immediate by (1) since $m \ge 6$ and $R(C_m, C_m) = 3m/2 - 1$.

Case 2. m is odd. Then $\mathrm{mr}(C_m, C_n) \leq R(C_m, C_m) = 2m-1$. If n is odd or $n \geq 2m$, then the red-blue coloring of K_{2m-2} inducing a red $2K_{m-1}$ and a blue $K_{m-1,m-1}$ shows that $\mathrm{mr}(C_m, C_n) \geq 2m-1$. If n is even and $m+1 \leq n \leq 2m-2$, on the other hand, then the red-blue coloring of $K_{m+n/2-2}$ inducing a red $K_{m-1} + K_{n/2-1}$ and a blue $K_{m-1,n/2-1}$ shows that $\mathrm{mr}(C_m, C_n) \geq m + n/2 - 1$, while $\mathrm{mr}(C_m, C_n) \leq R(C_n, C_n) = 3n/2 - 1$. ■

Let us also consider $\mathrm{mr}(C_m, P_n)$. Note that $\mathrm{mr}(C_m, P_n) \leq \mathrm{mr}(C_m, C_n)$.

Theorem 8. *For integers $m \geq 5$ and $n \geq 4$,*

$$\mathrm{mr}(C_m, P_n) = \begin{cases} \lfloor 3n/2 \rfloor - 1 & \textit{if } n \leq m, \\ 3m/2 - 1 & \textit{if } n \geq m+1 \textit{ and } m \textit{ is even}, \\ 2m-1 & \textit{if } n \geq 2m-1 \textit{ and } m \textit{ is odd}. \end{cases}$$

If m is odd and $m+1 \leq n \leq 2m-2$, then

$$m + \lfloor n/2 \rfloor - 1 \leq \mathrm{mr}(C_m, P_n) \leq \min\{2m-1, \lfloor 3n/2 \rfloor - 1\}. \tag{3}$$

Proof. If $n \leq m$, then $\mathrm{mr}(C_m, P_n) = R(P_n, P_n) = \lfloor 3n/2 \rfloor - 1$. Hence, let us suppose that $5 \leq m < n$. If m is even, then the red-blue coloring of $K_{3m/2-2}$ inducing a red $K_{m/2-1} + K_{m-1}$ and a blue $K_{m/2-1,m-1}$ shows that $\mathrm{mr}(C_m, P_n) > 3m/2 - 2$. Then the result follows since $\mathrm{mr}(C_m, P_n) \leq \mathrm{mr}(C_m, C_n) = 3m/2 - 1$. If m is odd, then $\mathrm{mr}(C_m, P_n) \leq R(C_m, C_m) = 2m-1$. In the red-blue coloring of K_{2m-2} inducing a red $2K_{m-1}$ and a blue $K_{m-1,m-1}$, a longest monochromatic path is of order $2m-2$ and a longest monochromatic odd cycle is of order $m-2$. Therefore, if m is odd and $n \geq 2m-1$, then $\mathrm{mr}(C_m, P_n) = 2m-1$.

It remains to consider the case where m is odd and $m+1 \leq n \leq 2m-2$. In the red-blue coloring of $K_{m+\lfloor n/2 \rfloor -2}$ resulting in a red $K_{m-1} + K_{\lfloor n/2 \rfloor -1}$ and a blue $K_{m-1,\lfloor n/2 \rfloor -1}$, a longest monochromatic path is of order $n-1$ and a longest monochromatic odd cycle is of order $m-2$. Hence, $m + \lfloor n/2 \rfloor - 1 \leq \mathrm{mr}(C_m, P_n) \leq \min\{2m-1, \lfloor 3n/2 \rfloor - 1\}$ by (1). ■

It turns out that the values $\mathrm{mr}(C_m, C_n)$ and $\mathrm{mr}(C_m, P_n)$ equal the lower bounds in (2) and (3), respectively, for $m = 5$.

Theorem 9. $\mathrm{mr}(C_5, P_n) = \lfloor n/2 \rfloor + 4$ *for* $6 \leq n \leq 8$.

Proof. First we prove that $\mathrm{mr}(C_5, P_7) \leq 7$ by verifying that every red-blue coloring of K_7 induces a monochromatic C_5 or a monochromatic P_7. Assume, to the contrary, that there is a red-blue coloring of K_7 with $V(K_7) = \{v_1, v_2, \ldots, v_7\}$ with neither a monochromatic C_5 nor a monochromatic P_7. We may assume that $(v_1, v_2, v_3, v_4, v_1)$ is a red 4-cycle since $R(C_4, C_4) = 6$. Assume further that v_1v_5 is a red edge since there is no blue P_7. Then v_2v_5 and v_4v_5 are blue edges to avoid red 5-cycles.

If v_6v_7 is red, then v_2v_6, v_2v_7 and v_4v_7 must be all blue edges. However then, a red P_7 or a blue C_5 results regardless of the color of the edge v_5v_6, which is a contradiction. Hence, v_6v_7 must be a blue edge. Then at least one of v_2v_6 and v_4v_7, say the former, must be red since a blue C_5 results otherwise. This in turn implies that the edges v_1v_6, v_3v_6 and v_5v_7 are all blue. Since a red C_5 or a blue P_7 is produced if v_3v_7 and v_4v_7 are assigned the same color, let us assume, by the symmetry, that v_3v_7 is red and v_4v_7 is blue. Then both v_4v_6 and v_5v_6 are blue to avoid red Hamiltonian paths. However then, there is a monochromatic C_5 regardless of the color of the edge v_2v_7. Thus, there is no such coloring and the result follows by Observation 2(b) and (3).

To verify that $\mathrm{mr}(C_5, P_8) \leq 8$, let there be given a red-blue coloring of K_8 with $V(K_8) = \{v_1, v_2, \ldots, v_8\}$ avoiding monochromatic 5-cycles. From the previous result, we may assume that $(v_1, v_2, \ldots, v_7)$ is a red P_7. If this cannot be extended to a red P_8, then v_1v_8 and v_7v_8 are blue. Also, the edges v_1v_5, v_2v_6 and v_3v_7 are blue, which in turn implies that v_3v_5 is red and v_1v_6 is blue. However then, this still produces a monochromatic C_5 regardless of the color of v_2v_7. This implies that there must be a red P_8, that is, $\mathrm{mr}(C_5, P_8) \leq 8$ and so $\mathrm{mr}(C_5, P_8) = 8$ by (3). ■

Theorem 10. $\mathrm{mr}(C_5, C_n) = n/2 + 4$ *for* $n = 6, 8$.

Proof. Let us first show that $\mathrm{mr}(C_5, C_6) \leq 7$. Consider a red-blue coloring of K_7 with $V(K_7) = \{v_1, v_2, \ldots, v_7\}$ containing no monochromatic 5-cycles. Then we may assume that $(v_1, v_2, \ldots, v_7)$ is a red P_7 by Theorem 9. If there is no red C_6, then v_1v_6 and v_2v_7 are blue edges. Also, v_1v_5, v_2v_6 and v_3v_7 are blue since there is no red C_5. Then there is a blue C_6 or the edge v_3v_5 is red. If the latter occurs, however, then a monochromatic C_5 results no matter how the edges v_1v_4 and v_4v_7 are colored. This verifies the claim and so $\mathrm{mr}(C_5, C_6) = 7$ by (2).

To show that $\mathrm{mr}(C_5, C_8) \leq 8$, consider a red-blue coloring of K_8 with $V(K_8) = \{v_1, v_2, \ldots, v_8\}$ containing no monochromatic 5-cycles. Assume further that $(v_1, v_2, \ldots, v_8)$ is a red P_8 and the edges v_1v_5, v_1v_8, v_2v_6, v_3v_7 and v_4v_8 are all blue. We consider the following two cases.

Case 1. v_1v_3 or v_6v_8, say the former, is a red edge. Then v_2v_5 is a blue edge, which implies that v_6v_8 is red and v_4v_7 is blue. Thus, $(v_3, v_7, v_4, v_8, v_1, v_5, v_2, v_6)$ is a blue P_8. Let us therefore assume that v_3v_6 is red. Then either both v_2v_4 and v_5v_7 are red, creating a red C_8, or there is a blue C_5.

Case 2. Both v_1v_3 and v_6v_8 are blue edges. Then in order to avoid a monochromatic C_5, observe that v_2v_5 is red and v_1v_7 is blue, forcing v_3v_6 to be red. This then implies that v_2v_4 and v_5v_7 are blue. Thus, both v_1v_4 and v_5v_8 must be red, creating a red C_8.

The result is now immediate again by (2). ■

4. Common Monochromatic Subgraphs: Definitions and Observations

Let G be a nonempty graph without isolated vertices. The Ramsey number $R(G,G)$ is the smallest integer n such that *every* red-blue coloring of K_n results in a monochromatic subgraph that is isomorphic to G (and we have seen that $R(G,G) = \mathrm{mr}(G,G)$).

There is another approach to study red-blue colorings of K_n and resulting monochromatic subgraphs. For a fixed integer $n \geq 3$, consider an arbitrary red-blue coloring of K_n. Then for each nonempty subgraph G of K_n without isolated vertices, either there is a monochromatic G or there is no monochromatic G. A natural question to ask is, then, which nonempty subgraph $G \subseteq K_n$ without isolated vertices has the property that *every* red-blue coloring of K_n results in a monochromatic G?

Let G be a nonempty graph without isolated vertices. For a fixed integer $n \geq 3$, the graph G is a *common monochromatic subgraph* of K_n if *every* red-blue coloring of K_n results in a monochromatic copy of G. The following is therefore an immediate observation.

Observation 11. *A nonempty graph G without isolated vertices is a common monochromatic subgraph of K_n if and only if $R(G,G) \leq n$.*

Suppose that G and G' are nonempty graphs without isolated vertices with $G' \subseteq G$. Then certainly $R(G',G') \leq R(G,G)$. Hence, Observation 11 implies the following.

Observation 12. *Suppose that G and G' are nonempty graphs without isolated vertices with $G' \subseteq G$. If G is a common monochromatic subgraph of K_n, then so is G'.*

Observation 12 suggests an additional important concept. For a fixed integer $n \geq 3$, a nonempty graph G without isolated vertices is a *maximal common monochromatic subgraph* of K_n if G is a common monochromatic subgraph of K_n that is not a proper subgraph of any common monochromatic subgraph of K_n.

For each integer $n \geq 3$, let us denote the sets of common monochromatic subgraphs and maximal common monochromatic subgraphs of K_n by $\mathcal{S}(n)$ and $\mathcal{S}^*(n)$, respectively. Thus, $\mathcal{S}^*(n) \subseteq \mathcal{S}(n)$. Also, $\mathcal{S}(n) \neq \emptyset$ since P_2 is a common monochromatic subgraph of K_3 and

$$\mathcal{S}(3) \subseteq \mathcal{S}(4) \subseteq \mathcal{S}(5) \subseteq \cdots \subseteq \mathcal{S}(n) \subseteq \mathcal{S}(n+1) \subseteq \cdots . \tag{4}$$

(We will soon see that the sets $\mathcal{S}^*(n)$ for $n \geq 3$ do *not* have a similar nesting property.) Note also that a nonempty graph G without isolated vertices belongs to the set $\mathcal{S}(n)$ if and only if G is a subgraph of a graph belonging to $\mathcal{S}^*(n)$. Therefore,

$\mathcal{S}^*(n) \neq \emptyset$. Moreover, once one of the sets $\mathcal{S}(n)$ and $\mathcal{S}^*(n)$ is determined, the other can be obtained as well.

For each $n \geq 3$, there are red-blue colorings of K_n with $\lceil n(n-1)/4 \rceil$ red edges and $\lfloor n(n-1)/4 \rfloor$ blue edges. Therefore, if G is a common monochromatic subgraph of K_n, then $|E(G)| \leq \lceil n(n-1)/4 \rceil$. From this, $\mathcal{S}(3) \subseteq \{P_2, P_3\}$ and it is straightforward to verify that each of P_2 and P_3 is a common monochromatic subgraph of K_3. Thus $\mathcal{S}(3) = \{P_2, P_3\}$ and $\mathcal{S}^*(3) = \{P_3\}$. Furthermore, by examining the red-blue colorings of K_4 in Figure 1, it follows that $\mathcal{S}(3) = \mathcal{S}(4) = \{P_2, P_3\}$ and $\mathcal{S}^*(3) = \mathcal{S}^*(4) = \{P_3\}$.

5. Common Monochromatic Subgraphs of Complete Graphs of Small Order

In order to discuss the sets $\mathcal{S}(n)$ and $\mathcal{S}^*(n)$ for $n \geq 5$, let us state a few more results on common monochromatic subgraphs of K_n. First, recall for every integer $m \geq 2$ that

$$R(K_{1,m}, K_{1,m}) = \begin{cases} 2m-1 & \text{if } m \text{ is even,} \\ 2m & \text{if } m \text{ is odd.} \end{cases}$$

For K_n, therefore,

(i) $K_{1,n/2}$ is a common monochromatic subgraph while $K_{1,n/2+1}$ is not when n is even,

(ii) $K_{1,(n-1)/2}$ is a common monochromatic subgraph while $K_{1,(n+1)/2}$ is not when $n \equiv 1 \pmod 4$ and

(iii) $K_{1,(n+1)/2}$ is a common monochromatic subgraph while $K_{1,(n+3)/2}$ is not when $n \equiv 3 \pmod 4$. Also recall for every integer $m \geq 3$ that

$$\begin{aligned} R(P_m, P_m) &= \lfloor 3m/2 \rfloor - 1, \\ R(C_m, C_m) &= \begin{cases} 6 & \text{if } m = 3, 4, \\ 2m-1 & \text{if } m \text{ is odd and } m \geq 5, \\ 3m/2 - 1 & \text{if } m \text{ is even and } m \geq 6. \end{cases} \end{aligned}$$

Thus, the following is a consequence of Observation 12, providing us with some necessary conditions for a graph to be a common monochromatic subgraph of K_n.

Proposition 13. *Suppose that $n \geq 3$ and $G \in \mathcal{S}(n)$.*
(a) *The maximum degree $\Delta(G)$ of G satisfies the following.*

$$\Delta(G) \leq \begin{cases} \lceil n/2 \rceil & \text{if } n \equiv 3 \pmod 4, \\ \lfloor n/2 \rfloor & \text{otherwise.} \end{cases}$$

(b) *Every path in G is of order at most $2n/3+1$. Also, every subgraph of $P_{\lfloor 2n/3+1\rfloor}$ without isolated vertices belongs to $\mathcal{S}(n)$.*

(c) *G is acyclic for $3 \le n \le 5$. For $n \ge 6$, every odd cycle in G is of order at most $(n+1)/2$ while every even cycle in G is of order at most $(2n+2)/3$. Also, every odd cycle of order at most $(n+1)/2$ and every even cycle of order at most $(2n+2)/3$ belong to $\mathcal{S}(n)$.*

Let $n \ge 4$. Consider the red-blue coloring of K_n resulting in a red K_1+K_{n-1} and a blue $K_{1,n-1}$. Also, for each integer i with $1 \le i \le \lfloor n/2 \rfloor$, consider the red-blue coloring of K_n resulting in a red $K_i + K_{n-i}$ and a blue $K_{i,n-i}$. These colorings of K_n give us the following.

Proposition 14. *Suppose that $n \ge 4$ and $G \in \mathcal{S}(n)$.*

(a) *G is not a spanning subgraph of K_n.*

(b) *If G is not bipartite, then $G \subseteq K_i + K_{n-i}$ for $1 \le i \le \lfloor n/2 \rfloor$.*

We have seen that $\mathcal{S}(3) = \mathcal{S}(4) = \{P_2, P_3\}$ and $\mathcal{S}^*(3) = \mathcal{S}^*(4) = \{P_3\}$. We next determine the sets $\mathcal{S}(5)$ and $\mathcal{S}^*(5)$. Note first that $P_4 \in \mathcal{S}(5)$. Also, if $G \in \mathcal{S}(5)$, then G is acyclic and $\Delta(G) \le 2$ by Proposition 13. It then follows by Observation 12 that

$$\{P_2, P_3, P_4, 2P_2\} \subseteq \mathcal{S}(5) \subseteq \{P_2, P_3, P_4, P_5, 2P_2, P_2 + P_3\}.$$

Therefore, $\mathcal{S}(5) = \{P_2, P_3, P_4, 2P_2\}$ by Proposition 14(a). Furthermore, every graph in $\mathcal{S}(5)$ is a subgraph of P_4 and so $\mathcal{S}^*(5) = \{P_4\}$.

Thus far we have found that $\mathcal{S}^*(3) = \mathcal{S}^*(4) = \{P_3\}$ and $\mathcal{S}^*(5) = \{P_4\}$ (and so $\mathcal{S}^*(4) \not\subseteq \mathcal{S}^*(5)$). Let us next consider $\mathcal{S}^*(6)$ and $\mathcal{S}^*(7)$. In particular, we show that $\mathcal{S}^*(6) = \{C_3, G_1\}$ and $\mathcal{S}^*(7) = \{C_3 + P_2, G_2, G_3\}$, where the graphs G_1, G_2 and G_3 are shown in Figure 4. We first show that both C_3 and G_1 are common

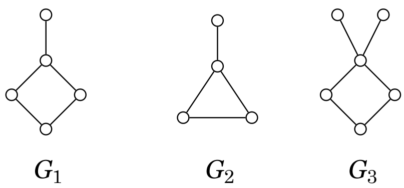


Figure 4. The graphs G_1, G_2 and G_3.

monochromatic subgraphs of K_6.

Lemma 15. $\{C_3, G_1\} \subseteq \mathcal{S}(6)$.

Proof. That $C_3 \in \mathcal{S}(6)$ is immediate since $R(C_3, C_3) = 6$. To see that $G_1 \in \mathcal{S}(6)$, consider an arbitrary red-blue coloring of K_6 with $V(K_6) = \{v_1, v_2, \ldots, v_6\}$. Since

$R(C_4, C_4) = 6$, we may assume, without loss of generality, that $(v_1, v_2, v_3, v_4, v_1)$ is a red 4-cycle, say. Then either there is a red G_1 or the edges v_iv_5 and v_iv_6 for $1 \le i \le 4$ are all blue, creating a blue G_1. ■

Theorem 16. $\mathcal{S}^*(6) = \{C_3, G_1\}$.

Proof. First we show that $C_3 \in \mathcal{S}^*(6)$. If this is not the case, then there is a common monochromatic subgraph G of K_6 containing C_3 as a proper subgraph. Then G is certainly not bipartite and so $G \subseteq 2K_3$ by Proposition 14(b). Furthermore, G is not a spanning subgraph of K_6 by Proposition 14(a). Thus, $G = C_3 + P_2$. However, this is impossible as the red-blue coloring of K_6 inducing a red $2K_1 + K_4$ and a blue $K_{1,1,4}$ contains no monochromatic copy of $C_3 + P_2$. Thus, C_3 is a maximal common monochromatic subgraph of K_6. In fact, C_3 is the only common monochromatic subgraph of K_6 that is not bipartite.

Next we show that $G_1 \in \mathcal{S}^*(6)$ by verifying that G_1 is the only common monochromatic subgraph of K_6 containing C_4 as a proper subgraph. Since C_4+P_2 is a spanning subgraph of K_6 while $K_{1,1,2}$ is not bipartite, if $G \in \mathcal{S}(6)$ and G contains C_4 as a proper subgraph, then G must contain G_1 as a spanning subgraph and G must be bipartite. Therefore, if $G \neq G_1$, then $G = K_{2,3}$. However, the red-blue coloring of K_6 inducing a red $P_2 \Box C_3$ (the cartesian product of P_2 and C_3) and a blue C_6 shows that this is impossible. Thus, C_4 and G_1 are the only common monochromatic subgraphs of K_6 that are bipartite and not acyclic. Furthermore, $G_1 \in \mathcal{S}^*(6)$.

Finally, every forest of order at most 5 without isolated vertices is either $K_{1,4}$, which is not a common monochromatic subgraph of K_6 by Proposition 13(a), or a proper subgraph of G_1. Hence, no forest can be a maximal common monochromatic subgraph of K_6. This completes the proof. ■

By Theorem 16, we see that $G \in \mathcal{S}(6)$ if and only if G is a nonempty graph without isolated vertices and either $G \subseteq C_3$ or $G \subseteq G_1$. Thus,

$$\mathcal{S}(6) = \mathcal{S}^*(6) \cup \{P_2, P_3, P_4, P_5, 2P_2, P_2 + P_3, K_{1,3}, S_{2,3}, C_4\},$$

where $S_{2,3}$ is the double star of order 5. (A *double star* is a tree whose diameter equals 3.)

As before, in order to show that $\mathcal{S}^*(7) = \{C_3 + P_2, G_2, G_3\}$, we first show that the three graphs are common monochromatic subgraphs of K_7.

Lemma 17. $\{C_3 + P_2, G_2, G_3\} \subseteq \mathcal{S}(7)$.

Proof. Let there be given a red-blue coloring of K_7 with $V(K_7) = \{v_1, v_2, \ldots, v_7\}$. Since $R(C_3, C_3) = 6$, assume that (v_1, v_2, v_3, v_1) is a red 3-cycle.

If there is no red $C_3 + P_2$, then the set $\{v_4, v_5, v_6, v_7\}$ induces a blue K_4. If one of v_1v_4 and v_2v_4 is blue, then a blue $C_3 + P_2$ results. Otherwise, both v_1v_4

and v_2v_4 are red and a monochromatic C_3+P_2 is produced regardless of the color of the edge v_3v_5.

Next, if there is no red G_2, then at least one of the edges joining two vertices in the set $\{v_4, v_5, v_6, v_7\}$, say v_4v_5, is a blue edge. If one of v_1v_4 and v_1v_5 is red, then there is a red G_2. Otherwise, both v_1v_4 and v_1v_5 are blue and a monochromatic G_2 is produced regardless of the color of v_2v_4.

Finally, we show that G_3 is a common monochromatic subgraph of K_7. Let there be given a red-blue coloring of K_7. Suppose first that there is a monochromatic $K_{1,5}$, say v_iv_6 is red for $1 \le i \le 5$. If the set $\{v_1, v_2, \ldots, v_5\}$ induces a blue K_5, then either both v_1v_7 and v_2v_7 are red, producing a red G_3, or there is a blue G_3. Otherwise, we may assume that v_1v_2 is red. If there is no red G_3, then (i) at least one of v_1v_7 and v_2v_7 is blue and (ii) v_1v_i and v_2v_i are blue for $3 \le i \le 5$. This produces a blue G_3.

Since there is no 3-regular graph of order 7, we may now assume that v_iv_7 is red for $1 \le i \le 4$ while v_5v_7 and v_6v_7 are blue. If every edge joining a vertex in $\{v_1, v_2, v_3, v_4\}$ and a vertex in $\{v_5, v_6\}$ is blue, then a blue G_3 is produced. Hence, assume that v_1v_5 is red. Then v_iv_5 is blue for $2 \le i \le 4$ if there is no red G_3. This in turn implies that either one of v_2v_6 and v_3v_6 is blue, producing a blue G_3, or there is a red G_3. ■

Theorem 18. $\mathcal{S}^*(7) = \{C_3 + P_2, G_2, G_3\}$.

Proof. First note that $3P_2 \notin \mathcal{S}(7)$ as the red-blue coloring of K_7 inducing a red $2K_1+K_5$ and a blue $K_{1,1,5}$ shows. Therefore, there is no common monochromatic subgraph of K_7 containing $3P_2$ as a subgraph by Observation 12. Also, this coloring shows that $C_3 + P_3 \notin \mathcal{S}(7)$.

Let us first assume that G is a common monochromatic subgraph of K_7 that is not bipartite. Therefore, $C_3 \subseteq G \subseteq K_3 + K_4$, $G \neq C_3 + P_3$ and $3P_2 \not\subseteq G$. Also, G is not a spanning subgraph of K_7 and so either $G \in \{C_3, C_3+P_2, G_2\}$ or $K_{1,1,2} \subseteq G$. However, the latter case is impossible as the red-blue coloring of K_7 in Figure 5 shows. Hence, Lemma 17 implies that G is a common monochromatic

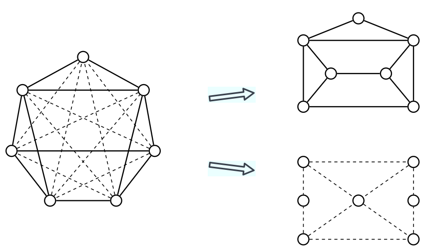


Figure 5. A red-blue coloring of K_7.

subgraph of K_7 that is not bipartite if and only if $G \in \{C_3, C_3+P_2, G_2\}$, that is,

$\{C_3 + P_2, G_2\} \subseteq \mathcal{S}^*(7)$.

Next we assume that G is a common monochromatic subgraph of K_7 that is bipartite. If G is not acyclic, then C_4 is a subgraph of G. Since G is a bipartite graph of order at most 6 and $3P_2 \not\subseteq G$, it follows that either (i) $G \in \{C_4, G_1, G_3\}$ or (ii) $K_{2,3} \subseteq G$ or (iii) $G = G_4$, which is the graph shown in Figure 6. However,

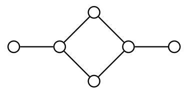


Figure 6. The graph G_4.

the red-blue coloring of K_7 in Figure 5 shows that (ii) does not occur. Similarly, (iii) does not occur as the red-blue coloring of K_7 inducing a red $K_1 \vee 2K_3$ (the join of K_1 and $2K_3$) and a blue $K_1 + K_{3,3}$ shows. Thus, G is a common monochromatic subgraph of K_7 containing C_4 as a subgraph if and only if $G \in \{C_4, G_1, G_3\}$, which implies that $G_3 \in \mathcal{S}^*(7)$.

Finally, let G be a forest of order at most 6 without isolated vertices. Therefore, if $G \in \mathcal{S}(7)$, then $\Delta(G) \leq 4$ and $3P_2 \not\subseteq G$ and so either $G \subseteq G_3$ or $G = S_{3,3}$ (the double star of order 6 whose maximum degree equals 3). However, the red-blue coloring of K_7 inducing a red $K_2 + K_5$ and a blue $K_{2,5}$ contains no monochromatic copy of $S_{3,3}$, which implies that $G \subseteq G_3$. Therefore, there is no acyclic maximal common monochromatic subgraph of K_7. This completes the proof. ■

We see from Proposition 13 that $P_6, C_6 \in \mathcal{S}(8)$ while $P_7, C_7 \notin \mathcal{S}(8)$. For K_8, in fact, while there is no common monochromatic subgraph of order 8 by Proposition 14(a), there is no *connected* common monochromatic subgraph of order 7. The following two results show why.

Proposition 19. *For each integer $n \geq 8$, no tree of order $n-1$ is a common monochromatic subgraph of K_n.*

Proof. If T is a tree of order $n-1 \geq 7$, then there is a unique integer i with $1 \leq i \leq \lfloor (n-1)/2 \rfloor$ such that T is a spanning subgraph of $K_{i,n-1-i}$. If $1 \leq i \leq \lfloor n/2 \rfloor - 2$, then the red-blue coloring of K_n with a red $K_{\lfloor n/2 \rfloor} + K_{\lceil n/2 \rceil}$ and a blue $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ shows that $T \notin \mathcal{S}(n)$. Similarly, if $\lfloor n/2 \rfloor - 1 \leq i \leq \lfloor (n-1)/2 \rfloor$, then $T \notin \mathcal{S}(n)$ as the red-blue coloring of K_n with a red $K_2 + K_{n-2}$ and a blue $K_{2,n-2}$ shows. ■

We therefore obtain the following by Observation 12 and Propositions 14(a) and 19 for $n \geq 8$.

Corollary 20. *Suppose that $n \geq 8$ and $G \in \mathcal{S}(n)$. If G is connected, then $|V(G)| \leq n-2$.*

We are prepared to determine the set $\mathcal{S}^*(8)$. We first present two lemmas.

Lemma 21. $\{C_3 + P_3, K_{1,3} + P_3, G_2 + P_2, G_3\} \subseteq \mathcal{S}(8)$.

Proof. Clearly $G_3 \in \mathcal{S}(7) \subseteq \mathcal{S}(8)$. Let there be given an arbitrary red-blue coloring of K_8 with $V(K_8) = \{v_1, v_2, \ldots, v_8\}$. Since $R(C_6, C_6) = 8$, we may assume that $C = (v_1, v_2, \ldots, v_6, v_1)$ is a red 6-cycle.

If there is a red edge joining a vertex belonging to C and one of v_7 and v_8, then there is a red $K_{1,3} + P_3$. Otherwise, there is a blue $K_{1,3} + P_3$. Also, if there is neither a red $C_3 + P_3$ nor a red $G_2 + P_2$, then (v_1, v_3, v_5, v_1) and (v_2, v_4, v_6, v_2) are blue triangles. Furthermore, we may assume that v_1v_7 is blue. Therefore, blue copies of $C_3 + P_3$ and $G_2 + P_2$ result. ∎

Lemma 22. $\{P_2 \Box P_3, G_4\} \subseteq \mathcal{S}(8)$.

Proof. Let there be given a red-blue coloring of K_8 with $V(K_8) = \{v_1, v_2, \ldots, v_8\}$, where $C = (v_1, v_2, \ldots, v_6, v_1)$ is a red 6-cycle.

Assume, to the contrary, that there is no monochromatic $P_2 \Box P_3$. Hence, v_1v_4, v_2v_5 and v_3v_6 are blue. Furthermore, we may assume that v_1v_3 is red. We consider the following three cases.

Case 1. v_2v_4 or v_2v_6, say the former, is red. Then v_2v_6 and v_3v_5 are blue.

Subcase 1.1. v_1v_5 is red and v_4v_6 is blue. If both v_4v_7 and v_4v_8 are red, then a red $P_2 \Box P_3$ is produced or the edges v_2v_7, v_2v_8, v_3v_7, v_6v_7 and v_6v_8 are blue, producing a blue $P_2 \Box P_3$. Hence, assume that v_4v_7 is blue. Then there is a blue $P_2 \Box P_3$ or both v_2v_7 and v_3v_7 are red, producing a red $P_2 \Box P_3$. This is a contradiction.

Subcase 1.2. v_1v_5 and v_4v_6 are red. For $i = 7, 8$, note that at most one of v_2v_i, v_3v_i, v_5v_i and v_6v_i is red. We may therefore assume that v_2v_7, v_3v_7, v_5v_7 and v_2v_8 are blue. Then there exists a blue $P_2 \Box P_3$ if v_5v_8 is blue and so assume further that v_3v_8 and v_6v_8 are blue and v_5v_8 and v_6v_7 are red. Then either both v_1v_7 and v_1v_8 are blue, creating a blue $P_2 \Box P_3$, or a red $P_2 \Box P_3$ results. This cannot occur.

Subcase 1.3. v_1v_5 and v_4v_6 are blue. Let $i = 7, 8$. If v_2v_i or v_3v_i is red, then v_5v_i and v_6v_i are blue. Also, if v_2v_i or v_3v_i is blue, then v_1v_i and v_4v_i are red.

If v_2v_7, v_2v_8, v_3v_7 and v_3v_8 are all blue, therefore, then v_1v_7, v_1v_8, v_4v_7 and v_4v_8 are red. This creates a monochromatic copy of $P_2 \Box P_3$ regardless of the colors of v_5v_7 and v_5v_8.

Thus, let us assume that v_2v_7 is red and so v_5v_7 and v_6v_7 are blue. If v_2v_8 is red, then v_5v_8 and v_6v_8 are blue. Then either v_1v_7 or v_4v_8 is blue and there is a blue $P_2 \Box P_3$ or there is a red $P_2 \Box P_3$. On the other hand, if v_2v_8 is blue, then v_1v_8 is red and there is a monochromatic copy of $P_2 \Box P_3$ regardless of the color of v_7v_8.

Case 2. v_1v_5 or v_3v_5, say the former, is red. By Case 1, assume that (v_2, v_4, v_6, v_2) is a blue triangle.

Subcase 2.1. v_3v_5 is red. If v_1v_7, v_3v_7 and v_5v_7 are blue, then v_2v_7, v_4v_7 and v_6v_7 must be red. However, this creates a red $P_2\Box P_3$. Therefore, assume that v_1v_7 is red. Then v_2v_7, v_4v_7 and v_6v_7 are blue. Similarly, v_2v_8, v_4v_8 and v_6v_8 are also blue. Then either v_3v_7 or v_5v_7 is blue and there is a blue $P_2\Box P_3$ or a red $P_2\Box P_3$ results.

Subcase 2.2. v_3v_5 is blue. If v_1v_7 or v_1v_8, say the former, is red, then v_2v_7 and v_6v_7 are blue. Then both v_3v_7 and v_5v_7 are red, producing a red $P_2\Box P_3$, or there is a blue $P_2\Box P_3$. Hence, assume next that v_1v_7 and v_1v_8 are blue. Thus, at least one of v_2v_7 and v_6v_8, say v_2v_7, is red. Then either v_3v_7 or v_4v_7 is red and there is a red $P_2\Box P_3$ or a blue $P_2\Box P_3$ results.

Case 3. Neither Case 1 nor Case 2 occurs. Then one can verify that a blue $P_2\Box P_3$ is produced.

Therefore, none of the three cases is possible and so $P_2\Box P_3$ must be a common monochromatic subgraph of K_8.

To show that $G_4 \in \mathcal{S}(8)$, assume that v_1v_4 is red so that the cycle C and v_1v_4 form a red $P_2\Box P_3$. Then either there is a red G_4 or each edge joining a vertex in $\{v_2, v_3, v_5, v_6\}$ and a vertex in $\{v_7, v_8\}$ is blue, producing a blue G_4. ■

Theorem 23. $\mathcal{S}^*(8) = \{P_2\Box P_3, C_3 + P_3, K_{1,3} + P_3, G_2 + P_2, G_3, G_4\}$.

Proof. Observe that $P_3 + 2P_2 \notin \mathcal{S}(8)$ as the red-blue coloring of K_8 with a red $2K_1 + K_6$ and a blue $K_{1,1,6}$ shows. Also, neither $K_{1,1,2}$ nor $K_{2,3}$ belongs to $\mathcal{S}(8)$ as the red-blue coloring of K_8 in Figure 7(a) shows.

If $G \in \mathcal{S}(8)$ and G is not bipartite, then $C_3 \subseteq G \subseteq 2K_4$. Since neither $P_3 + 2P_2$ nor $K_{1,1,2}$ is a subgraph of G, it follows that either (i) G is a subgraph of $C_3 + P_3$ or $G_2 + P_2$ or (ii) $G \in \{C_3 + K_{1,3}, 2C_3\}$. However, $2C_3 \notin \mathcal{S}(8)$ as the red-blue coloring of K_8 resulting in a red $P_3 + K_5$ and a blue $\overline{P_3 + K_5}$ shows.

Similarly, $C_3 + K_{1,3} \notin \mathcal{S}(8)$ as the red-blue coloring of K_8 resulting in a red $2K_1 + K_6$ and a blue $K_{1,1,6}$ shows. Therefore, (ii) does not occur and so $C_3 + P_3$ and $G_2 + P_2$ are the maximal common monochromatic subgraphs of K_8 that are not bipartite.

Next assume that $G \in \mathcal{S}(8)$ and G is bipartite. If G contains a 6-cycle as a subgraph, then $|V(G)| = 6$ by Corollary 20. Furthermore, $K_{2,3} \nsubseteq G$ and so $G \in \{C_6, P_2\Box P_3\}$.

If G is bipartite and $C_6 \nsubseteq G$, then either $C_4 \subseteq G$ or G is a forest. First suppose that $C_4 \subseteq G$. Recall that neither $P_3 + 2P_2$ nor $K_{2,3}$ can be a subgraph of G. Thus, either $G = C_4 + P_2 \subseteq P_2\Box P_3$ or G is connected. In particular, if G is connected, then $|V(G)| \leq 6$ and G must be a subgraph of one of $P_2\Box P_3$,

G_3 and G_4. Hence, $P_2 \square P_3$, G_3 and G_4 are the maximal common monochromatic subgraphs of K_8 containing even cycles.

Finally, suppose that G is an acyclic common monochromatic subgraph of K_8. The red-blue coloring of K_8 with a red $2K_4$ and a blue $K_{4,4}$ shows that $G \neq K_{1,4} + P_2$. Since $\Delta(G) \leq 4$ and $P_3 + 2P_2 \nsubseteq G$, it follows that either $G = K_{1,3} + P_3$ or G is a subgraph of one of $P_2 \square P_3$ and G_3. Furthermore, we have already seen that neither $C_3 + K_{1,3}$ nor $G_2 + P_3$ belongs to $\mathcal{S}(8)$. Therefore, $K_{1,3} + P_3$ is the only acyclic maximal common monochromatic subgraph of K_8. This completes the proof. ■

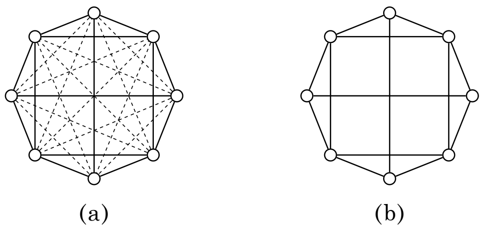


Figure 7. A red-blue coloring of K_8 and the monochromatic subgraph.

(The resulting red subgraph and blue subgraph are both isomorphic to the graph in Figure 7(b).)

The Figures 8–12 summarize the sets $\mathcal{S}^*(n)$ for $3 \leq n \leq 8$.

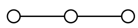


Figure 8. The member of $\mathcal{S}^*(3) = \mathcal{S}^*(4)$.

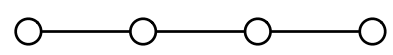


Figure 9. The member of $\mathcal{S}^*(5)$.

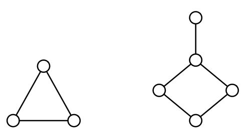


Figure 10. The members of $\mathcal{S}^*(6)$.

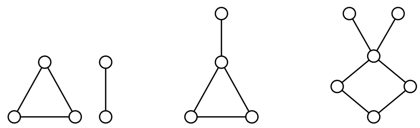


Figure 11. The members of $\mathcal{S}^*(7)$.

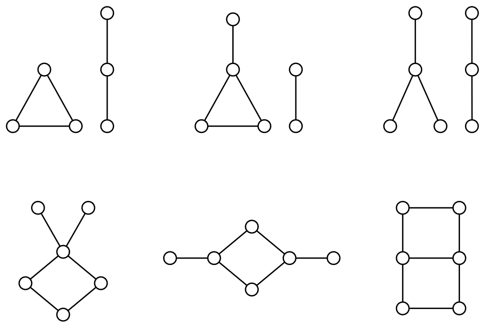


Figure 12. The members of $\mathcal{S}^*(8)$.

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