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TREE-LIKE PARTIAL HAMMING GRAPHS

TANJA GOLOGRANC

Institute of Mathematics, Physics and Mechanics Jadranska 19, 1000 Ljubljana, Slovenia e-mail: tanja.gologranc@imfm.si

Abstract

Tree-like partial cubes were introduced in [B. Brešar, W. Imrich, S. Klavžar, *Tree-like isometric subgraphs of hypercubes*, Discuss. Math. Graph Theory, 23 (2003), 227–240] as a generalization of median graphs. We present some incorrectnesses from that article. In particular we point to a gap in the proof of the theorem about the dismantlability of the cube graph of a tree-like partial cube and give a new proof of that result, which holds also for a bigger class of graphs, so called tree-like partial Hamming graphs. We investigate these graphs and show some results which imply previously-known results on tree-like partial cubes. For instance, we characterize tree-like partial Hamming graphs and prove that every tree-like partial Hamming graph G contains a Hamming graph that is invariant under every automorphism of G. The latter result is a direct consequence of the result about the dismantlability of the intersection graph of maximal Hamming graphs of a tree-like partial Hamming graph.

Keywords: partial Hamming graph, expansion procedure, dismantlable graph, gated subgraph, intersection graph.

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1. INTRODUCTION

Median and quasi-median graphs are well studied classes of graphs, cf. [3, 4, 11, 15, 16, 17, 18, 19, 24]. One of the well-known characterizations of median graphs is that they constitute the class of retracts of hypercubes, see Bandelt [1]. On the other hand, regular median graphs are precisely hypercubes [18]. For a survey of many different aspects of median graphs, the reader is referred to [16]. Quasi-median graphs have been introduced by Mulder [19] as a natural nonbipartite extension of median graphs. Chung, Graham, and Saks [11] and independently Wilkeit [24] proved that they are the weak retracts of Hamming graphs. On the

other hand, Hamming graphs are the regular quasi-median graphs [19]. In [3] a survey of characterizations of quasi-median graphs is given including some new ones.

Partial cubes, that is, isometric subgraphs of hypercubes, have been first investigated by Graham and Pollak [12], see also [25]. A nonbipartite extension of this class are isometric subgraphs of Hamming graphs, called partial Hamming graphs, see [10, 13, 23]. Since (weak) retracts are isometric subgraphs, quasimedian graphs are partial Hamming graphs and median graphs are partial cubes.

Median graphs have many interesting properties, cf. [4, 16, 17, 18], but not a lot of them can be extended to partial cubes. Brešar, Imrich and Klavžar [8] introduced a class of tree-like partial cubes which lies between median graphs and partial cubes and possesses many of the properties of median graphs. The authors characterized tree-like partial cubes and listed several properties which are shared with median graphs.

Tree-like partial Hamming graphs which we introduce in this paper are defined with an expansion procedure. There are also many other classes of graphs defined or characterized with a certain type of expansion. The most investigated such classes are median graphs, quasi-median graphs, partial cubes and partial Hamming graphs [10, 19]. But there are also several subclasses of partial cubes and partial Hamming graphs with nice (maybe just partial) results using expansions [7, 14]. Because of those nice results one could ask whether graphs characterized with an expansion procedure have also other interesting properties.

In this paper we consider tree-like partial cubes and their generalizations. In the next section we introduce tree-like partial Hamming graphs and recall some well-known definitions and results. We follow with a section in which we detect a mistake in the proof of the result from [8] about dismantlability of the cube graph of a tree-like partial cube. We also present a counterexample of the assertion from [8] that convex subgraphs of a tree-like partial cubes are tree-like partial cubes. We continue with a section in which we extend some results on tree-like partial cubes to a bigger class of tree-like partial Hamming graphs. In particular we show that Hamming graphs are the only regular tree-like partial Hamming graphs and that any gated subgraph of a graph from this class is also in this class, which implies a characterization of tree-like partial Hamming graphs. Finally we prove a result about dismantlability of the intersection graph of maximal Hamming graph s of a tree-like partial Hamming graph which gives a corrected proof of the result from [8] about dismantlability of the cube graph of a tree-like partial cube.

2. NOTATION AND PRELIMINARY RESULTS

All graphs G = (V, E) occurring in this paper are undirected and without loops or multiple edges. The *distance* $d(u, v) = d_G(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path, and the *interval* I(u, v) between u and vconsists of all vertices on shortest (u, v)-paths, that is, of all vertices (metrically) between u and v: $I(u, v) = \{x \in V : d(u, x) + d(x, v) = d(u, v)\}$. For a set U of vertices of a graph G we denote with $\langle U \rangle$ the subgraph of G induced with the vertices of U. A subgraph H of G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. An induced subgraph of G (or the corresponding vertex set) is called *convex* if it includes the interval of G between any pair of its vertices. An induced subgraph H of a graph G is said to be *gated* if for every vertex xoutside H there exists a vertex x' (the *gate* of x) in H such that each vertex yof H is connected with x by a shortest path passing through the gate x' (i.e., $x' \in I(x, y)$). Clearly gated subgraphs are convex and convex subgraphs are isometric.

The Cartesian product [15] $G = G_1 \Box \cdots \Box G_n$ of n graphs G_1, \ldots, G_n has the *n*-tuples (x_1, \ldots, x_n) as its vertices (with vertex x_i from G_i) and an edge between two vertices $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ if and only if, for some i, the vertices x_i and y_i are adjacent in G_i , and $x_j = y_j$ for the remaining $j \neq i$. The subgraph G_i^u induced by all vertices that differ from a given vertex uonly in the *i*th coordinate is isomorphic to G_i and called the G_i -layer through u. The Cartesian product of k copies of K_2 is a hypercube or k-cube Q_k . If all the factors in a Cartesian product are complete graphs then G is called Hamming graph. Isometric subgraphs of hypercubes are called partial cubes and isometric subgraphs of Hamming graphs are partial Hamming graphs.

A graph G is a *median graph* if there exists a unique vertex x to every triple of vertices u, v, and w such that x lies simultaneously on a shortest u, v-path, a shortest u, w-path, and a shortest w, v-path. Median graphs are partial cubes, cf. [19, 15].

Binary expansion was first defined in [17] and a generalization of binary expansion using more covering sets was first introduced in [19]. We will use the definition of general expansion introduced by Chepoi [10] in the following way.

Let G be a connected graph and let W_1, W_2, \ldots, W_n be subsets of V(G) such that:

- 1. $W_i \cap W_j \neq \emptyset$ for all $i, j \in \{1, \ldots, n\}$;
- 2. $\bigcup_{i=1}^{n} W_i = V(G);$
- 3. there are no edges between sets $W_i \setminus W_j$ and $W_j \setminus W_i$ for all $i, j \in \{1, \ldots, n\}$;
- 4. subgraphs $\langle W_i \rangle$, $\langle W_i \cup W_j \rangle$ are isometric in G for all $i, j \in \{1, \ldots, n\}$.

Then to each vertex $x \in V(G)$ we associate a set $\{i_1, i_2, \ldots, i_t\}$ of all indices i_j , where $x \in W_{i_j}$. A graph G' is called an expansion of G relative to the sets W_1, W_2, \ldots, W_n if it is obtained from G in the following way:

- 1. replace each vertex x of G with a clique with vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_t}$;
- 2. if an index i_s belongs to both sets $\{i_1, \ldots, i_t\}, \{i'_1, \ldots, i'_l\}$ corresponding to adjacent vertices x and y in G then let $x_{i_s}y_{i_s} \in E(G')$.

If $U = W_i \cap W_j$ is convex in G for all $i, j \in \{1, ..., n\}$, we speak of a *convex* expansion and if the intersection is isometric in G, then the expansion is called *isometric*. Contraction is the operation inverse to the expansion. If n = 2 then the expansion is called *binary expansion*.

Let U be an isometric subset of a graph G and $n \ge 2$. If $W_1 = V(G), W_2 = W_3 = \cdots = W_n = U$ then the expansion is called *peripheral expansion* of G along U (see Figure 1). Peripheral expansion was first introduced in [20] under the name *extremal expansion*. In this case G' consists of the union of graphs induced by $V(G), \underbrace{U, \ldots, U}_{n-1}$ where the copies of U (one such copy is contained also

in V(G) induce a subgraph isomorphic to $K_n \Box U$. We say that a graph G is a *tree-like partial Hamming graph* if it can be obtained from K_1 by a sequence of peripheral expansions. If n = 2 in each step of the expansion procedure then G is called a *tree-like partial cube* introduced in [8]. Thus every tree-like partial cube is also a tree-like partial Hamming graph.



Figure 1. Peripheral expansion of G along U.

Partial cubes were characterized as graphs that can be obtained from K_1 by a sequence of binary expansions [10] and median graphs are graphs that can be obtained from K_1 by a sequence of binary convex expansions [17, 19]. Moreover, by a result of Mulder [20], these expansions can be assumed to be peripheral. Hence, by definition, every median graph is also a tree-like partial cube and every tree-like partial cube is also a partial cube. From the result of Chepoi [10], who proved that partial Hamming graphs are exactly the graphs that can be obtained from K_1 by a sequence of expansions, it follows that every tree-like partial Hamming graph.

Let G = (V, E) be a connected graph and ab an edge of G. Then we use the following notation:

 $W_{ab} = \{ w \in V : d_G(a, w) < d_G(b, w) \},\$

 $U_{ab} = \{ w \in W_{ab} : w \text{ has a neighbor in } W_{ba} \},\$

 $F_{ab} = \{ e \in E : e \text{ is an edge between } W_{ab} \text{ and } W_{ba} \}.$

As in [23] we denote for a subgraph H of a graph G,

$$W(H) = \{x \in V(G) : \text{ for each } a \in H, d(a, x) = d(H, x)\}.$$

Note that in bipartite graphs W_{ab} and W_{ba} are disjoint, $V = W_{ab} \cup W_{ba}$ and $W(\langle \{a, b\} \rangle) = \emptyset$ for any edge ab in G.

A graph G is an *amalgam* of two subgraphs G' and G'' if $G' \cup G'' = G, G' \cap G'' \neq \emptyset$, and there are no edges between $G' \setminus G''$ and $G'' \setminus G'$. We also say that G is obtained by an *amalgamation* along the common subgraph $G' \cap G''$ of G' and G''. The amalgamation is called *isometric* if the intersection $G' \cap G''$ is an isometric subgraph of G' and G''.

A subgraph V' of G is called *peripheral* if there exist graphs G', V, U such that G is an isometric amalgam of G' and V along U, where $V \cong K_n \Box U$ for some $n \ge 2$ and $V' = V \setminus U$. It is clear that $V' \cong K_{n-1} \Box U$. The corresponding vertex set of V' is called *periphery*. Peripheral subgraphs were first introduced in [20] under the name *extremal subgraphs*. A peripheral subgraph was also used by Brešar [6], where the amalgamation was gated instead of isometric. To simplify the notation let U denote also the corresponding vertex set of U and let V' denote also the corresponding vertex set of V'.

Every tree-like partial Hamming graph G can be obtained with an expansion procedure. Therefore we will use the following notation. Let G be obtained by peripheral expansion from a tree-like partial Hamming graph G' along U and let V' be the subgraph of G obtained in this expansion step. Then G is also isometric amalgam of G' and the graph induced with the vertices of $V' \cup U$ along U. Since $V = K_n \Box U$ and $V' = V \setminus U$, V' is peripheral subgraph of G. Thus every tree-like partial Hamming graph contains a periphery. Note also that $U = U_{ab}$ and $\langle U_{ba} \cup W(\langle \{a, b\} \rangle) \rangle = V'$ for any edge ab between G' and V'.

3. TREE-LIKE PARTIAL CUBES

Here is the main characterization of tree-like partial cubes proved in [8].

Theorem 1 [8]. A partial cube G is tree-like if and only if every gated subgraph of G contains a periphery.

The authors of [8] remarked that Theorem 1 implies that convex subgraphs of tree-like partial cubes are tree-like partial cubes. We claim that this is not always true. Indeed if H is convex subgraph of G then a gated subgraph of H is not necessary gated in G. We reject the result also with the counterexample depicted on Figure 2, where the outer six-cycle (periphery U_{ab}) is convex but it is not a tree-like partial cube.



Figure 2. Tree-like partial cube with convex C_6 .

We continue with pointing to an error in the proof of the following theorem from [8].

Theorem 2 [8]. Every weak retract of a tree-like partial cube is a tree-like partial cube.

In the proof of this theorem the authors used that a periphery U of a tree-like partial cube G is a tree-like partial cube which is not always true. Furthermore let u and x be two adjacent vertices of a periphery U of a tree-like partial cube G and let v and y be their unique neighbors in $G \setminus U$, respectively. In the proof of Theorem 2 the authors also claimed that the subgraph of G induced by $G \setminus (W_{vu} \cap W_{vy})$ is a tree-like partial cube which is again not necessarily true. A counterexample is depicted on Figure 3. Therefore the question is whether weak retracts of tree-like partial cubes are tree-like partial cubes?



Figure 3. Tree-like partial cube G, two adjacent vertices u and x of a periphery U of G (the outer C_6) and a subgraph of G induced by $G \setminus (W_{vu} \cap W_{vy})$ which is not a tree-like partial cube (it contains a gated subgraph (the outer C_6) without a periphery).

On the other hand, if H is a gated subgraph of G and H' is a gated subgraph of H then H' is gated in G. Thus Theorem 1 directly implies that gated subgraphs of tree-like partial cubes are tree-like partial cubes.

Cube graphs are the intersection graphs of maximal hypercubes. The intersection graph of maximal Hamming graphs of G is a graph H, in symbols H = Q(G), in which the vertices are the maximal Hamming subgraphs of G and two vertices in H are adjacent whenever the corresponding Hamming graphs in G intersect. Note that the only Hamming graphs in partial cubes are hypercubes. Thus the intersection graph of maximal Hamming graphs of a partial cube G is exactly the cube graph of G. Furthermore, for a partial cube G, let G^{Δ} denote the graph obtained from a graph G that has the same vertex set as G and in that two vertices are adjacent whenever they are in the same hypercube of G [5]. The clique graph of a graph G is the intersection graph of maximal cliques in G.

Dismantlable graphs are defined by an elimination procedure, that is a generalization of the elimination of simplicial vertices in chordal graphs. We say that a vertex u in a graph G is *dominated* by its neighbor v if all neighbors of uexcept v are also neighbors of v. If G can be reduced to the one-vertex graph by successive removal of dominated vertices then G is called a *dismantlable graph*. Dismantlable graphs were investigated in [2, 21, 9].

The authors of [8] proved that the cube graph Q(G) of a tree-like partial cube G is dismantlable. They used the argument that the cube graph of a tree-like partial cube G coincides with the clique graph of G^{Δ} , which is not true. For example, let $G = Q_3^-$, which is a graph obtained from Q_3 with the removal of one vertex. Then the cube graph of Q_3^- is isomorphic to K_3 and the clique graph of $(Q_3^-)^{\Delta}$ is isomorphic to K_4 (see Figure 4). We give a new proof of this result using different accession in Section 4.



Figure 4. Graphs Q_3^- and $(Q_3^-)^{\Delta}$.

4. TREE-LIKE PARTIAL HAMMING GRAPHS

In this section we list some properties of tree-like partial Hamming graphs which generalize the results on tree-like partial cubes. In particular we characterize treelike partial Hamming graphs and show that the intersection graph of maximal Hamming graphs of a tree-like partial Hamming graph is dismantlable, which corrects and generalizes the proof about dismantlability of the cube graph of a tree-like partial cube from [8]. From the latter result we deduce that every tree-like partial Hamming graph G contains a Hamming graph that is invariant under every automorphism of G.

There are many properties of tree-like partial Hamming graphs that can be extended from tree-like partial cubes. Here is the characterization of regular tree-like partial Hamming graphs.

Theorem 3. Regular tree-like partial Hamming graphs are precisely Hamming graphs.

Proof. Let G be a regular tree-like partial Hamming graph and let G be obtained by peripheral expansion along U from a tree-like partial Hamming graph G'. Let V' be the subgraph of G obtained in the last expansion step, that is, V' a is peripheral subgraph of G, and let V be the subgraph of G induced with $U \cup V'$. Furthermore, let n be the number of copies of U in G, that is $V = K_n \Box U$. Since G is regular and there is no edge from V' to $G' \setminus U$ every vertex x from V' has the same degree as its unique neighbor x' in U. Therefore all vertices from U have degrees k + n - 1, where $k = deg_U(x')$. Thus the vertices from U have no neighbors in $G' \setminus U$ which means that $G = K_n \Box U$. Since G is regular and every vertex from U has n - 1 neighbors in $G \setminus U$, also U is a regular tree-like partial Hamming graph. Using induction assumption we get that U is a Hamming graph and thus so is G.

This result clearly implies the previously-known result for tree-like partial cubes.

Corollary 4. Regular tree-like partial cubes are hypercubes.

For the next theorem we need the following well-known result.

Lemma 5 [22]. Let $G = G_1 \Box G_2$ be a Cartesian product of connected graphs. Then H is gated in G if and only if $H = H_1 \Box H_2$, where H_1 (resp. H_2) is gated in G_1 (resp. G_2).

We already mentioned that gated subgraph of a tree-like partial cube is a tree-like partial cube. The extension of this result to the tree-like partial Hamming graphs gives a useful characterization of these graphs.

Theorem 6. Every gated subgraph of a tree-like partial Hamming graph is a tree-like partial Hamming graph.

Proof. The proof is by induction on the number of vertices of a tree-like partial Hamming graph. Let G_1 be a gated subgraph of a tree-like partial Hamming graph G and let G be obtained by a peripheral expansion along U from a tree-like partial Hamming graph G'. Let V' be the subgraph of G obtained in the

last expansion step and let V be the subgraph of G induced with the vertices of $U \cup V'$, that is $V \cong K_n \Box U$. If G_1 is contained in G' we infer from the isometry of G' that G_1 is also gated in G', which is a smaller tree-like partial Hamming graph. By induction assumption G_1 is a tree-like partial Hamming graph.

Assume now that $G_1 \cap G' \neq \emptyset$ and $G_1 \cap V' \neq \emptyset$. To complete the proof of this case we need the following two claims.

Claim 7. $G_1 \cap V$ is gated in V.

Proof. Since V is an isometric subgraph of G it is enough to see that the gate of $v \in V$ in G_1 is from V. Let v be an arbitrary vertex from $V \setminus G_1$ and suppose that the gate g of v in G_1 is from $G \setminus V$. First let $v \in V'$. Since $G_1 \cap V' \neq \emptyset$ there exists $x \in G_1 \cap V'$. Clearly g cannot lie on the interval between v and x, which gives a contradiction. Thus we may assume that $v \in U$. Since there are at least two U-layers of V that have nonempty intersection with G_1 , there exists a U-layer U_1 of V different from U which has nonempty intersection with G_1 . Let $y'' \in G_1 \cap U_1$ and let y be the copy of v in U_1 . Since $g \in G \setminus V$ is the gate of v in $G_1, y \notin G_1$. Let y' be the gate of y in G_1 . Then $y' \in U_1$, otherwise y' cannot lie on a shortest y, y''-path of G. Now let v' be the copy of y' in Uand note that $v' \in G_1$. Indeed if $v' \notin G_1$ then y' is the gate of v' in G_1 which implies that $G_1 \cap U = \emptyset$, a contradiction. Thus $v' \in G_1$. Since g is the gate of v in G_1 , d(y, y') = d(v, v') = d(v, g) + d(g, v') and since y' is the gate of y in G_1 , d(y,g) = d(y,y') + d(y',g) > d(v,v'). On the other hand we can find y, g-path in G of length $1 + d(v, g) \le d(v, v')$, a contradiction.

Claim 8. $G_1 \cap G'$ is gated in G.

Proof. Note that if y is a gate for $x \in G \setminus G'$ in G_1 then the unique neighbor y' of y in U is a gate for x in $G_1 \cap G'$.

G' is an isometric subgraph of G and $G_1 \cap G'$ is gated in G, $G_1 \cap G'$ is also gated in G' and hence it is a tree-like partial Hamming graph by the induction assumption. From the structure of $G_1 \cap V$ it follows that G_1 is obtained from $G_1 \cap G'$ by peripheral expansion along $G_1 \cap U$ which implies that G_1 is a tree-like partial Hamming graph.

Finally let G_1 be contained in V', that is G_1 is contained in a Cartesian product $K_{n-1} \square U$. Using Lemma 5 we get that $G_1 = H \square H'$, where H is gated in K_{n-1} and H' is isomorphic to gated subgraph of U. Therefore H is either K_1 or K_{n-1} . Since G_1 is gated in G it is clear that $H \neq K_{n-1}$ (unless n = 2), otherwise there is no gate of a vertex $x \in U \cap H'$ in G_1 . Thus $G_1 = K_1 \square H'$ is contained in one U-layer of V'. Now we consider the subgraph G_2 of $U \subseteq G'$ isomorphic to G_1 , induced by vertices that correspond to the copy of G_1 in U. We claim that G_2 is gated in G'. Indeed, the distance from any vertex x of G' to a vertex of G_1 is exactly 1 plus the distance from x to the corresponding vertex of G_2 . Hence the gatedness of G_2 clearly follows from the gatedness of G_1 . Using the induction assumption we get that G_2 is a tree-like partial Hamming graph and therefore so is G_1 .

Theorem 9. A partial Hamming graph G is tree-like if and only if every gated subgraph of G contains a periphery.

Proof. Let G be a tree-like partial Hamming graph and let G_1 be an arbitrary gated subgraph of G. Then it follows from Theorem 6 that G_1 is a tree-like partial Hamming graph and hence it contains a periphery.

For the converse suppose that G is a partial Hamming graph in which every gated subgraph contains a periphery. Since G is gated in G it contains a periphery and thus one can obtain G by a peripheral expansion from a graph G'. If G' would contain a gated subgraph G_1 without periphery then G_1 would be gated also in G. By induction on the number of vertices we get that G' is a tree-like partial Hamming graph, and thus so is G.

Corollary 10. For any periphery U of a tree-like partial Hamming graph G, $G \setminus U$ is a tree-like partial Hamming graph.

Our next goal is to prove that the intersection graph of maximal Hamming graphs of any tree-like partial Hamming graph is dismantlable.

In the rest of the paper we will use the following notation. Let G be obtained by peripheral expansion from a tree-like partial Hamming graph G' along U. Then the graph obtained in this expansion step is a peripheral subgraph V' of Gisomorphic to $K_{n-1} \Box U$ for some $n \geq 2$. We denote with V the subgraph of Ginduced with the vertices of $U \cup V'$, that is $V \cong K_n \Box U$.

Note that in Cartesian products complete graphs lie in layers. Therefore the proof of the following result is obvious and thus we skip it.

Lemma 11. Let H be a Hamming subgraph of a Cartesian product $G = U \Box K_n$. Then there exists a Hamming graph H' in U such that $H \cong K_m \Box H'$ for some $m \leq n$ and every U-layer of G has either empty intersection with H or the intersection is isomorphic to H'.

Lemma 12. Let G be a tree-like partial Hamming graph and let H be a Hamming subgraph of G having nonempty intersection with $G \setminus G'$. Then H is contained in V.

Proof. Let $H = K_{n_1} \Box \cdots \Box K_{n_k}$ and suppose that $H \nsubseteq G \setminus G'$, which means that H intersects $G \setminus G'$ and G'. Let $H_1 = K_{n_1} \Box \cdots \Box K_{n_{k-1}}$, that is $H = H_1 \Box K_{n_k}$ and let H' be one H_1 -layer of H, which means that H' is a subgraph of H. First

note that every vertex $x \in G \setminus G'$ has exactly one neighbor x' in G' and x' is contained in U.

First let $H' \subseteq G'$. Since $H \cap (G \setminus G') \neq \emptyset$ there exists $x \in H \cap (G \setminus G')$ which is contained in $H \setminus H'$, since $H' \subseteq G'$. Let x' be the unique neighbor of x in $H' \subseteq G'$. Since x has just one neighbor in G', all the neighbors of x in $H \setminus H'$ are from $G \setminus G'$. Every such vertex has a unique neighbor in $H' \cap G'$. We conclude that $H \setminus H'$ is contained in $G \setminus G'$ and thus H' is contained in U which implies that H is contained in V.

Finally let H' intersect $G \setminus G'$. Then using the induction, we get $H' \subseteq V$. If $H' \subseteq G \setminus G'$, then it is clear that H is contained in V. Thus we may assume that $H' \cap G', H' \cap (G \setminus G') \neq \emptyset$. From Lemma 11 it follows that there exists a Hamming graph H'' in U such that $H' \cong K_m \Box H''$ for some $m \leq n$ and every U-layer of V has either empty intersection with H' or the intersection is isomorphic to H''. For the purpose of contradiction suppose that there exists $z \in H \setminus H'$ such that $z \notin V$. Let H'_i be the H_1 -layer of $H = H_1 \Box K_{n_k}$ that contains z and let z' be the unique neighbor of z in H'. Since z is adjacent to $z' \in H' \subset V$ and $z \notin V$ it is clear that $z' \in U$. From the structure of $H' (H' \cong K_m \Box H'')$ and since $H' \cap (G \setminus G') \neq \emptyset$, there exists $y' \in H' \cap (G \setminus G')$ that lies in the same K_n -layer of V as z'. Let y be the neighbor of y' in $H'_i \subseteq H \setminus H'$ and thus y is adjacent to z. Since every vertex from $G \setminus G'$ has just one neighbor in G', we get that $y \in G \setminus G'$, which contradicts the fact that y is adjacent to $z \in G \setminus V$. Thus H is contained in V.

From Lemma 11 and Lemma 12 we get the following result.

Corollary 13. Let G be a tree-like partial Hamming graph and let H be a Hamming subgraph of G having nonempty intersection with $G \setminus G'$. Then there exists Hamming graph H' in U such that $H \cong K_m \Box H'$ for some $m \leq n$ and every U-layer of V has either empty intersection with H or the intersection is isomorphic to H'.

Let $\mathcal{Q}^{(G)}$ be the function which maps maximal Hamming subgraphs of G to vertices of Q(G), such that two vertices x and y of Q(G) are adjacent if and only if the Hamming graphs $(\mathcal{Q}^{(G)})^{-1}(x)$ and $(\mathcal{Q}^{(G)})^{-1}(y)$ have nonempty intersection in G. For every Hamming graph H of G we denote the image of H with respect to $\mathcal{Q}^{(G)}$ with $x_H^{(G)}$, that is $\mathcal{Q}^{(G)}(H) = x_H^{(G)}$.

Theorem 14. For any tree-like partial Hamming graph G, the graph Q(G) is dismantlable.

Proof. The proof is by induction on the number of vertices of a tree-like partial Hamming graph. Let U, V, G' be the subgraphs of G defined above. Then G' is a tree-like partial Hamming graph and hence Q(G') is dismantlable by induction

assumption. Clearly Q(G') is induced subgraph of Q(G). Therefore, to complete the proof, it is enough to see that the vertices of $Q(G) \setminus Q(G')$ are dominated in Q(G). Note that every vertex of $Q(G) \setminus Q(G')$ corresponds to the maximal Hamming subgraph H of G such that $H \cap G' (= H \cap U$, using Lemma 12) is not a maximal Hamming graph of G' and $H \cap (G \setminus G') \neq \emptyset$. We will prove that every such vertex of Q(G) is dominated in Q(G). Therefore let H be such maximal Hamming graph of G, that is $x_H^{(G)} \in Q(G) \setminus Q(G')$. Since H is a maximal, $H \cap U \neq \emptyset$ and it follows from Corollary 13 that $H' = H \cap G'$ is a Hamming graph such that $H \cong K_n \Box H'$. Since $x_H^{(G)} \in Q(G) \setminus Q(G')$, H' is not a maximal Hamming graph of G'. Let K be a maximal Hamming graph in G', containing H'. Then, using Lemma 12, we get that K is also maximal Hamming graph of G. Let H_1, \ldots, H_n be maximal Hamming subgraphs of G which have nonempty intersection with $(G \setminus G') \cap H$ and let $H'_i = H_i \cap U = H_i \cap G'$, where the last equality holds because of Lemma 12. Corollary 13 implies that $H'_i \cap H' \neq \emptyset$ for every $i \in \{1, \ldots, n\}$. Furthermore let K_1, \ldots, K_m be maximal Hamming graphs in G' having nonempty intersection with H', such that $K_j \neq H'_i$ for all $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$ and $K_j \neq K$ for all $j \in \{1, \ldots, m\}$. Clearly these Hamming graphs are also maximal in G and the neighbors of the vertex $x_H^{(G)}$ in Q(G) are exactly the vertices $x_{H_1}^{(G)}, \ldots, x_{H_n}^{(G)}, x_{H_1}^{(G)}, \ldots, x_{K_m}^{(G)}$. Therefore the vertex $x_H^{(G)}$ is dominated by its neighbor $x_K^{(G)}$ in Q(G), which completes the proof.

Corollary 15. Let G be a tree-like partial cube. Then the cube graph of G is dismantlable.

Dismantlability of the intersection graph of maximal Hamming graphs of a treelike partial Hamming graph implies the following result.

Corollary 16. Let G be a tree-like partial Hamming graph. Then G contains a Hamming graph that is invariant under every automorphism of G.

Proof. First note that every automorphism of a tree-like partial Hamming graph G induces an automorphism of Q(G). Observe also that automorphisms of dismantlable graphs always fix a complete subgraph (see also [8]) and thus it follows from Theorem 14 that Q(G) contains a complete subgraph K that is invariant under every automorphism of Q(G). Since vertices of K are pairwise intersecting Hamming graphs of G, their intersection is a Hamming graph which is invariant under all automorphisms of G.

5. Concluding Remarks

This paper has much in common with [8]. We explain that convex subgraphs of tree-like partial cubes are not necessary tree-like partial cubes as the authors from [8] asserted. Moreover we correct the proof of the theorem from [8] which says that the cube graph of a tree-like partial cube is dismantlable. Beside that we also generalize the mentioned result. Finally we point to a gap in the proof of the theorem from [8] that weak retracts of tree-like partial cubes are tree-like partial cubes, but it remains open whether the result holds or not.

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