

NOTE

A DIFFERENT SHORT PROOF OF BROOKS' THEOREM

LANDON RABERN

Arizona State University
School of Mathematical & Statistical Sciences

e-mail: landon.rabern@gmail.com

Abstract

Lovász gave a short proof of Brooks' theorem by coloring greedily in a good order. We give a different short proof by reducing to the cubic case.

Keywords: coloring, clique number, maximum degree.

2010 Mathematics Subject Classification: 05C15.

In [5] Lovász gave a short proof of Brooks' theorem by coloring greedily in a good order. Here we give a different short proof by reducing to the cubic case. One interesting feature of the proof is that it does not use any connectivity concepts. Our notation follows Diestel [2] except we write K_t instead of K^t for the complete graph on t vertices.

Theorem 1 (Brooks [1]). *Every graph G with $\chi(G) = \Delta(G) + 1 \geq 4$ contains $K_{\Delta(G)+1}$.*

Proof. Suppose the theorem is false and choose a counterexample G minimizing $|G|$. Put $\Delta := \Delta(G)$. Using minimality of $|G|$, we see that $\chi(G - v) \leq \Delta$ for all $v \in V(G)$. In particular, G is Δ -regular.

First, suppose $\Delta \geq 4$. Pick $v \in V(G)$ and let w_1, \dots, w_Δ be v 's neighbors. Since $K_{\Delta+1} \not\subseteq G$, by symmetry we may assume that w_2 and w_3 are not adjacent. Choose a $(\Delta+1)$ -coloring $\{\{v\}, C_1, \dots, C_\Delta\}$ of G where $w_i \in C_i$ so as to maximize $|C_1|$. Then C_1 is a maximal independent set in G and in particular, with $H := G - C_1$, we have $\chi(H) = \chi(G) - 1 = \Delta = \Delta(H) + 1 \geq 4$. By minimality of $|G|$, we get $K_\Delta \subseteq H$. But $\{\{v\}, C_2, \dots, C_\Delta\}$ is a Δ -coloring of H , so any K_Δ in H must contain v and hence w_2 and w_3 , a contradiction.

Therefore G is 3-regular. Since G is not a forest it contains an induced cycle C . Put $T := N(C)$. Then $|T| \geq 2$ since $K_4 \not\subseteq G$. Take different $x, y \in T$ and put

$H_{xy} := G - C$ if x is adjacent to y and $H_{xy} := (G - C) + xy$ otherwise. Then, by minimality of $|G|$, either H_{xy} is 3-colorable or adding xy created a K_4 in H_{xy} .

Suppose the former happens. Then we have a 3-coloring of $G - C$ where x and y receive different colors. We can easily extend this partial coloring to all of G since each vertex of C has a set of two available colors and some pair of vertices in C get different sets.

Whence adding xy created a K_4 , call it A , in H_{xy} . We conclude that T is independent and each vertex in T has exactly one neighbor in C . Hence $|T| \geq |C| \geq 3$. Pick $z \in T - \{x, y\}$. Then x is contained in a K_4 , call it B , in H_{xz} . Since $d(x) = 3$, we must have $A - \{x, y\} = B - \{x, z\}$. But then any $w \in A - \{x, y\}$ has degree at least 4, a contradiction. ■

We note that the reduction to the cubic case is an immediate consequence of more general lemmas on hitting all maximum cliques with an independent set (see [4], [6] and [3]). Tverberg pointed out that this reduction was also demonstrated in his paper [7].

REFERENCES

- [1] R.L. Brooks, *On colouring the nodes of a network*, in: Math. Proc. Cambridge Philos. Soc. **37** Cambridge Univ. Press (1941) 194–197.
- [2] R. Diestel, *Graph Theory* (Fourth Ed., Springer Verlag, 2010).
- [3] A.D. King, *Hitting all maximum cliques with a stable set using lopsided independent transversals*, J. Graph Theory **67** (2011) 300–305.
doi:10.1002/jgt.20532
- [4] A.V. Kostochka, *Degree, density, and chromatic number*, Metody Diskret. Anal. **35** (1980) 45–70 (in Russian).
- [5] L. Lovász, *Three short proofs in graph theory*, J. Combin. Theory (B) **19** (1975) 269–271.
doi:10.1016/0095-8956(75)90089-1
- [6] L. Rabern, *On hitting all maximum cliques with an independent set*, J. Graph Theory **66** (2011) 32–37.
doi:10.1002/jgt.20487
- [7] H. Tverberg, *On Brooks' theorem and some related results*, Math. Scand. **52** (1983) 37–40.

Received 28 May 2012
Accepted 3 January 2013