# ON THE DETERMINANT OF $q$-DISTANCE MATRIX OF A GRAPH 

Hong-Hai Li ${ }^{1}$, Li Su ${ }^{2}$ and Jing Zhang<br>College of Mathematics and Information Science<br>Jiangxi Normal University<br>Nanchang, 330022, P.R. China<br>e-mail: suli@jxnu.edu.cn<br>lhh@mail.ustc.edu.cn<br>zhangjing3611@163.com


#### Abstract

In this note, we show how the determinant of the $q$-distance matrix $D_{q}(T)$ of a weighted directed graph $G$ can be expressed in terms of the corresponding determinants for the blocks of $G$, and thus generalize the results obtained by Graham et al. [R.L. Graham, A.J. Hoffman and H. Hosoya, On the distance matrix of a directed graph, J. Graph Theory 1 (1977) 85-88]. Further, by means of the result, we determine the determinant of the $q$-distance matrix of the graph obtained from a connected weighted graph $G$ by adding the weighted branches to $G$, and so generalize in part the results obtained by Bapat et al. [R.B. Bapat, S. Kirkland and M. Neumann, On distance matrices and Laplacians, Linear Algebra Appl. 401 (2005) 193209]. In particular, as a consequence, determinantal formulae of $q$-distance matrices for unicyclic graphs and one class of bicyclic graphs are presented.


Keywords: $q$-distance matrix, determinant, weighted graph, directed graph.

2010 Mathematics Subject Classification: 05C50,15A18.

[^0]
## 1. Introduction

We consider graphs which have no loops or parallel edges. A weighted graph is a (directed or undirected) graph in which each edge or arc is assigned a weight, which is a positive number. An unweighted graph, or simply a graph, is thus a weighted graph with each of the edges or arcs bearing weight 1 . Let $G$ be a weighted directed graph with vertex set $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$. For vertices $v_{i}, v_{j}$ of $G$, the distance from $v_{i}$ to $v_{j}$, denoted by $d_{i j}$, is defined to be the minimum weight of all paths from $v_{i}$ to $v_{j}$, where the weight of a path is the sum of the weights of the arcs in that path. We shall assume that $G$ is strongly connected so that $d_{i j}$ always exists. The distance matrix $D(G)$ of $G$ is an $n \times n$ matrix which has $d_{i j}$ as its $(i, j)$ entry.

Some $q$-analogs of the distance for a tree were considered in $[2,9]$. Now we generalize the notion for a general weighted directed graph. Let $G$ be a weighted directed graph and suppose that the distance from $u$ to $v$ is $\alpha$. Define the $q$ distance from $u$ to $v$ to be $[\alpha]$, where

$$
[\alpha]=\left\{\begin{array}{lc}
\frac{1-q^{\alpha}}{1-q}, & \text { if } q \neq 1 \\
\alpha, & \text { otherwise }
\end{array}\right.
$$

By definition, $[0]=0$ and $[\alpha]=1+q+q^{2}+\cdots+q^{\alpha-1}$ if $\alpha$ is a positive integer. The $q$-distance matrix $D_{q}(G)$ of $G$ is the square matrix which has as its $(i, j)$ entry the $q$-distance from $v_{i}$ to $v_{j}$. If $q=1$ then $D_{q}(G)$ is the distance matrix $D(G)$ of a graph $G$. Hence the distance matrix $D(G)$ is a special case of the $q$-distance matrix $D_{q}(G)$.

Distance matrices of graphs, particularly trees, have been extensively investigated in the literature. A classical result concerning the determinant of the distance matrix of a tree, due to Graham and Pollak [4], asserted that for a tree $T_{n}$ on $n$ vertices, $\operatorname{det}\left(T_{n}\right)=(-1)^{n-1}(n-1) 2^{n-2}$. Thus, $\operatorname{det}\left(T_{n}\right)$ is a function dependent on $n$ only, independent of the structure of $T_{n}$. Graham, Hoffman and Hosoya [5] studied further and obtained the following result. For a square matrix $A$, let $\operatorname{cof}(A)$ denote the sum of cofactors of $A$.

Theorem 1 [5]. If $G$ is a strongly connected directed graph with blocks $G_{1}, G_{2}, \ldots$, $G_{r}$, then

$$
\begin{gathered}
\operatorname{cof}(D(G))=\prod_{i=1}^{r} \operatorname{cof}\left(D\left(G_{i}\right)\right) \\
\operatorname{det}(D(G))=\sum_{i=1}^{r} \operatorname{det}\left(D\left(G_{i}\right)\right) \prod_{j \neq i} \operatorname{cof}\left(D\left(G_{j}\right)\right)
\end{gathered}
$$

Graham and Pollack determinantal formula for tree has been extended by Bapat et al. to the weighted case [1] and further by Yan et al. to the $q$-distance matrix of weighted tree [9]. We are not aware of Sivasubramanian's work [7] until we have
finished the paper. It is worth pointing that our Theorem 2 is a generalization of Theorem 2 of $[7]$ to the weighted case.

In this paper we show how the determinant of the $q$-distance matrix $D_{q}(T)$ of a weighted directed graph $G$ can be expressed in terms of the corresponding determinants for the blocks of $G$. Our proof is basically the same as Graham's proof, but this indeed generalizes Graham et al.'s result to $q$-distance matrix case, and further, by applying the result we determine the determinant of the $q$-distance matrix of the graph obtained from a connected weighted graph $G$ by adding the weighted branches to $G$, and so generalize in part the results obtained by Bapat et al. in [1]. In particular, as a consequence, determinantal formulae of $q$-distance matrices for unicyclic graphs and one class of bicyclic graphs are presented.

## 2. Main Results

We begin with some notation and definition. Let $G$ be a strongly connected directed graph and $D$ be its $q$-distance matrix which has its following form:

$$
D=\left(\begin{array}{c|ccc}
0 & {\left[\alpha_{1}\right]} & \cdots & {\left[\alpha_{n-1}\right]} \\
\hline\left[\beta_{1}\right] & & & \\
\vdots & & D_{1} & \\
{\left[\beta_{n-1}\right]} & & &
\end{array}\right) .
$$

Denote by $\xi(D)$ the cofactor in position $(1,1)$ of the matrix obtained by subtracting the first row from all other rows, then $p^{\alpha_{i}}$ times the first column from the $(i+1)$ th column of $D$ for $i=1, \ldots, n-1$. Observe that $\xi(D)=\operatorname{det}\left(D_{1}-M\right)$, where $M$ is the $(n-1) \times(n-1)$ matrix with $\left[\beta_{i}+\alpha_{j}\right]$ as its $(i, j)$ entry (since $\left.[\alpha+\beta]=p^{\beta}[\alpha]+[\beta]=[\alpha]+p^{\alpha}[\beta]\right)$.

A block of a graph is defined to be a maximal subgraph having no cut vertices.
Theorem 2. If $G$ is a strongly connected directed graph with blocks $G_{1}, G_{2}, \ldots$, $G_{r}$, then

$$
\begin{gather*}
\xi\left(D_{q}(G)\right)=\prod_{i=1}^{r} \xi\left(D_{q}\left(G_{i}\right)\right)  \tag{1a}\\
\operatorname{det}\left(D_{q}(G)\right)=\sum_{i=1}^{r} \operatorname{det}\left(D_{q}\left(G_{i}\right)\right) \prod_{j \neq i} \xi\left(D_{q}\left(G_{j}\right)\right) . \tag{1b}
\end{gather*}
$$

Proof. We proceed by induction on $r$, the number of blocks of $G$. The theorem is trivial for $r=1$ as $G$ itself is a block in this case. Assume that it holds for all strongly connected directed graphs with fewer than $r$ blocks, and let $G$ be a strongly connected directed graph with $r$ blocks. Then $G$ is not a block and has
at least one block which contains exactly one cut vertex of $G$, say $G_{1}$ with the unique cut vertex labeled by 0 . Let $G_{1}^{*}=G-\left(G_{1}-\{0\}\right)$ be the remainder of $G$. Assume that $V\left(G_{1}\right)=\{0,1, \ldots, m\}$ and $V\left(G_{1}^{*}\right)=\{0, m+1, \ldots, m+n\}$. Let
$D_{q}\left(G_{1}\right)=\left(\begin{array}{c|ccc}0 & {\left[a_{1}\right]} & \cdots & {\left[a_{m}\right]} \\ \hline\left[b_{1}\right] & & & \\ \vdots & & E & \\ {\left[b_{m}\right]} & & & \end{array}\right), D_{q}\left(G_{1}^{*}\right)=\left(\begin{array}{cccc}0 & {\left[f_{1}\right]} & \cdots & {\left[f_{n}\right]} \\ \hline\left[g_{1}\right] & & \\ \vdots & & H & \\ {\left[g_{n}\right]} & & & \end{array}\right)$.
Thus we have

$$
D_{q}(G)=\left(\begin{array}{c|c|c}
0 & \bar{a} & \bar{f} \\
\hline \bar{b} & E & \left(\left[b_{i}+f_{j}\right]\right) \\
\hline \bar{g} & \left(\left[g_{i}+a_{j}\right]\right) & H
\end{array}\right)
$$

where $\bar{a}=\left(\left[a_{1}\right], \ldots,\left[a_{m}\right]\right), \bar{b}=\left(\left[b_{1}\right], \ldots,\left[b_{m}\right]\right)^{T}, \bar{f}=\left(\left[f_{1}\right], \ldots,\left[f_{n}\right]\right)$ and $\bar{g}=$ $\left(\left[g_{1}\right], \ldots,\left[g_{n}\right]\right)^{T}$. Subtract $q^{a_{i}}\left(q^{f_{j}}\right.$, resp. $)$ times the first column from the $(i+1)$ th $((j+m+1)$ th, resp. $)$ column of $D_{q}(G)$, for $i=1, \ldots, m$ and $j=1, \ldots, n$; and also subtract the first row from every other row of $D_{q}(G)$. Then

$$
\begin{aligned}
\operatorname{det}\left(D_{q}(G)\right) & =\operatorname{det}\left(\begin{array}{c|c|c}
0 & \bar{a} & \bar{f} \\
\hline \bar{b} & E-\left(\left[b_{i}+a_{j}\right]\right) & 0 \\
\hline \bar{g} & 0 & H-\left(\left[g_{i}+f_{j}\right]\right)
\end{array}\right) \\
& =\operatorname{det}\left(\begin{array}{c|c}
0 & \bar{a} \\
\hline \bar{b} & E-\left(\left[b_{i}+a_{j}\right]\right)
\end{array}\right) \operatorname{det}\left(H-\left(\left[g_{i}+f_{j}\right]\right)\right) \\
& +\operatorname{det}\left(\begin{array}{c|c}
0 & \bar{f} \\
\hline \bar{g} & H-\left(\left[g_{i}+f_{j}\right]\right)
\end{array}\right) \operatorname{det}\left(E-\left(\left[b_{i}+a_{j}\right]\right)\right) \\
& =\operatorname{det}\left(D_{q}\left(G_{1}\right)\right) \xi\left(D_{q}\left(D_{1}^{*}\right)\right)+\operatorname{det}\left(D_{q}\left(G_{1}^{*}\right)\right) \xi\left(D_{q}\left(D_{1}\right)\right)
\end{aligned}
$$

where the second equality follows by Laplace expansion of determinants. Also we note that

$$
\begin{aligned}
\xi\left(D_{q}(G)\right) & =\operatorname{det}\left(\begin{array}{c|c}
E-\left(\left[b_{i}+a_{j}\right]\right) & 0 \\
0 & H-\left(\left[g_{i}+f_{j}\right]\right)
\end{array}\right) \\
& =\operatorname{det}\left(E-\left(\left[b_{i}+a_{j}\right]\right)\right) \operatorname{det}\left(H-\left(\left[g_{i}+f_{j}\right]\right)\right) \\
& =\xi\left(D_{q}\left(D_{1}\right)\right) \xi\left(D_{q}\left(D_{1}^{*}\right)\right)
\end{aligned}
$$

By the induction hypothesis, the assertion (1a) and (1b) follow immediately.
Let $\vec{T}$ be a directed graph obtained from a tree of order $n$ by replacing each undirected edge $f_{i}=\{u, v\}$ with two arcs (oppositely oriented edges) $e_{i}=(u, v)$ and $e_{i}^{\prime}=(v, u)$. Let $u_{i}>0$ and $v_{i}>0$ be the weights of the $\operatorname{arcs} e_{i}$ and $e_{i}^{\prime}$,
respectively. Note that $\vec{T}$ is a strongly connected graph consisting of $n-1$ blocks, denoted by $G_{1}, G_{2}, \ldots, G_{n-1}$. Observe that each $G_{i}$ actually consists of two opposite arcs, say $e_{i}$ and $e_{i}^{\prime}$. As $D_{q}\left(G_{i}\right)=\left(\begin{array}{cc}0 & {\left[u_{i}\right]} \\ {\left[v_{i}\right]} & 0\end{array}\right), \operatorname{det}\left(D_{q}\left(G_{i}\right)\right)=-\left[u_{i}\right]\left[v_{i}\right]$ and $\xi\left(D_{q}\left(G_{i}\right)\right)=-\left[u_{i}+v_{i}\right]$. Applying Theorem 2 to $\vec{T}$, we have $\xi\left(D_{q}(\vec{T})\right)=$ $\prod_{i=1}^{n-1}\left(-\left[u_{i}+v_{i}\right]\right)$ and the following result.
Theorem 3. Let $\vec{T}$ be the directed graph on $n$ vertices constructed as above. Then

$$
\operatorname{det}\left(D_{q}(\vec{T})\right)=(-1)^{n-1} \prod_{i=1}^{n-1}\left(\left[u_{i}+v_{i}\right]\right) \sum_{i=1}^{n-1} \frac{\left[u_{i}\right]\left[v_{i}\right]}{\left[u_{i}+v_{i}\right]} .
$$

Theorem 3 yields the following generalization of results of Yan and Yeh [9] and also Bapat and Rekhi [3]. This can easily be seen if we replace each undirected edge in a tree by two arcs of opposite orientations and then apply Theorem 3 to the obtained directed graph.
Corollary 4 [3]. Let $T$ be a weighted tree with $n$ vertices and weights $\alpha_{1}, \alpha_{2}, \ldots$, $\alpha_{n-1}$. Then

$$
\operatorname{det}\left(D_{q}(T)\right)=(-1)^{n-1} \prod_{i=1}^{n-1}\left[2 \alpha_{i}\right] \sum_{i=1}^{n-1} \frac{\left[\alpha_{i}\right]}{1+q^{\alpha_{i}}} .
$$

In particular, by letting $q=1$ in Corollary 4 , we obtain the following result.
Corollary 5 [1]. Let $T$ be a weighted tree with $n$ vertices and weights $\alpha_{1}, \alpha_{2}, \ldots$, $\alpha_{n-1}$. Then

$$
\operatorname{det}(D(T))=(-1)^{n-1} 2^{n-2}\left(\prod_{i=1}^{n-1} \alpha_{i}\right)\left(\sum_{i=1}^{n-1} \alpha_{i}\right)
$$

Next we shall focus ourself on undirected graphs. Let $G$ be a weighted graph, and suppose that we have a collection of weighted trees $B_{1}, \ldots, B_{k}$. Let $\bar{G}$ be the graph obtained from $G$ and $B_{1}, \ldots, B_{k}$ by adding, for each $i=1, \ldots, k$, a weighted edge between some vertex of $B_{i}$ and some vertex of $G$. We say that the new graph $\bar{G}$ is constructed by adding the weighted branches $B_{1}, \ldots, B_{k}$ to $G$. Note that trees, unicyclic graphs and bicyclic graphs all can be constructed in this way. Let $\varepsilon_{n}$ be the $n$th standard unit basis vector in $\mathbb{R}^{n}, \mathbf{1}$ be the all ones vector in $\mathbb{R}^{n}$ and $J$ be the all-ones matrix of dimension $n$. In order to discuss the determinant and inertia properties of distance matrices of weighted trees and unicyclic graphs, Bapat et al. [1] obtained a key observation as follows:

Theorem 6 [1]. Let $G$ be a connected weighted graph on $n$ vertices with distance matrix $D$, and suppose that $D \mathbf{1}=d \mathbf{1}$. Form $\bar{G}$ from $G$ by adding weighted branches to $G$ on a total of $m$ vertices, with positive weights $\alpha_{1}, \ldots, \alpha_{m}$ on the new edges. Let $\bar{D}$ be the distance matrix for $\bar{G}$. Then for each $x \in \mathbb{R}, \operatorname{det}(\bar{D}+x J)=$ $(-2)^{m} \operatorname{det}(D)\left(\prod_{i=1}^{m} \alpha_{i}\right)\left(1+\frac{n x}{d}+\frac{n}{2 d} \sum_{i=1}^{m} \alpha_{i}\right)$.

Now we determine the determinant of the $q$-distance matrix of $\bar{G}$ and thus generalize in part the above result.

Theorem 7. Let $G$ be a connected weighted graph on $n$ vertices with distance matrix $D_{q}(G)$, and suppose that $D_{q}(G) \mathbf{1}=d \mathbf{1}$. Let $\bar{G}$ be the graph obtained from $G$ by adding weighted branches to $G$ on a total of $m$ vertices, with positive weights $\alpha_{1}, \ldots, \alpha_{m}$ on the new edges. Then

$$
\begin{equation*}
\operatorname{det}\left(D_{q}(\bar{G})\right)=\prod_{i=1}^{m}\left(-\left[2 \alpha_{i}\right]\right)\left(1+\left(\frac{n}{d}+q-1\right) \sum_{j=1}^{m} \frac{\left[\alpha_{j}\right]}{1+q^{\alpha_{j}}}\right) \operatorname{det}\left(D_{q}(G)\right) \tag{2}
\end{equation*}
$$

Proof. According to the formation of $\bar{G}, G$ can not be a proper subgraph of any block of $\bar{G}$. By Theorem 2, we have

$$
\begin{align*}
\operatorname{det}\left(D_{q}(\bar{G})\right) & =\prod_{i=1}^{m}\left(-\left[2 \alpha_{i}\right]\right) \operatorname{det}\left(D_{q}(G)\right) \\
& +\xi\left(D_{q}(G)\right) \prod_{i=1}^{m}\left(-\left[2 \alpha_{i}\right]\right) \sum_{j=1}^{m} \frac{\left[\alpha_{j}\right]^{2}}{\left[2 \alpha_{j}\right]} \tag{3}
\end{align*}
$$

Now we will determine $\xi\left(D_{q}(G)\right)$ in terms of $\operatorname{det}\left(D_{q}(G)\right)$.
Let $G^{\prime}$ denote the graph obtained from $G$ by adding a pendant vertex. Without loss of generality, assume that the vertex $n+1$ is pendant, adjacent to vertex $n$, and that the weight of the corresponding pendant edge is $\alpha$. Add $-q^{\alpha}$ times the $n$th row and $n$th column to the last row and last column, respectively. Then

$$
D_{q}\left(G^{\prime}\right)=\left(\begin{array}{c|c}
I & 0 \\
\hline q^{\alpha} \varepsilon_{n}^{T} & 1
\end{array}\right)\left(\begin{array}{c|c}
D_{q}(G) & {[\alpha] \mathbf{1}} \\
\hline[\alpha] \mathbf{1}^{T} & -2 q^{\alpha}[\alpha]
\end{array}\right)\left(\begin{array}{c|c}
I & q^{\alpha} \varepsilon_{n} \\
\hline 0 & 1
\end{array}\right) .
$$

And so

$$
\operatorname{det}\left(D_{q}\left(G^{\prime}\right)\right)=\operatorname{det}\left(\begin{array}{c|c}
D_{q}(G) & {[\alpha] \mathbf{1}} \\
\hline[\alpha] \mathbf{1}^{T} & -2 q^{\alpha}[\alpha]
\end{array}\right)=\left(-2 q^{\alpha}[\alpha]\right) \operatorname{det}\left(D_{q}(G)+\frac{[\alpha]}{2 q^{\alpha}} J\right)
$$

where the second equality follows from Schur's formula. Note that the eigenvalues of $D_{q}(G)$ may be written as $d, \lambda_{2}, \ldots, \lambda_{n}$, while the eigenvalues of $D_{q}(G)+\frac{[\alpha]}{2 q^{\alpha}} J$ are $d+\frac{n[\alpha]}{2 q^{\alpha}}$ and $\lambda_{2}, \ldots, \lambda_{n}$. Then it follows from the preceding equation that

$$
\begin{align*}
\operatorname{det}\left(D_{q}\left(G^{\prime}\right)\right) & =\left(-2 q^{\alpha}[\alpha]\right)\left(d+\frac{n[\alpha]}{2 q^{\alpha}}\right) \prod_{j=2}^{n} \lambda_{j} \\
& =-\left(2 q^{\alpha}[\alpha]+\frac{n[\alpha]^{2}}{d}\right) \operatorname{det}\left(D_{q}(G)\right) \tag{4}
\end{align*}
$$

On the other hand, by Theorem 2, we have

$$
\begin{align*}
\operatorname{det}\left(D_{q}\left(G^{\prime}\right)\right) & =\operatorname{det}\left(D_{q}(G)\right) \xi\left(D_{q}\left(P_{2}\right)\right)+\operatorname{det}\left(D_{q}\left(P_{2}\right)\right) \xi\left(D_{q}(G)\right) \\
& =(-[2 \alpha]) \operatorname{det}\left(D_{q}(G)\right)+\left(-[\alpha]^{2}\right) \xi\left(D_{q}(G)\right) \tag{5}
\end{align*}
$$

Combining (4) and (5), we have

$$
\begin{equation*}
\xi\left(D_{q}(G)\right)=\left(\frac{n}{d}+q-1\right) \operatorname{det}\left(D_{q}(G)\right) \tag{6}
\end{equation*}
$$

and substitution in (3) implies that the assertion (2) holds.
Note that Corollary 4 can also be obtained in view of Theorem 7. The next two results deal with the determinant of $q$-distance matrix of the unicyclic graphs and one class of bicyclic graphs. We first recall some facts on circulant matrix.

A circulant matrix $C$ is a special kind of Toeplitz matrix having the form

$$
C=\left(\begin{array}{cccccc}
c_{0} & c_{1} & c_{2} & & \cdots & c_{n-1} \\
c_{n-1} & c_{0} & c_{1} & c_{2} & & \vdots \\
& c_{n-1} & c_{0} & c_{1} & \ddots & \\
\vdots & \ddots & \ddots & \ddots & & c_{2} \\
c_{1} & \cdots & & & c_{n-1} & c_{1} \\
c_{0}
\end{array}\right),
$$

where each row vector is rotated one element to the right relative to the preceding row vector. Note that a circulant matrix is fully specified by one vector and then is denoted by $\operatorname{Circ}\left(c_{0}, c_{1}, \ldots, c_{n-1}\right)$ by convention. The eigenvalues of a circulant matrix $C=\operatorname{Circ}\left(c_{0}, c_{1}, \ldots, c_{n-1}\right)$ are given by $\left\{f_{C}\left(\zeta^{j}\right) \mid j=0,1, \ldots, n-1\right\}$, where $f_{C}(x)=\sum_{i=0}^{n-1} c_{i} x^{i}$ and $\zeta=e^{\frac{2 \pi}{n} \mathrm{i}}$. Consequently, the determinant of circulant matrix $C$ can be determined as in the following result.
Lemma 8 [ 8$]$. Let $C=\operatorname{Circ}\left(c_{0}, c_{1}, \ldots, c_{n-1}\right)$ and $f_{C}(x)=\sum_{i=0}^{n-1} c_{i} x^{i}$. Then

$$
\operatorname{det}(C)=\prod_{j=0}^{n-1} f_{C}\left(\zeta^{j}\right)
$$

where $\zeta$ is the nth root of unity $e^{\frac{2 \pi}{n} \mathrm{i}}$.
As usual, the path and cycle of order $n$ are denoted by $P_{n}$ and $C_{n}$, respectively.
Theorem 9. Let $G$ be a unicyclic graph with $n+m$ vertices and cycle length $n$. Then

$$
\begin{equation*}
\operatorname{det}\left(D_{q}(G)\right)=(-1)^{m}(1+q)^{m-1}\left(1+q+m\left(\frac{n}{d}+q-1\right)\right) \operatorname{det}\left(D_{q}\left(C_{n}\right)\right) \tag{7}
\end{equation*}
$$

with
(8) $\operatorname{det}\left(D_{q}\left(C_{n}\right)\right)= \begin{cases}\prod_{s=0}^{2 k}\left(\sum_{r=1}^{k} 2[r] \cos \frac{2 r s s}{2 k+1}\right), & \text { if } n=2 k+1 ; \\ \prod_{s=0}^{2 k-1}\left(\sum_{r=1}^{k-1} 2[r] \cos \frac{2 r s \pi}{2 k}+(-1)^{s}[k]\right), & \text { if } n=2 k .\end{cases}$

Proof. Observe that $D_{q}\left(C_{n}\right)=\operatorname{Circ}(0,[1], \ldots,[k],[k], \ldots,[1])$ or $D_{q}\left(C_{n}\right)=\operatorname{Circ}$ $(0,[1], \ldots,[k],[k-1], \ldots,[1])$ depending on whether $n=2 k+1$ or $n=2 k$. Then the hypothesis of Theorem 7 applies to $G$, and so (7) follows immediately. By Lemma 8, the statement (8) holds obviously.

A bicyclic graph is a connected graph in which the number of edges equals the number of vertices plus one. Let $C_{p}$ and $C_{q}$ be two vertex-disjoint cycles. Suppose that $a_{1}$ is a vertex of $C_{p}$ and $a_{l}$ is a vertex of $C_{q}$. Joining $a_{1}$ and $a_{l}$ by a path $a_{1} a_{2} \cdots a_{l}$ of length $l-1$ results in a graph to be called an $\infty$-graph, where $l \geq 1$ and $l=1$ means identifying $a_{1}$ with $a_{l}$. Let $P_{r+1}, P_{s+1}$ and $P_{t+1}$ be three vertexdisjoint paths, where $r, s, t \geq 1$ and at most one of them is 1 . Identifying the three initial vertices and terminal vertices of them respectively results in a graph to be called a $\theta$-graph. The bicyclic graphs consist of two types of graphs: one type, denoted by $\mathcal{B}_{\infty}$, are those graphs each of which is an $\infty$-graph with trees attached; the other type, denoted by $\mathcal{B}_{\theta}$, are those graphs each of which is a $\theta$-graph with trees attached (one can see [6] for the details). Note that for any $G \in \mathcal{B}_{\infty}$, two of the blocks of $G$ are cycles and the remainder are $P_{2}$ 's.

Theorem 10. Let $G \in \mathcal{B}_{\infty}$ with $n$ vertices and two cycle blocks of $G$ be $C_{r}$ and $C_{s}$. Then
(9) $\operatorname{det}\left(D_{q}(G)\right)=(-(1+q))^{a}\left(c_{r}+c_{s}+\frac{a}{1+q} c_{r} c_{s}\right) \operatorname{det}\left(D_{q}\left(C_{r}\right)\right) \operatorname{det}\left(D_{q}\left(C_{s}\right)\right)$,
where $a=n+1-r-s, c_{r}=\frac{r}{d_{r}}+q-1, c_{s}=\frac{s}{d_{s}}+q-1$ and $d_{r}, d_{s}$ denote the row sum of $D_{q}\left(C_{r}\right), D_{q}\left(C_{s}\right)$ respectively.
Proof. Applying Theorem 2 to $G$, we have

$$
\begin{align*}
\operatorname{det}\left(D_{q}(G)\right) & =(-(1+q))^{a}\left(\operatorname{det}\left(D_{q}\left(C_{r}\right)\right) \xi\left(D_{q}\left(C_{s}\right)\right)+\operatorname{det}\left(D_{q}\left(C_{s}\right)\right) \xi\left(D_{q}\left(C_{r}\right)\right)\right) \\
0) & +(-1)^{a} a(1+q)^{a-1} \xi\left(D_{q}\left(C_{r}\right)\right) \xi\left(D_{q}\left(C_{s}\right)\right) . \tag{10}
\end{align*}
$$

Letting $G=C_{n}$ in (6), we have

$$
\begin{equation*}
\xi\left(D_{q}\left(C_{n}\right)\right)=\left(\frac{n}{d}+q-1\right) \operatorname{det}\left(D_{q}\left(C_{n}\right)\right) \tag{11}
\end{equation*}
$$

and substitutions in (10) for $n=r, s$ yield the conclusion (9).
According to a result of Bapat et al. [1],

$$
\operatorname{det}\left(D\left(C_{n}\right)\right)= \begin{cases}0, & \text { if } n \equiv 0(\bmod 2) ; \\ \left\lceil\frac{n}{2}\right\rceil\left\lfloor\frac{n}{2}\right\rfloor, & \text { otherwise }\end{cases}
$$

Now letting $q=1$ particularly in the above theorem, we obtain the following corollary, whose proof we omit.

Corollary 11. Let $G \in \mathcal{B}_{\infty}$ with $n$ vertices and two cycle blocks of $G$ be $C_{r}$ and $C_{s}$. Then

$$
\operatorname{det}(D(G))= \begin{cases}0, & \text { if } r s \equiv 0(\bmod 2) \\ (-2)^{a}\left(r\left\lceil\frac{s}{2}\right\rceil\left\lfloor\frac{s}{2}\right\rfloor+s\left\lceil\frac{r}{2}\right\rceil\left\lfloor\frac{r}{2}\right\rfloor+\frac{a}{2} r s\right), & \text { otherwise }\end{cases}
$$

where $a=n+1-r-s$.

## References

[1] R.B. Bapat, S. Kirkland and M. Neumann, On distance matrices and Laplacians, Linear Algebra Appl. 401 (2005) 193-209. doi:10.1016/j.laa.2004.05.011
[2] R.B. Bapat, A.K. Lal and S. Pati, A q-analogue of the distance matrix of a tree, Linear Algebra Appl. 416 (2006) 799-814. doi:10.1016/j.laa.2005.12.023
[3] R.B. Bapat and Pritha Rekhi, Inverses of $q$-distance matrices of a tree, Linear Algebra Appl. 431 (2009) 1932-1939. doi:10.1016/j.laa.2009.06.032
[4] R.L. Graham and H.O. Pollak, On the addressing problem for loop switching, Bell. System Tech. J. 50 (1971) 2495-2519.
[5] R.L. Graham, A.J. Hoffman and H. Hosoya, On the distance matrix of a directed graph, J. Graph Theory 1 (1977) 85-88. doi:10.1002/jgt. 3190010116
[6] S.G. Guo, The spectral radius of unicyclic and bicyclic graphs with $n$ vertices and $k$ pendant vertices, Linear Algebra Appl. 408 (2005) 78-85. doi:10.1016/j.laa.2005.05.022
[7] S. Sivasubramanian, A q-analogue of Graham, Hoffman and Hosoya's result, Electron. J. Combin. 17 (2010) \#21.
[8] P. Lancaster, Theory of Matrices (Academic Press, NY, 1969).
[9] W. Yan, Y.-N. Yeh, The determinants of q-distance matrices of trees and two quantities relating to permutations, Adv. in Appl. Math. 39 (2007) 311-321. doi:10.1016/j.aam.2006.04.002


[^0]:    ${ }^{1}$ Supported by National Natural Science Foundation of China (No. 11201198 and No. 11026143), Natural Science Foundation of Jiangxi Province (No.20132BAB201013), the Sponsored Program for Cultivating Youths of Outstanding Ability in Jiangxi Normal University. Corresponding author.
    ${ }^{2}$ Supported by the Provincial Natural Science Foundation of Jiangxi (No.2010GZS0093), the Young Growth Foundation of Jiangxi Normal University (No. 4555).

