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ON THE DETERMINANT OF q-DISTANCE MATRIX OF A GRAPH

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Abstract

In this note, we show how the determinant of the q-distance matrix $D_q(T)$ of a weighted directed graph G can be expressed in terms of the corresponding determinants for the blocks of G, and thus generalize the results obtained by Graham *et al.* [R.L. Graham, A.J. Hoffman and H. Hosoya, *On the distance matrix of a directed graph*, J. Graph Theory **1** (1977) 85–88]. Further, by means of the result, we determine the determinant of the q-distance matrix of the graph obtained from a connected weighted graph G by adding the weighted branches to G, and so generalize in part the results obtained by Bapat *et al.* [R.B. Bapat, S. Kirkland and M. Neumann, *On distance matrices and Laplacians*, Linear Algebra Appl. **401** (2005) 193–209]. In particular, as a consequence, determinantal formulae of q-distance matrices for unicyclic graphs and one class of bicyclic graphs are presented.

Keywords: q-distance matrix, determinant, weighted graph, directed graph.

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1. Introduction

We consider graphs which have no loops or parallel edges. A *weighted graph* is a (directed or undirected) graph in which each edge or arc is assigned a weight, which is a positive number. An unweighted graph, or simply a graph, is thus a weighted graph with each of the edges or arcs bearing weight 1. Let G be a weighted directed graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. For vertices v_i, v_j of G, the *distance* from v_i to v_j, denoted by d_{ij}, is defined to be the minimum weight of all paths from v_i to v_j, where the weight of a path is the sum of the weights of the arcs in that path. We shall assume that G is strongly connected so that d_{ij} always exists. The distance matrix $D(G)$ of G is an $n \times n$ matrix which has d_{ij} as its (i,j) entry.

Some q-analogs of the distance for a tree were considered in [2,9]. Now we generalize the notion for a general weighted directed graph. Let G be a weighted directed graph and suppose that the distance from u to v is α. Define the *q-distance* from u to v to be $[\alpha]$, where

$$[\alpha] = \begin{cases} \frac{1-q^\alpha}{1-q}, & \text{if } q \neq 1; \\ \alpha, & \text{otherwise.} \end{cases}$$

By definition, $[0] = 0$ and $[\alpha] = 1 + q + q^2 + \cdots + q^{\alpha-1}$ if α is a positive integer. The *q-distance matrix* $D_q(G)$ of G is the square matrix which has as its (i,j) entry the q-distance from v_i to v_j. If $q = 1$ then $D_q(G)$ is the distance matrix $D(G)$ of a graph G. Hence the distance matrix $D(G)$ is a special case of the q-distance matrix $D_q(G)$.

Distance matrices of graphs, particularly trees, have been extensively investigated in the literature. A classical result concerning the determinant of the distance matrix of a tree, due to Graham and Pollak [4], asserted that for a tree T_n on n vertices, $\det(T_n) = (-1)^{n-1}(n-1)2^{n-2}$. Thus, $\det(T_n)$ is a function dependent on n only, independent of the structure of T_n. Graham, Hoffman and Hosoya [5] studied further and obtained the following result. For a square matrix A, let $\mathrm{cof}(A)$ denote the sum of cofactors of A.

Theorem 1 [5]**.** *If G is a strongly connected directed graph with blocks $G_1, G_2, \ldots, G_r$, then*

$$\mathrm{cof}(D(G)) = \prod_{i=1}^{r} \mathrm{cof}(D(G_i)),$$

$$\det(D(G)) = \sum_{i=1}^{r} \det(D(G_i)) \prod_{j \neq i} \mathrm{cof}(D(G_j)).$$

Graham and Pollack determinantal formula for tree has been extended by Bapat *et al.* to the weighted case [1] and further by Yan *et al.* to the q-distance matrix of weighted tree [9]. We are not aware of Sivasubramanian's work [7] until we have finished the paper. It is worth pointing that our Theorem 2 is a generalization of Theorem 2 of [7] to the weighted case.

In this paper we show how the determinant of the q-distance matrix $D_q(T)$ of a weighted directed graph G can be expressed in terms of the corresponding determinants for the blocks of G. Our proof is basically the same as Graham's proof, but this indeed generalizes Graham *et al.'s* result to q-distance matrix case, and further, by applying the result we determine the determinant of the q-distance matrix of the graph obtained from a connected weighted graph G by adding the weighted branches to G, and so generalize in part the results obtained by Bapat *et al.* in [1]. In particular, as a consequence, determinantal formulae of q-distance matrices for unicyclic graphs and one class of bicyclic graphs are presented.

2. Main Results

We begin with some notation and definition. Let G be a strongly connected directed graph and D be its q-distance matrix which has its following form:

$$D = \left(\begin{array}{c|ccc} 0 & [\alpha_1] & \cdots & [\alpha_{n-1}] \\ \hline [\beta_1] & & & \\ \vdots & & D_1 & \\ [\beta_{n-1}] & & & \end{array}\right).$$

Denote by $\xi(D)$ the cofactor in position $(1,1)$ of the matrix obtained by subtracting the first row from all other rows, then p^{α_i} times the first column from the $(i+1)$th column of D for $i = 1, \ldots, n-1$. Observe that $\xi(D) = \det(D_1 - M)$, where M is the $(n-1) \times (n-1)$ matrix with $[\beta_i + \alpha_j]$ as its (i,j) entry (since $[\alpha + \beta] = p^{\beta}[\alpha] + [\beta] = [\alpha] + p^{\alpha}[\beta]$).

A *block* of a graph is defined to be a maximal subgraph having no cut vertices.

Theorem 2. *If G is a strongly connected directed graph with blocks $G_1, G_2, \ldots, G_r$, then*

$$\xi(D_q(G)) = \prod_{i=1}^{r} \xi(D_q(G_i)), \tag{1a}$$

$$\det(D_q(G)) = \sum_{i=1}^{r} \det(D_q(G_i)) \prod_{j \neq i} \xi(D_q(G_j)). \tag{1b}$$

Proof. We proceed by induction on r, the number of blocks of G. The theorem is trivial for $r = 1$ as G itself is a block in this case. Assume that it holds for all strongly connected directed graphs with fewer than r blocks, and let G be a strongly connected directed graph with r blocks. Then G is not a block and has at least one block which contains exactly one cut vertex of G, say G_1 with the unique cut vertex labeled by 0. Let $G_1^* = G - (G_1 - \{0\})$ be the remainder of G. Assume that $V(G_1) = \{0, 1, \ldots, m\}$ and $V(G_1^*) = \{0, m+1, \ldots, m+n\}$. Let

$$D_q(G_1) = \left(\begin{array}{c|ccc} 0 & [a_1] & \cdots & [a_m] \\ \hline [b_1] & & & \\ \vdots & & E & \\ [b_m] & & & \end{array}\right), \quad D_q(G_1^*) = \left(\begin{array}{c|ccc} 0 & [f_1] & \cdots & [f_n] \\ \hline [g_1] & & & \\ \vdots & & H & \\ [g_n] & & & \end{array}\right).$$

Thus we have

$$D_q(G) = \left(\begin{array}{c|c|c} 0 & \bar{a} & \bar{f} \\ \hline \bar{b} & E & ([b_i + f_j]) \\ \hline \bar{g} & ([g_i + a_j]) & H \end{array}\right),$$

where $\bar{a} = ([a_1], \ldots, [a_m])$, $\bar{b} = ([b_1], \ldots, [b_m])^T$, $\bar{f} = ([f_1], \ldots, [f_n])$ and $\bar{g} = ([g_1], \ldots, [g_n])^T$. Subtract q^{a_i} (q^{f_j}, resp.) times the first column from the $(i+1)$th ($(j+m+1)$th, resp.) column of $D_q(G)$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$; and also subtract the first row from every other row of $D_q(G)$. Then

$$\begin{aligned}
\det(D_q(G)) &= \det \left(\begin{array}{c|c|c} 0 & \bar{a} & \bar{f} \\ \hline \bar{b} & E - ([b_i + a_j]) & 0 \\ \hline \bar{g} & 0 & H - ([g_i + f_j]) \end{array}\right) \\
&= \det \left(\begin{array}{c|c} 0 & \bar{a} \\ \hline \bar{b} & E - ([b_i + a_j]) \end{array}\right) \det(H - ([g_i + f_j])) \\
&+ \det \left(\begin{array}{c|c} 0 & \bar{f} \\ \hline \bar{g} & H - ([g_i + f_j]) \end{array}\right) \det(E - ([b_i + a_j])) \\
&= \det(D_q(G_1))\xi(D_q(D_1^*)) + \det(D_q(G_1^*))\xi(D_q(D_1)),
\end{aligned}$$

where the second equality follows by Laplace expansion of determinants. Also we note that

$$\begin{aligned}
\xi(D_q(G)) &= \det \left(\begin{array}{c|c} E - ([b_i + a_j]) & 0 \\ \hline 0 & H - ([g_i + f_j]) \end{array}\right) \\
&= \det(E - ([b_i + a_j])) \det(H - ([g_i + f_j])) \\
&= \xi(D_q(D_1))\xi(D_q(D_1^*)).
\end{aligned}$$

By the induction hypothesis, the assertion (1a) and (1b) follow immediately. ■

Let $\vec{T}$ be a directed graph obtained from a tree of order n by replacing each undirected edge $f_i = \{u, v\}$ with two arcs (oppositely oriented edges) $e_i = (u, v)$ and $e_i' = (v, u)$. Let $u_i > 0$ and $v_i > 0$ be the weights of the arcs e_i and e_i', respectively. Note that $\vec{T}$ is a strongly connected graph consisting of $n-1$ blocks, denoted by $G_1, G_2, \ldots, G_{n-1}$. Observe that each G_i actually consists of two opposite arcs, say e_i and e_i'. As $D_q(G_i) = \begin{pmatrix} 0 & [u_i] \\ [v_i] & 0 \end{pmatrix}$, $\det(D_q(G_i)) = -[u_i][v_i]$ and $\xi(D_q(G_i)) = -[u_i + v_i]$. Applying Theorem 2 to $\vec{T}$, we have $\xi(D_q(\vec{T})) = \prod_{i=1}^{n-1}(-[u_i + v_i])$ and the following result.

Theorem 3. *Let $\vec{T}$ be the directed graph on n vertices constructed as above. Then*

$$\det(D_q(\vec{T})) = (-1)^{n-1} \prod_{i=1}^{n-1}([u_i + v_i]) \sum_{i=1}^{n-1} \frac{[u_i][v_i]}{[u_i + v_i]}.$$

Theorem 3 yields the following generalization of results of Yan and Yeh [9] and also Bapat and Rekhi [3]. This can easily be seen if we replace each undirected edge in a tree by two arcs of opposite orientations and then apply Theorem 3 to the obtained directed graph.

Corollary 4 [3]**.** *Let T be a weighted tree with n vertices and weights $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. Then*

$$\det(D_q(T)) = (-1)^{n-1} \prod_{i=1}^{n-1} [2\alpha_i] \sum_{i=1}^{n-1} \frac{[\alpha_i]}{1 + q^{\alpha_i}}.$$

In particular, by letting $q = 1$ in Corollary 4, we obtain the following result.

Corollary 5 [1]**.** *Let T be a weighted tree with n vertices and weights $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. Then*

$$\det(D(T)) = (-1)^{n-1} 2^{n-2} \left(\prod_{i=1}^{n-1} \alpha_i\right) \left(\sum_{i=1}^{n-1} \alpha_i\right).$$

Next we shall focus ourself on undirected graphs. Let G be a weighted graph, and suppose that we have a collection of weighted trees $B_1, \ldots, B_k$. Let $\bar{G}$ be the graph obtained from G and $B_1, \ldots, B_k$ by adding, for each $i = 1, \ldots, k$, a weighted edge between some vertex of B_i and some vertex of G. We say that the new graph $\bar{G}$ is *constructed by adding the weighted branches $B_1, \ldots, B_k$ to G*. Note that trees, unicyclic graphs and bicyclic graphs all can be constructed in this way. Let ε_n be the nth standard unit basis vector in $\mathbb{R}^n$, $\mathbf{1}$ be the all ones vector in $\mathbb{R}^n$ and J be the all-ones matrix of dimension n. In order to discuss the determinant and inertia properties of distance matrices of weighted trees and unicyclic graphs, Bapat *et al.* [1] obtained a key observation as follows:

Theorem 6 [1]**.** *Let G be a connected weighted graph on n vertices with distance matrix D, and suppose that $D\mathbf{1} = d\mathbf{1}$. Form $\bar{G}$ from G by adding weighted branches to G on a total of m vertices, with positive weights $\alpha_1, \ldots, \alpha_m$ on the new edges. Let $\bar{D}$ be the distance matrix for $\bar{G}$. Then for each $x \in \mathbb{R}$, $\det(\bar{D} + xJ) = (-2)^m \det(D)(\prod_{i=1}^{m} \alpha_i)(1 + \frac{nx}{d} + \frac{n}{2d} \sum_{i=1}^{m} \alpha_i)$.*

Now we determine the determinant of the q-distance matrix of $\bar{G}$ and thus generalize in part the above result.

Theorem 7. *Let G be a connected weighted graph on n vertices with distance matrix $D_q(G)$, and suppose that $D_q(G)\mathbf{1} = d\mathbf{1}$. Let $\bar{G}$ be the graph obtained from G by adding weighted branches to G on a total of m vertices, with positive weights $\alpha_1, \ldots, \alpha_m$ on the new edges. Then*

$$\det(D_q(\bar{G})) = \prod_{i=1}^{m}(-[2\alpha_i])\left(1+\left(\frac{n}{d}+q-1\right)\sum_{j=1}^{m}\frac{[\alpha_j]}{1+q^{\alpha_j}}\right)\det(D_q(G)). \tag{2}$$

Proof. According to the formation of $\bar{G}$, G can not be a proper subgraph of any block of $\bar{G}$. By Theorem 2, we have

$$\begin{aligned}\det(D_q(\bar{G})) =& \prod_{i=1}^{m}(-[2\alpha_i])\det(D_q(G))\\ &+\xi(D_q(G))\prod_{i=1}^{m}(-[2\alpha_i])\sum_{j=1}^{m}\frac{[\alpha_j]^2}{[2\alpha_j]}.\end{aligned} \tag{3}$$

Now we will determine $\xi(D_q(G))$ in terms of $\det(D_q(G))$.

Let G' denote the graph obtained from G by adding a pendant vertex. Without loss of generality, assume that the vertex $n+1$ is pendant, adjacent to vertex n, and that the weight of the corresponding pendant edge is α. Add $-q^{\alpha}$ times the nth row and nth column to the last row and last column, respectively. Then

$$D_q(G') = \left(\begin{array}{c|c} I & 0 \\ \hline q^{\alpha}\varepsilon_n^T & 1\end{array}\right)\left(\begin{array}{c|c} D_q(G) & [\alpha]\mathbf{1} \\ \hline [\alpha]\mathbf{1}^T & -2q^{\alpha}[\alpha]\end{array}\right)\left(\begin{array}{c|c} I & q^{\alpha}\varepsilon_n \\ \hline 0 & 1\end{array}\right).$$

And so

$$\det(D_q(G')) = \det\left(\begin{array}{c|c} D_q(G) & [\alpha]\mathbf{1} \\ \hline [\alpha]\mathbf{1}^T & -2q^{\alpha}[\alpha]\end{array}\right) = (-2q^{\alpha}[\alpha])\det(D_q(G)+\frac{[\alpha]}{2q^{\alpha}}J),$$

where the second equality follows from Schur's formula. Note that the eigenvalues of $D_q(G)$ may be written as $d, \lambda_2, \ldots, \lambda_n$, while the eigenvalues of $D_q(G)+\frac{[\alpha]}{2q^{\alpha}}J$ are $d+\frac{n[\alpha]}{2q^{\alpha}}$ and $\lambda_2, \ldots, \lambda_n$. Then it follows from the preceding equation that

$$\begin{aligned}\det(D_q(G')) =& (-2q^{\alpha}[\alpha])\left(d+\frac{n[\alpha]}{2q^{\alpha}}\right)\prod_{j=2}^{n}\lambda_j\\ =& -\left(2q^{\alpha}[\alpha]+\frac{n[\alpha]^2}{d}\right)\det(D_q(G)).\end{aligned} \tag{4}$$

On the other hand, by Theorem 2, we have

$$\begin{aligned}\det(D_q(G')) =& \det(D_q(G))\xi(D_q(P_2)) + \det(D_q(P_2))\xi(D_q(G))\\ =& (-[2\alpha])\det(D_q(G)) + (-[\alpha]^2)\xi(D_q(G)).\end{aligned} \tag{5}$$

Combining (4) and (5), we have

$$\xi(D_q(G)) = \left(\frac{n}{d} + q - 1\right) \det(D_q(G)) \tag{6}$$

and substitution in (3) implies that the assertion (2) holds. ∎

Note that Corollary 4 can also be obtained in view of Theorem 7. The next two results deal with the determinant of q-distance matrix of the unicyclic graphs and one class of bicyclic graphs. We first recall some facts on circulant matrix.

A *circulant matrix* C is a special kind of Toeplitz matrix having the form

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & c_2 & & \vdots \\ & c_{n-1} & c_0 & c_1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & & c_2 \\ & & & & & c_1 \\ c_1 & \cdots & & & c_{n-1} & c_0 \end{pmatrix},$$

where each row vector is rotated one element to the right relative to the preceding row vector. Note that a circulant matrix is fully specified by one vector and then is denoted by $\mathrm{Circ}(c_0, c_1, \ldots, c_{n-1})$ by convention. The eigenvalues of a circulant matrix $C = \mathrm{Circ}(c_0, c_1, \ldots, c_{n-1})$ are given by $\{f_C(\zeta^j) | j = 0, 1, \ldots, n-1\}$, where $f_C(x) = \sum_{i=0}^{n-1} c_i x^i$ and $\zeta = e^{\frac{2\pi}{n}\mathrm{i}}$. Consequently, the determinant of circulant matrix C can be determined as in the following result.

Lemma 8 [8]**.** *Let* $C = \mathrm{Circ}(c_0, c_1, \ldots, c_{n-1})$ *and* $f_C(x) = \sum_{i=0}^{n-1} c_i x^i$. *Then*

$$\det(C) = \prod_{j=0}^{n-1} f_C(\zeta^j),$$

where ζ *is the nth root of unity* $e^{\frac{2\pi}{n}\mathrm{i}}$.

As usual, the path and cycle of order n are denoted by P_n and C_n, respectively.

Theorem 9. *Let G be a unicyclic graph with $n + m$ vertices and cycle length n. Then*

$$\det(D_q(G)) = (-1)^m (1+q)^{m-1} \left(1 + q + m\left(\frac{n}{d} + q - 1\right)\right) \det(D_q(C_n)) \tag{7}$$

with

$$\det(D_q(C_n)) = \begin{cases} \prod_{s=0}^{2k} \left(\sum_{r=1}^{k} 2[r] \cos \frac{2rs\pi}{2k+1}\right), & \text{if } n = 2k+1; \\ \prod_{s=0}^{2k-1} \left(\sum_{r=1}^{k-1} 2[r] \cos \frac{2rs\pi}{2k} + (-1)^s [k]\right), & \text{if } n = 2k. \end{cases} \tag{8}$$

Proof. Observe that $D_q(C_n) = \text{Circ}(0, [1], \ldots, [k], [k], \ldots, [1])$ or $D_q(C_n) = \text{Circ}$ $(0, [1], \ldots, [k], [k-1], \ldots, [1])$ depending on whether $n = 2k+1$ or $n = 2k$. Then the hypothesis of Theorem 7 applies to G, and so (7) follows immediately. By Lemma 8, the statement (8) holds obviously. ■

A bicyclic graph is a connected graph in which the number of edges equals the number of vertices plus one. Let C_p and C_q be two vertex-disjoint cycles. Suppose that a_1 is a vertex of C_p and a_l is a vertex of C_q. Joining a_1 and a_l by a path $a_1 a_2 \cdots a_l$ of length $l-1$ results in a graph to be called an ∞-graph, where $l \geq 1$ and $l = 1$ means identifying a_1 with a_l. Let P_{r+1}, P_{s+1} and P_{t+1} be three vertex-disjoint paths, where $r, s, t \geq 1$ and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them respectively results in a graph to be called a θ-graph. The bicyclic graphs consist of two types of graphs: one type, denoted by $\mathcal{B}_\infty$, are those graphs each of which is an ∞-graph with trees attached; the other type, denoted by $\mathcal{B}_\theta$, are those graphs each of which is a θ-graph with trees attached (one can see [6] for the details). Note that for any $G \in \mathcal{B}_\infty$, two of the blocks of G are cycles and the remainder are P_2's.

Theorem 10. *Let $G \in \mathcal{B}_\infty$ with n vertices and two cycle blocks of G be C_r and C_s. Then*

$$\det(D_q(G)) = (-(1+q))^a \left(c_r + c_s + \frac{a}{1+q} c_r c_s\right) \det(D_q(C_r)) \det(D_q(C_s)), \tag{9}$$

where $a = n+1-r-s$, $c_r = \frac{r}{d_r} + q - 1$, $c_s = \frac{s}{d_s} + q - 1$ and d_r, d_s denote the row sum of $D_q(C_r), D_q(C_s)$ respectively.

Proof. Applying Theorem 2 to G, we have

$$\begin{aligned}\det(D_q(G)) = (-(1+q))^a (&\det(D_q(C_r))\xi(D_q(C_s)) + \det(D_q(C_s))\xi(D_q(C_r))) \\ &+ (-1)^a a(1+q)^{a-1} \xi(D_q(C_r))\xi(D_q(C_s)).\end{aligned} \tag{10}$$

Letting $G = C_n$ in (6), we have

$$\xi(D_q(C_n)) = \left(\frac{n}{d} + q - 1\right) \det(D_q(C_n)) \tag{11}$$

and substitutions in (10) for $n = r, s$ yield the conclusion (9). ■

According to a result of Bapat *et al.* [1],

$$\det(D(C_n)) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2; \\ \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor, & \text{otherwise.} \end{cases}$$

Now letting $q = 1$ particularly in the above theorem, we obtain the following corollary, whose proof we omit.

Corollary 11. *Let $G \in \mathcal{B}_\infty$ with n vertices and two cycle blocks of G be C_r and C_s. Then*

$$\det(D(G)) = \begin{cases} 0, & \text{if } rs \equiv 0 \,(\bmod 2); \\ (-2)^a \left(r\lceil \frac{s}{2}\rceil\lfloor \frac{s}{2}\rfloor + s\lceil \frac{r}{2}\rceil\lfloor \frac{r}{2}\rfloor + \frac{a}{2}rs\right), & \text{otherwise,} \end{cases}$$

where $a = n + 1 - r - s$.

References

[1] R.B. Bapat, S. Kirkland and M. Neumann, *On distance matrices and Laplacians*, Linear Algebra Appl. **401** (2005) 193–209. doi:10.1016/j.laa.2004.05.011

[2] R.B. Bapat, A.K. Lal and S. Pati, *A q-analogue of the distance matrix of a tree*, Linear Algebra Appl. **416** (2006) 799–814. doi:10.1016/j.laa.2005.12.023

[3] R.B. Bapat and Pritha Rekhi, *Inverses of q-distance matrices of a tree*, Linear Algebra Appl. **431** (2009) 1932–1939. doi:10.1016/j.laa.2009.06.032

[4] R.L. Graham and H.O. Pollak, *On the addressing problem for loop switching*, Bell. System Tech. J. **50** (1971) 2495–2519.

[5] R.L. Graham, A.J. Hoffman and H. Hosoya, *On the distance matrix of a directed graph*, J. Graph Theory **1** (1977) 85–88. doi:10.1002/jgt.3190010116

[6] S.G. Guo, *The spectral radius of unicyclic and bicyclic graphs with n vertices and k pendant vertices*, Linear Algebra Appl. **408** (2005) 78–85. doi:10.1016/j.laa.2005.05.022

[7] S. Sivasubramanian, *A q-analogue of Graham, Hoffman and Hosoya's result*, Electron. J. Combin. **17** (2010) #21.

[8] P. Lancaster, Theory of Matrices (Academic Press, NY, 1969).

[9] W. Yan, Y.-N. Yeh, *The determinants of q-distance matrices of trees and two quantities relating to permutations*, Adv. in Appl. Math. **39** (2007) 311–321. doi:10.1016/j.aam.2006.04.002

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